Analysis of Stratified Lake using an Eddy Diffusion and a Mixed-layer Models

Kim, Kyung-Sub*

ABSTRACT : A one-dimensional eddy diffusion model and a mixed-layer model are developed and applied to simulate the vertical temperature profiles in lakes. Also the running result of each method are compared and analyzed. In an eddy diffusion model, molecular diffusivity is neglected and eddy diffusivity which does not need lake-specific fitting parameter and constant lake's level are applied. The heat exchanges at the water surface and the bottom are formulated by the energy balance and zero energy gradient, respectively. In a mixed-layer model, two layers approach which has a constant thickness is adopted. The application of these models which use explicit finite difference and Runge-Kutta methods respectively demonstrates that the models simulate water temperatures efficiently.

1. Introduction

The water temperature is particularly significant, because the variation of the water temperature affects the aquatic ecosystem directly or indirectly. Appropriate growth and reproduction of plants in agricultural sites depend on the temperature of irrigation water. The rates of chemical and biological reactions are influenced by the water temperature. Thus, the water temperature, which is relevant to the modeling of the fate of pollutants, is an important water-quality parameter. In dimictic lakes, water circulates freely twice a year in spring and fall and are directly stratified in summer and inversely stratified in winter. The vertical temperature profiles in lakes show a characteristic yearly cycle.

Many scientists and engineers have studied mathematical models to predict the vertical temperature profiles in lakes. Two general classes are in use : mixed-layer models and eddy diffusion models. The mixed-layer models use a mechanical energy balance to predict the thickness of the epilimnion. This thick surface layer is then modeled as a well-mixed segment such as a CSTR(Continuous Stirred Tank Reactor) (Chapra and Reckhow, 1983; Chapra, 1996). A simple approach uses two layers-surface and bottom. The eddy diffusion models have been used successfully to simulate thermal stratification in a variety of limnological studies (Babajimopoulos

* Assistant Professor, Dept. of Env. Engr. , Anseong National University

and Papadopoulos, 1986; Hostetler and Bartlein, 1990). This approach requires specifying values of the eddy diffusivities, which are highly variable in the vertical direction. In the past, these values were determined as the neutral eddy diffusivity multiplied by some function of a stability parameter. Several formulations for the neutral eddy diffusivity were presented by Henderson-Sellers (1976), which requires the velocity profiles in lakes. Later Henderson-Sellers (1985) developed another method to determine the eddy diffusivity, which requires no lake-specific fitting parameters.

The purpose of this paper is to present a one-dimensional eddy diffusion and mixed-layer models to predict lake temperatures. This realistic model to adequately simulate the temperature profiles for all seasons can be used to base subsequent integration of biological and chemical mechanisms including eutrophication process.

2. An Eddy Diffusion Model and a Mixed-layer Model

2. 1 An Eddy Diffusion Model

The nonlinear heat transfer equation assuming horizontal homogeneity is given by

$$\frac{\partial T}{\partial t} = \frac{1}{A} \frac{\partial}{\partial z} [A(\mu+K) \frac{\partial T}{\partial z}] - \frac{1}{\rho c A} \frac{\partial (HA)}{\partial z} \tag{1}$$

where T is temperature (℃); t is time (day); z is the vertical coordinate increased downward from the water surface (m); A is the cross-sectional area of the lake (m^2); K is the eddy diffusivity ($m^2\ d^{-1}$); μ is molecular diffusivity of water ($m^2\ d^{-1}$); ρ is density of water (Kg ; c is specific heat of water ($J\ Kg^{-1}\ ℃^{-1}$); and H is heat source term ($J\ m^{-2}\ d^{-1}$), which is the heat flux rate generated from internal absorption of solar radiation. This term at any depth z is determined by the following equation

$$H = (1-\beta)(H_s - H_{sr})e^{-\eta z} \tag{2}$$

where β is the portion of the solar radiation absorbed in the surface layer which is generally supposed as 0.6 m in thickness; H_s is shortwave solar radiation ($J\ m^{-2}\ d^{-1}$); H_{sr} is reflected shortwave solar radiation ($J\ m^{-2}\ d^{-1}$); and η is extinction coefficient (m^{-1}). If molecular diffusion is neglected, i. e. turbulent diffusion is mainly responsible for the vertical heat transfer, Eq. (1) is slightly changed.

2.1.1 Initial and Boundary Conditions

One initial and two boundary conditions are essential to solve Eq. (1). The initial condition used in this paper is provided by specifying the uniform temperature, that is :

$$T(z, 0) = T_o \tag{3}$$

This condition holds on early spring season in dimictic lakes.

Two boundary conditions are specificed at the bottom and top of the lake. The boundary condition at the bottom is assumed that the heat flux is zero:

$$-\rho c(K\frac{\partial T}{\partial z})_{z=b} = 0 \tag{4}$$

where b is the depth of the lake. When the solar radiation reaches the bottom of the shallow lake, the different boundary condition should be incorporated. Another boundary condition, describing the heat exchange between the lake surface and the air, is written as

$$-\rho c(K\frac{\partial T}{\partial z})_{z=0} = [\beta(H_s - H_{sr}) + H_a - H_{ar}] - (H_{br} \pm H_c \pm H_e) \tag{5}$$

where H_a is longwave solar radiation (J m^{-2} d^{-1}); H_{ar} is the reflected longwave radiation (J m^{-2} d^{-1}); H_{br} is longwave radiation emitted by the water (J m^{-2} d^{-1}); and H_c and H_e are heat fluxes from the water surface due to conduction and evaporation (J m^{-2} d^{-1}). The first group of terms in brackets on the right-hand side of Eq. (5) is independent of the water temperature and the second group of terms depends in various ways on the water temperature.

2.1.2 Eddy Diffusivity

The method of Henderson-Sellers (1985), which does not require lake-specific fitting parameters, is used. The eddy diffusivity is given by

$$K(z) = 86400\frac{kw^*z}{P_o}e^{-k^*z}(1+37R_i^2)^{-1} \tag{6}$$

where k is von Karman's constant (0.4); w^* is the surface value of the friction velocity (m s^{-1}); P_o is the neutral value of the turbulent Prandtl number (1.0); k^* is a latitudinally dependent parameter of the Ekman profile (m^{-1}); and R_i is the gradient Richardson number. The gradient

Richardson number is given by

$$R_i = [-1+(1+\frac{40N^2 k^2 z^2}{w^{*2} e^{-2k^* z}})^{0.5}]/20 \quad (7)$$

where the Brunt-Vaisala frequency N is specified as

$$N = (-\frac{g}{\rho}\frac{\partial \rho}{\partial z})^{0.5} \quad (8)$$

where g is gravity acceleration (9.8 m s^{-2}). More details are found in the foregoing paper. The temperature dependency of water density is approximated by a formula with an accuracy of 0.03% or better (Heggen, 1983) :

$$\rho = (1-1.9549\times 10^{-5} \mid T-4 \mid ^{1.68})10^3 \quad (9)$$

The specific heat of water is a weak function of temperature given by

$$c(T) = 4187.6[0.99716 + 3.979\times 10^{-4} f(T)] \quad (10)$$

where

$$f(T) = e^{(r/10.6)} + e^{(-r/10.6)} \quad (11)$$

in which

$$r = 34.5-T \qquad T < 34.5℃ \quad (12)$$
$$r = 2.08(T-34.5)^{0.67} \qquad T \geq 34.5℃ \quad (13)$$

2.2 A Mixed-layer Model

A simple model for this case consists of two well-mixed layers of constant thickness separated by an interface across which diffusive transport occurs. The metalimnion(thermocline) acts as the interface between the surface and the bottom layers. For such a system, each of the layers is assumed as a CSTR and heat balances can be written for each layer as

$$V_e \rho_e c_e \frac{dT_e}{dt} = Q \rho_e c_e T_{in}(t) - Q \rho_e c_e T_e \pm A_s \Delta H + v_t A_t \rho_e c_e (T_h - T_e) \quad (14)$$

$$V_h \rho_h c_h \frac{dT_h}{dt} = v_t A_t \rho_h c_h (T_e - T_h) \quad (15)$$

where the subscripts e and h designate the epilimnion and hypolimnion, respectively ; Q is the volumetric flow rate entering the system ($m^3 s^{-1}$); $T_{in}(t)$ is the average inflow temperature ; A_s is the lake's surface area (m^2) ; H is surface heat flux ($J m^{-2} d^{-1}$); v_t is the thermocline heat transfer coefficient ($m s^{-1}$) ; and A_t is the thermocline area (m^2). Surface heat flux is given as

$$\Delta H = H_s - H_{sr} + H_a - H_{ar} - (H_{br} \pm H_c \pm H_e) \quad (16)$$

In a similiar fashion of an eddy diffusion model, initial condition given by Eq. (3) is needed to solve the governing equations.

2. 3 Surface Heat Fluxes

The overview of the surface heat fluxes is described. Empirical and theoretical equations were developed for each individual heat flux (Jobson, 1980; Henderson-Sellers, 1986). Edinger *et al.* (1974) provided an excellent and comprehensive report of this research. Thomann and Mueller (1987) have summarized the fundamental approach to water quality modeling.

2.3.1 Shortwave Radiation (Hs and Hsr)

The incoming solar radiation H_s is estimated by the equation

$$H_s = 86400(\bar{a} + \bar{b} \frac{n}{D}) R_\infty \quad (17)$$

where n is the number of sunshine hours; D is the maximum value of n : $\bar{a}$ and $\bar{b}$ are the parameters relating latitude ($0.18 \le \bar{a} \le 0.33$, $0.37 \le \bar{b} \le 0.62$); and R_∞ is the value of shortwave radiation flux (a daily total averaged over 24 hours) at the top of the atmosphere ($W m^{-2}$) and given by

$$R_\infty = 433.7477 \bar{d}^2 (d \sin\varphi \sin\delta + \cos\varphi \cos\delta \sin d) \quad (18)$$

where d is the half day length expressed as an angle (rad) ;$\bar{d}$ is the ratio of the average to the instantaneous earth-sun distance ; φ is the latitude (rad.); and δ is the solar decline (rad). The first and last two parameters are determined from the following equations

$$d = acos(-\tan\varphi \tan\delta) \quad (19)$$

$$\bar{d} = [1-0.01673 \cos(\frac{2\pi I}{365})]^{-1} \quad (20)$$

$$\delta = 0.4093 \sin[\frac{2\pi(I-79.75)}{365}] \quad (21)$$

where I is Julian date. The parameter n/D which is a measure of the duration of sunshine amount is determined from

$$n/D = 1-C \quad (22)$$

where C is the cloud cover fraction.

The reflected shortwave solar radiation H_{sr} is estimated by the equation

$$H_{sr} = a\, H_s \quad (23)$$

where a is the albedo of the water which varies between of 0.03 and 0.1. The average value of a =0.07 which Babajimopoulos and Papadopoulos (1986) selected is used in this paper.

2.3.2 Longwave Radiation(Ha and Har)

The longwave radiation H_a can be measured or esimated from the following equation

$$H_a = 4.45\times10^{-8}\,(T_a+273)^6\,(1+0.17C^2) \quad (24)$$

where T_a is the dry bulb air temperature (°C). The reflected longwave radiation H_{ar} is generally small; about 3% of the incoming longwave radiation.

2.3.3 Longwave Radiation Emitted by the Water (Hbr)

The longwave radiation H_{br} emitted by the water follows the Stefan-Boltzmann law for a black body and is given by

$$H_{br} = \varepsilon\,\sigma(T_a + 273)^4 \quad (25)$$

where ε is the emissivity of water (0.97); and σ is the Stefan-Boltzmann constant (4.899×10^{-3} J $m^{-2}\,d^{-1}\,°K^{-4}$).

2.3.4 Conductive Heat Transfer (H_c)

The rate of conductive heat transfer H_c depends on the temperature difference between the water and the air as well as the wind speed over the water

$$H_c = 41876\, c_1(19.0+0.95\, U_7^2)\ (T-T_a) \tag{26}$$

where c_1 is Bowen's coefficient (0.47 mmHg °C^{-1}) and U_7 is the wind speed measured at the hight of 7 m above the water (m s^{-1}). We know that H_c is positive when the water temperature is greater than the air temperature.

2.3.5 Evaporation (He)

Edinger *et al.* (1974) have suggested the following formula to compute the rate of heat loss by evaporation H_e

$$H_e = 41876(19.0+0.95\, U_7^2)\ (e_{sat}-e_a) \tag{27}$$

where e_{sat} is the saturated vapor pressure of water (mmHg); and e_a is the air-vapor pressure (mmHg).

The air-vapor pressure is defined as the multiplication of the saturated vapor pressure, e_{sat} at the air temperature and the fraction relative humidity. Raudkivi (1979) developed the following equation to calculate the e_{sat}

$$e_{sat} = 611e^{\left(\frac{17.27T_a}{237.3+T_a}\right)} \tag{28}$$

where the unit of e_{sat} is pascals. Generally, dew-point temperature is measured along with other meteorological variables. The following relationship can be used to calculate the reative humidity

$$R' = e^{\left(\frac{T_d-T_a}{15.7945}\right)} \tag{29}$$

where T_d is the dew-point temperature (°C). A detailed account of the above equation is found in Kim (1993).

2. 4 Numerical-Formulations

A fourth-order Runge-Kutta method and an explicit finite difference method are used for a mixed-layer model and an eddy diffusion model respectively to solve the governing equation

(Chapra and Canale, 1988). To meet the numerical stability criterion of an explicit finite difference method, the time step is restricted by the equation

$$\Delta t \leq \frac{\Delta z^2}{2K} \tag{30}$$

where Δt is the time step (day); and Δz is the space step (m).

3. Applications and Discussion

The models described in the foregoing section were applied to the Lake Calhoun data set in the P. C. machine (Pentium processor). Lake Calhoun is located in the Minneapolis-St. Paul metropolitan area in Minesota at a northern latitude of 45°(U.S.A.). It has the surface area of 1.71 km^2, the maxium depth of 25m, and the mean depth of 10 m. Fig. 1 shows cross-sectional area of this lake. It is known that in central Minesota the average annual loss to evaporation is balanced by precipitation. Thus, it is assumed that the level variations are equal to zero over simulation period. If the water budget is relatively important, another equation should be incororated to deal with level variations. The model was simulated for 214 days, beginning April 15, 1974 (the ice goes out), when the field data shows that the lake is isothermal of about 6.4℃, and ending November 15, 1974 (the lake starts to freeze). The meteorological input data such as air temperature, relative humidity, wind speed, and cloud cover were shown in Table 1 and were obtained from Ford (1976). Monthly average values of the meteorological data were fitted linearly to derive input values.

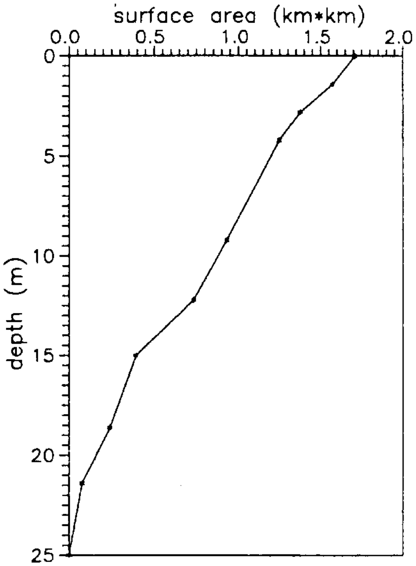

Fig. 1 Cross-sectional Area of Lake Calhoun

Table 1. Meteorogical Data

month	air temp.(°C)	rel. hun.(%)	wind speed(m/s)	cloud cover
4	8.4	58.8	5.3	0.7
5	12.4	64.5	5.1	0.8
6	18.6	61.3	4.3	0.6
7	24.8	58.8	4.5	0.5
8	19.6	69.0	4.5	0.6
9	12.9	65.0	4.1	0.5
10	9.9	66.5	4.2	0.6
11	0.9	78.0	4.4	0.7

3. 1 Eddy Diffusion Model

The space step was 1 m. The effect of the parameters $\bar{a}$, $\bar{b}$, β, and η were investigated by several different values. Compared with the measured temperature values, the parameters $\bar{a}$, $\bar{b}$, β, and η which produced the best results, were chosen as 0.33, 0.6, 0.5, and 0.3 respectively. Kim (1995) found that the temperature profiles are dependent upon the values of β and η, and higher values of β and η result in the thinner epilimnion depth. Also, the different values of β and η cause the change of metalimnion depth and the variation of β changes little the temperature values of epilimnion and hypolimnion layers. In Fig. 2, it is known that the seasonal cycle of the surface water temperature was simulated quite well. Fig. 3 shows the vertical profiles of temperature at specified dates.

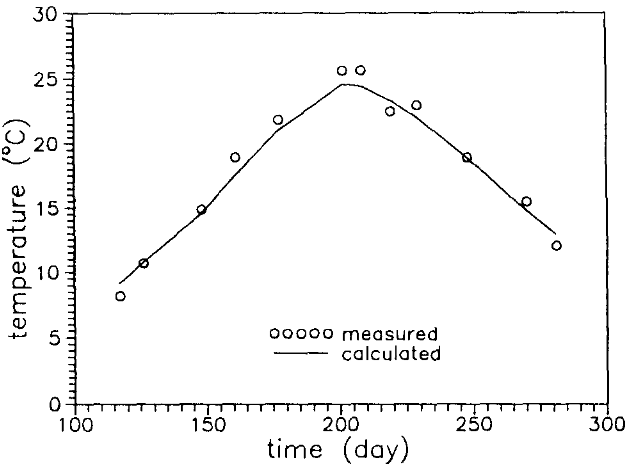

Fig. 2 Surface Water Temperature Distribution in Lake Calhoun

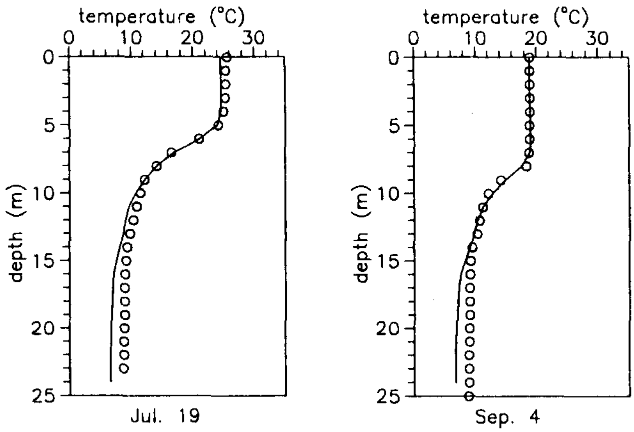

Fig. 3 Comparison of Measured and Caculated Temperature Profilles

3. 2 Mixed-layer Model

3.2.1 Thermocline Heat Transfer Coefficient

The thermocline heat transfer coefficient can be estimated by using temperature values in each layer. If the epilimnion temperature is constant during the stratified period, Eq. (15) can be written as

$$\frac{dT_h}{dt} + \lambda_h T_h = \lambda_h \overline{T}_e \tag{31}$$

where $\overline{T}$ is the constant epilimnion temperature and λ_h is the eigen value for the hypolimnion ($= v_t A_t / V_h$). If the hypolimnion temperature at the beginning of the summer period is $T_{h,i}$, Eq. (31) can be solved for

$$T_h = T_{h,i} e^{-\lambda_h t} + \overline{T}_e (1 - e^{-\lambda_h t}) \tag{32}$$

The above equation can be arranged to estimate the heat transfer coefficient across the thermocline

$$v_t = \frac{V_h}{A_t t_s} \ln \left(\frac{\overline{T}_e - T_{h,i}}{\overline{T}_e - T_{h,s}} \right) \tag{33}$$

where t_s is the time after the onset of stratification at which the hypolimnion temperature, $T_{h,s}$, is measured.

The lake Calhoun's temperature in the hypolimnion rises from 6.71 ℃ in April to 10.15℃ at September during the stratified period. The average epilimnetic temperature is approximately 16.52℃ and the epilimnion depth is assumed as 6 m. After substituting these values into Eq. (33), we found that the thermocline heat transfer coefficient becomes 0.0448 m/d. Fig. 4 shows the mean monthly epilimnetic and hypolimnetic temperatures to estimate heat transfer coefficient. The measured temperature values were compared with the running results of each method in Fig. 5. As we can see, the simulated temperature values in the epilimnion layer were in good agreement with the measured values, but the temperature values of an eddy diffusion model were slightly underpredicted in the bottom layer. Note that the temperature values of specified points (z = 3 m for epilimnion, z = 16 m for hypolimnion respectively) were used to plot Fig. 5.

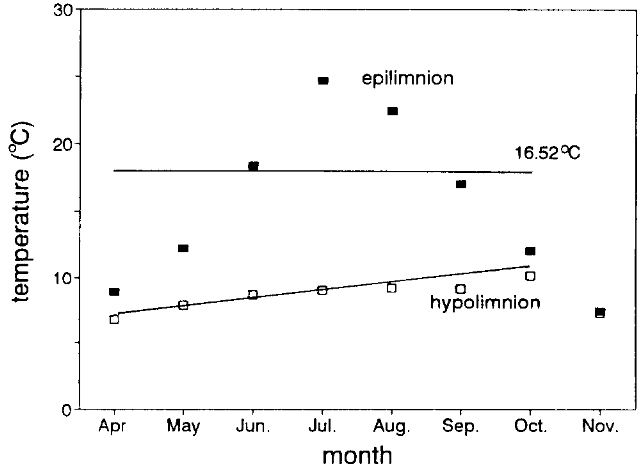

Fig. 4 Mean Monthly Epilimnetic and Hypolimnetic Temperatures

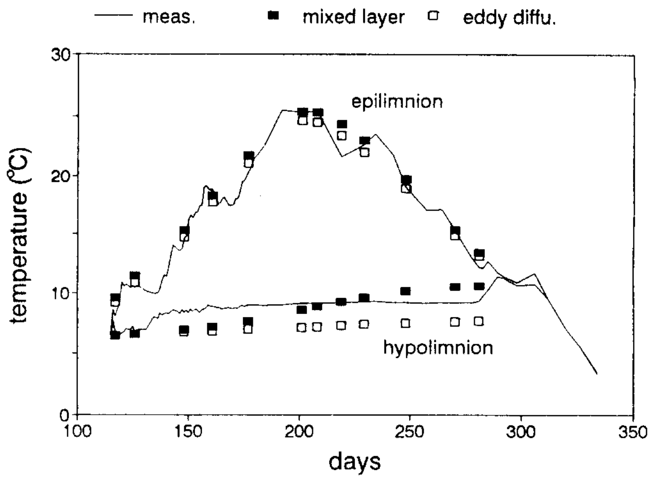

Fig. 5 Comparison of Measured and Calculated Temperature Values

The results revel the general trend of the temperature variation in Lake Calhoun specifically in the epilimnion layer. The peak temperature and the depth of epilimnion of the given environmental system are very important, because the water temperature affects the aquatic ecosystem directly or indirectly.

4. Conclusions

The models presented are simulating one-dimensional temperature profiles in lakes. Application of this model indicates that the model effectively simulates water temperatures. Especially, two well-mixed layers model which is very simple and easy to code for computer produces good results.

Several features are of note :

(1) Variations of water level.

The water level is changed by the mechanism of precipitation and evaporation.

If the water budgets are severely varied, the change of lake level should be incorporated in this model.

(2) Thermocline heat transfer coefficient.

In this paper, The thermocline heat transfer coefficient is estimated by the data from the direct field observation rather than from laboratory measurements and theory. We assumed that thermocline heat transfer coefficient occurs at the interface between surface and bottom layers. In real fields, thermocline is thick in the stratified period. More mixed layers are recommended to cover the disadvatages of two layers model.

(3) Shallow lakes.

In shallow lakes, the daylight heat reached to the lake's bottom contributes to another energy source of the ambient water temperature. Magnitude and directions of thermal energy between water-sediment interface are important to figure out the trend of the water temperature in daily or seasonal basis. More attention and research works are required in this area.

(4) Purpose.

An eddy diffusion and a mixed-layer models have a different characteristics. If we want to know temperature value at specified lake's point, the former is better than the latter. But the latter is very simple, easy to code and produce good results. Thus, which model should be selected is decided by the purpose of the project.

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