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A Study on the Tank-Attack Helicopter Duel

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Abstract

In this paper, we consider a two-person zero-sum game in which an attack helicopter with a missile wishes to destroy a tank. The tank has much small-caliber ammunition for protecting itself from the attack helicopter. And the attack helicopter possesses a missile for attacking the tank. We develop mdels for the behavior of the attack helicopter, in terms of missile launch time, and of the tank, in terms of ammunition firing rate, in several situations. In particular, we examine the Weiss-Gillman model.

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1. Introduction

This research is directed toward finding saddle point coordinate strategies for extensions of the classical unfair silent duel. The principal contribution of this research is the analysis of saddle point coordinate strategies for the Weiss-Gillman model. Prior work in this area includes the work of Weiss [8] who first examined the unfair silent duel, in the course of munition studies conducted at the Aberdeen Proving Ground. The interpretation of this problem as an advertising competition was considered by Gillman [3]. The work of Blackwell and Shiffman [1] also is related to the optimal strategies for the players. Karlin [5] also has addressed this porblem, but with a slightly different formulation.

Model Assumptions and Notation

Consider a situation where Player Ⅰ, an attack helicopter, is attacking a tank, Player Ⅱ, from initial range R. Player Ⅰ is armed with one missile, possessing lethality function F(r) decreasing on [0, R], presumably from near 1 to near 0. The tank is armed with a great weight(say, A ozs.) of armament, with lethality function p(r), also decreasing to near zero on [0, R]. p(r) is to be thought of in these terms: When da ounces of ammunition are expended by Player Ⅱ at range r, there results the probability p(r)da that the attack helicopter is killed.

A strategy for Player Ⅰ is a cumulative distribution function $\sigma(r)$ on [0, R] giving the probability distribution from which the helicopter initially selects its firing range. A strategy for Player Ⅱ is the ammunition distribution density $\tau(r)$ according to which Player Ⅱ plans to distribute the ammunition store A over [0, R], with $\tau(r)dr=da$, the number of ounces *da* allotted to the range interval dr, satisfying

$$\int_0^R \tau(r)dr \leq A.$$

The objective function $M(\sigma, \tau)$ is first taken to be the probability that Player Ⅱ is killed, computed as

$$M(\sigma, \tau) = \int_0^R F(r)\exp\left\{-\int_r^R \tau(s)p(s)ds\right\}d\sigma(r),$$

which is not an unreasonable assessment, since $\int_r^R \tau(s)p(s)ds$ is the expected number of potential kills gotten off by Player II up to range r. We make the assumption that the minimum closing range c=0, that $F(R)$, $p(R)>0$, and that there is an r_0 such that

$$\int_0^{r_0} -F'(s)/F(s)P(s)ds = A \tag{1}$$

$$r_0 \leq R \tag{2}$$

In essence, Equations (1) and (2) guaramtee that the candidate saddle point coordinate strategy $\tau^0(r) = -F'(r)/F(r)P(r)$ suggested by constancy-positivity is great enough to insure that the amount of ammunition A is expended on [0, R].

The Weiss-Gillman Model

Weiss [8] and Gillman [4] examined a somewhat more general model, with the minimum closing range c not necessarily zero, and $p(r_0)$ not necessarily greater than zero.

Helicopter strategy: σ(r) is the probability that the attack helicopter fires at a range when the range is less than or equal to r.

When $p(r_0) \neq 0$,

$\sigma^0(r)$: $P(r_0)/P(r)$ for $c \leq r \leq r_0$,

1 for $r > r_0$

When $p(r_0) = 0$,

$\sigma^0(r)$: 0 for $0 \leq r \leq r_0$,

1 for $r > r_0$

Tank strategy is pure: $\tau(r)$ is the rate of fire per unit range.

$\tau^0(r)$: $-F'(r)/F(r)P(r)$ for $c \leq r \leq r_0$,

0 for $r > r_0$

If both opponents adopt the optimal strategies, the probability of tank destruction is $F(r_0)$. That is,

$$M(\sigma^0, \tau^0) = F(r_0)$$

2. Analysis of the Weiss-Gillman Model

Introductory Remarks

This chapter deals with the optimal strategies suggested by Weiss-Gillman for the helicopter and tank. We re-derive these solutions, using the so-called constancy positivity principle, as well as the Lagrangian saddle point approach to constrained optimization. Both the constancy-positivity principle and Lagrangian saddle point approach are in their infinite-demensional versions. In particular, we shall first see how the tank's optimal strategy is suggested by the constancy-positivity principle, and then a sense in which the optimization problem involved in establishing a saddle point solution touches on Lagrangian optimization.

In this chapter we assume that the helicopter and tank open fire at the same range R which is at least as large as the range r_0, in accordance with the previous chapter, and the quantity $p(R)>0$, and $F(R)>0$. The first two sections below establish candidate saddle point coordinate strategies σ^0 and τ^0, under the assumption that τ^0 is derivable by the constancy-positivity principle. The third section below then verifies that the candidate strategies σ^0 and τ^0 do indeed constitute a saddle point for the game in which the duel starts at range R.

We will need to consider strategies for the helicopter that are cumulative distribution functions possessing both discrete and absolutely continuous parts.

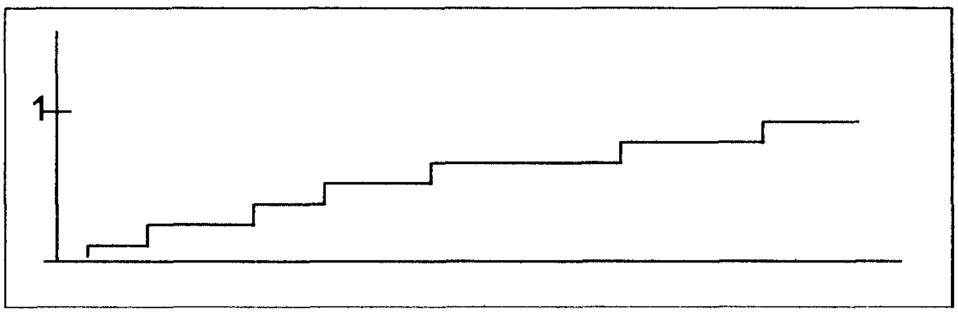

Figure 2.1: Step Function

Purely discrete cumulative distribution functions are step functions which are shown in Figure 2.1 indicating that the random variable being described takes on only a "countable" set of values, and each of them with a certain specified probability. Examples are the Poisson or the binomial cumulative distribution function. Average or expectation of sin X, X Poisson, is given by

$$\sum_{i=0}^{\infty} \sin(i)\{ \lambda^i \exp(-\lambda)/i! \}.$$

Purely absolutely continuous cumulative distribution functions are smooth functions which are shown in Figure 2.2 indicating that the random variable being described is capable of taking on a continuum of values, with any interval assigned probability given by the definite integral of a density function over
that interval. An example is the normal cumulative distribution function, with the familiar density

$$(1/\sqrt{2\pi})\exp(-X^2/2)$$

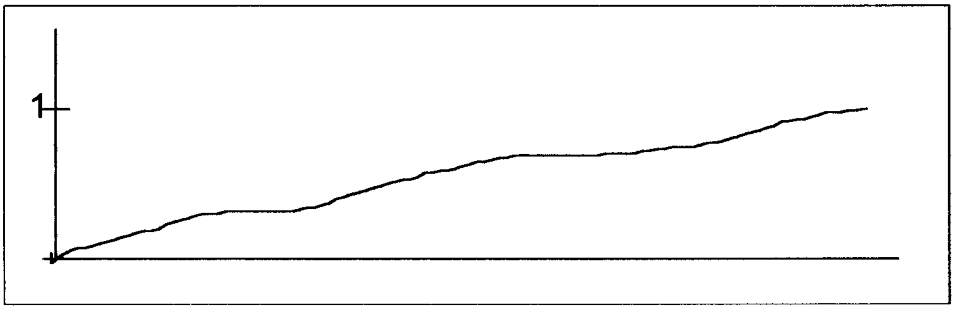

Figure 2.2 : Absolute-Continuous Distribution Function

Averages of expectations with respect to such cumulative distribution functions are expressed as integrals involving the density. Thus the expectation of sin X, X normal, is given by

$$\int_{-\infty}^{+\infty} [\sin(t)](1/\sqrt{2\pi})\exp(-t^2/2)dt.$$

We shall need to deal with cumulative distribution functions σ that are partly discrete and partly absolutely continuous. Such cumulative distribution functions may be thought of in at least these two ways:

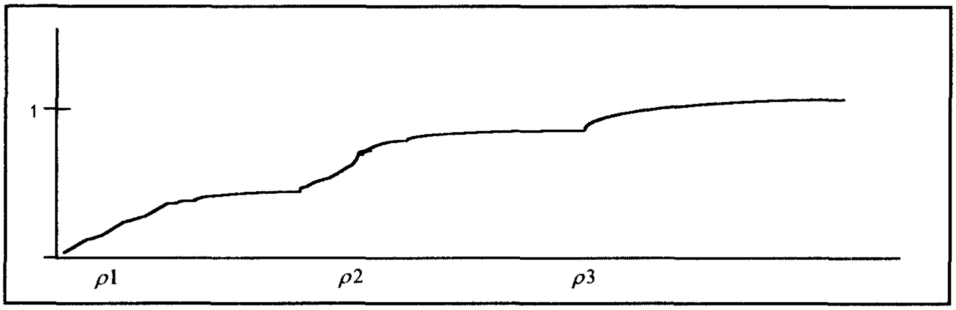

Figure 2.5: Continuous Distribution Function

Thus the expectation of sin X, X distributed as in Figure 2.3, is given by

$$\int_{-\infty}^{+\infty}[\sin(t)]g(t)dt+\sum_{i=0}^{3}\sin(\rho_i)]\Delta_i .$$

Such expectations commonly are given the Stieltjes integral designation, say

$$\int_{-\infty}^{+\infty}\sin(t)d\sigma(t).$$

We also note that, when $\sigma(t)=0$ for $t<0$, then

$$\sigma(r)=\int_{-\infty}^{r}d\sigma(t)=\int_{0}^{r}d\sigma(t) \quad \text{for any } r\geq 0 .$$

Candidate Optimal Tank Strategy by Constancy-Positivity Principle

In this section we will see how the tank's optimal strategy ($\tau^0(r)$) is suggested by the constancy-positivity principle.

As a cumulative distribution function with both steps and smooth portions which can be shown in Figure 2.3. Or, in more explicit fashion, as a cumulative distribution function equal to a weighted average

$$\sigma(t) = \theta\, \sigma_d(t) + (1-\theta)\, \sigma_{ac}(t)$$

of a discrete cumulative distribution function $\sigma_d(t)$ which can be seen in Figure 2.4, and an absolutely continuous cumulative distribution function $\sigma_{ac}(t)$ which can be described in Figure 2.5, with density function, namely, $s(t)$.
Averages or expectations with respect to such cumulative distribution functions are expressed as integrals with density $(1\ \theta)\, s(t) \equiv g(t)$, plus summations.

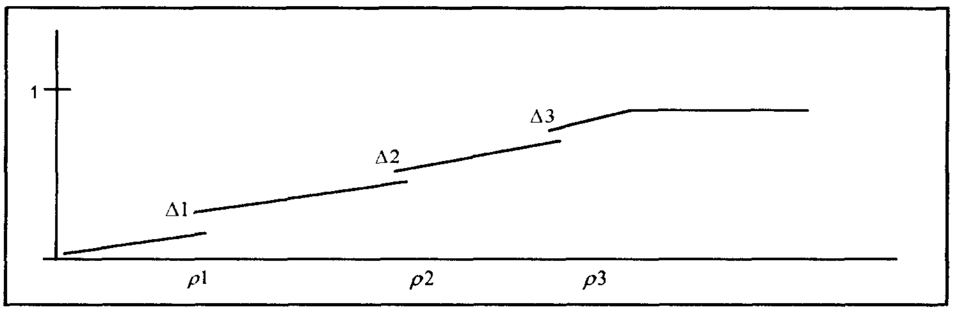

Figure 2.3 : CDF with Steps and Smooth Portions

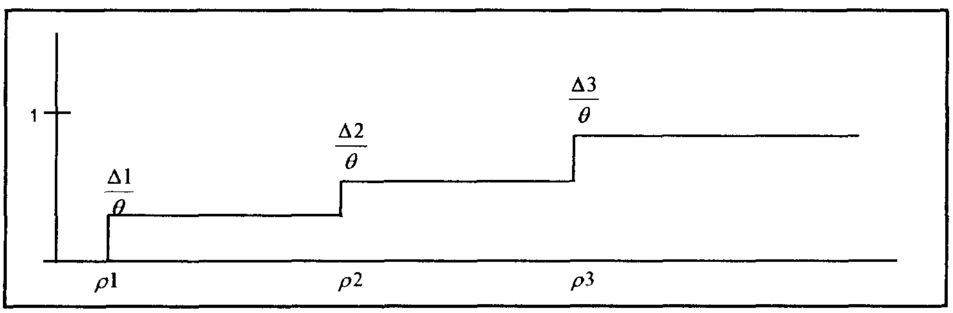

Figure 2.4: Discrete Distribution Function

We begin by recalling what is meant by a positive mixed strategy. When a matrix game is stochastically extended with respect to at least one of the players; i.e., whenever one of the players' strategies are in fact mixtures of available pure strategies, then a mixed strategy for that player is said to be positive if it puts mass on all of the player's pure strategies. Thus, when the player, say the helicopter, has a finite number of pure strategies, say $\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots, \sigma_m$, as the helicopter would in a finite version of the duel, then the helicopter's strategy is positive if ξ_i exceeds zero for all i, i=1, 2, ⋯, m. Analogously, when the helicopter has a continuum of pure strategies, as in the case of the duel studied in this paper, then a mixed strategy for the helicopter, as given by a cumulative distribution function σ(p) on [c, R], is said to be positive if no non-degenerate sub-interval of [c, R] is assigned zero probability under σ.

When a game is stochastically extended with respect to one of the players, say with respect to the helicopter, by the introduction of mixed strategies as above, and the helicopter possesses a positive saddle point coordinate strategy, say σ_0^+, then any saddle point coordinate stragegy τ_0 attains the saddle point payoff v in the presence of all (pure or mixed) strategies for the helicopter if, with r indexing the helicopter's pure strategies, M(r, τ_0) is continuous in r. This is shown by noting first that, if g(r) is non-positive and continuous,

$$[\int g(r)d\sigma_0^+(r)=0] \text{ implies } [g(r)=0]. \tag{2.1}$$

Then $[M(\sigma_0^+, \tau_0)-v=0]$ implies $[\int m(r, \tau_0)d\sigma_0^+(r)-\int vd\sigma_0^+(r)=0]$ implies $[\int[M(r, \tau_0)-v]d\sigma_0^+(r)=0]$, which implies that $M(r, \tau_0)-v=0$ by setting g(r)= $M(r, \tau_0)-v$ in Equation 2.1.

At any rate, then, if the helicopter does possess a positive saddle point coordinate strategy, then, given the required continuity, it must be that any τ_0 will satisfy

$$M(r,\ \tau_0) = v\ ,\ \text{for all } r \tag{2.2}$$

which should allow us to compute a candidate τ_0. This feature motivates us to start looking for a saddle point (σ_0 , τ_0) by searching for a τ_0 satisfying Equation 2.2. If that search is successful, σ_0 is then hunted down by looking for a σ such that τ_0 minimizes M in the presence of σ; i.e., a σ such that

$$M(\sigma,\ \tau_0) \le M(\sigma, \tau),\ \text{for all } \tau, \tag{2.3}$$

If the search for σ satisfying Equation 2.3 also is successful, and leads, say, to $\sigma = \sigma_0$, then (σ_0 , τ_0) is established as a saddle point, since we already have by Equation 2.2 that

$$M(\sigma_0,\ \tau_0) = M(\sigma,\ \tau_0),\ \text{for all } \sigma.$$

In the situation at hand (restricting r to the interval [c, r_0], with the hope that things will fall into place by themselves on [r_0, R]), relation Equation 2.2 gives

$$F(r)\exp[-\int_r^{r_0} p(s)\ \tau_0(s)ds] = v$$

$$\ln F(r) - \int_r^{r_0} p(s)\ \tau_0(s)ds = \ln v$$

$$[F'(r)/F(r)] + p(r)\ \tau_0(r) = 0$$

$$\tau_0(r) = -F'(r)/F(r)p(r) \equiv \tau^0(r)$$

We now define a candidate τ_0 by extending τ_0 to [c, R] by postulating that

$(r)=0$, $r_0 \le r \le R$, and apply Lagrangian saddle point methods to finding a σ_0 such that

$$M(\sigma_0, \tau_0) \le M(\sigma_0, \tau)$$

Optimal Helicopter Strategy by Lagrangian Saddle Point

In this section we will discuss how the helicopter's optimal strategy ($\sigma^0(r)$) is derived using Lagrangian saddle point methods.

Our task is to find a σ such that

$$M(\sigma, \tau) - M(\sigma, \tau_0) \ge 0$$

In other words, among the problems p_σ

$$\begin{aligned} &\text{Min} \quad M(\sigma, \tau) \\ &\tau \quad s.t. \int_C^R \tau(x)dx \le A \\ &\qquad \tau \ge 0 \quad \text{on } [C,R] \end{aligned}$$

parametrized by σ, find one, say p_{σ^0}, for which τ^0 is the minimizing τ:

$$\begin{aligned} &\text{MinM}(\sigma^0, \tau) = \text{M}(\sigma^0, \tau^0) \\ &\tau \quad s.t. \int_C^R \tau(x)dx \le A \\ &\qquad \tau \ge 0 \quad \text{on } [C,R] \end{aligned} \tag{2.4}$$

But the Lagrangian saddle point theory alerts us to the fact that it is sufficient for Equation 2.4 that τ^0 participate in a saddle point of the Lagrangian $L(\lambda, \tau)$ for the minimization problem $P\sigma^0$; i.e., that there be a $\lambda_0 \geq 0$ such that (λ_0, τ^0) is a saddle point of

$$L(\lambda, \tau) = M(\sigma^0, \tau) + \lambda[\int_C^R \tau(x)dx - A],$$

for all $\lambda \geq 0$ and $\tau \geq 0$ on $[c, R]$.

Thus σ^0 will satisfy Equation 2.4 if there is a $\lambda_0 \geq 0$ such that

$$\begin{aligned} & M(\sigma^0, \tau) + \lambda_0[\int_C^R \tau(x)dx - A] \\ \equiv & \int_c^R [F(r)e^{-\int_r^R p(s)\tau(s)ds}]d\sigma^0(r) + \lambda_0[\int_c^R \tau(x)dx - A] \\ \geq & M(\sigma^0, \tau^0) + \lambda_0[\int_C^R \tau^0(x)dx - A] \\ \equiv & \int_c^R [F(r)e^{-\int_r^R p(s)\tau^0(s)ds}]d\sigma^0(r) + \lambda_0[\int_c^R \tau^0(x)dx - A] \\ \geq & \int_c^R [F(r)e^{-\int_r^R p(s)\tau^0(s)ds}]d\sigma^0(r) + \lambda[\int_c^R \tau^0(x)dx - A], \end{aligned} \tag{2.5}$$

for all $\lambda \geq 0$ and $\tau \geq 0$ on $[c, R]$. But, since

$$\int_C^R \tau^0(x)dx = A,$$

the second inequality is automatically satisfied with equality.

We can change the order of integration when Stieltjes integration is involved. As expected from the usual calculus, for non-negative A(.) B(.) we have

$$\int_{r=c}^R B(r)\int_{x=r}^R A(x)dxd\sigma(r) = \int_{x=c}^R A(x)\int_{r=c}^x B(r)d\sigma(r)dx.$$

$$e^x \geq 1 + x.$$

$$e^{-z-z_0} - 1 \geq -(z - z_0).$$

$$e^{-z} - e^{-z_0} \geq -(z - z_0)e^{-z_0}.$$

$$e^{-\int_r^R a(s)ds} - e^{-\int_r^R a_0(s)ds} \geq - [e^{-\int_r^R a_0(s)ds}] [\int_r^R (a(x) - a_0(x))dx].$$

$$\int_c^R F(r)\, e^{-\int_r^R a(s)ds} d\sigma(r) - \int_c^R F(r)\, e^{-\int_r^R a_0(s)ds} d\sigma(r) \tag{2.6}$$

$$\geq - \int_c^R [F(r)\, e^{-\int_r^R a_0(s)ds}] [\int_r^R (a(x) - a_0(x))dx] d\sigma(r).$$

$$= - \int_c^R (a(x) - a_0(x)) \quad [\int_c^R F(r)\, e^{-\int_r^R a_0(s)ds} d\sigma(r)]dx,$$

where $B(r) = F(r)\, exp\{ - \int_r^R a_0(s)ds \}$ and $A(x) = \{ a(x) - a_0(x)\}$.

Equation 2.6 yields the following equation when $\sigma(r) = \sigma^0(r)$, and $a_0(x) = p(x)\, \tau^0(x)$, $a(x) = p(x)\, \tau(x)$, with $\tau(x) \geq 0$ and arbitary firing schedule for the tank over [c, R] , with $\int_c^R \tau(x) \leq A.$

$$\int_c^R [F(r)\;\; e^{-\int_r^R p(s)\tau(s)ds]}]\, d\sigma^0(r) + \lambda_0[\int_c^R \tau(x)dx - A]$$

$$- \int_c^R F(r)\;\; e^{-\int_r^R p(s)\tau^0(s)ds}]\, d\sigma^0(r) + \lambda_0[\int_c^R \tau^0(x)dx - A]$$

$$\geq \int_c^R [\tau(x) - \tau^0(x)]\, [-p(x) \int_c^x F(r)\, e^{-\int_r^R p(s)\tau^0(s)ds} d\sigma^0(r)]dx$$

$$+ \lambda_0[\int_c^R \tau(x) - \tau^0(x)]\, dx$$

$$= \int_c^R [\tau(x) - \tau^0(x)] \, [-p(x) \int_c^x F(r) \, e^{-\int_r^R p(s)\tau^0(s)ds} \, d\sigma^0(r) + \lambda_0] dx, \quad (2.7)$$

where, for later reference, we define

$$C(x) = -p(x) \int_c^x F(r) \; e^{-\int_r^R p(s)\tau^0(s)ds} \, d\sigma^0(r) + \lambda_0$$

Now we will examine the right hand side of Equation 2.7. The inequality of Equation 2.5 will be ensured by any σ^0 and λ_0 reducing C(x) to a function equal to zero on $[c, r_0]$ and greater than zero on $[r_0, R]$, say,

$$\sigma^0(x) \equiv p(r_0)/p(x) \qquad \text{on } [\, c, \; r_0 \,]$$

$$1 \qquad \text{on } [\; r_0, R \,] ,$$

and $\lambda_0 = \lambda^0 \equiv F(r_0)p(r_0) \geq 0,$

since, with $\lambda_0 = \lambda^0$, and repeating in part computations already done above, C(x) becomes , for $x \leq r_0$,

$$C(x) = -p(x) \cdot \int_c^x F(r) \, e^{-\int_r^{r_0} p(s)\tau^0(s)ds} \, d\sigma^0(r) + \lambda_0$$

$$= -p(x) \int_c^x F(r) \frac{[F(r_0)}{F(r)]} \, d\sigma^0(r) + \lambda_0$$

$$= -p(x) \, F(r_0) \, \sigma^0(x) + F(r_0)p(r_0)$$

$$= -p(x)F(r_0)[\frac{p(r_0)}{p(x)}] + F(r_0) \; p(r_0)$$

$$= -F(r_0) \; p(r_0) + F(r_0) \; p(r_0) = 0$$

and, for $x \geq r_0$,

$$C(x) = -p(x)\int_c^{r_0} F(r)\, e^{-\int_r^R p(s)\tau^0(s)ds}\, d\sigma^0(r)$$

$$-p(x)\int_{r_0}^{x} F(r)\, e^{-\int_r^R p(s)\tau^0(s)ds}\, d\sigma^0(r) + \lambda^0$$

$$= -p(x)\int_c^{R} F(r)\, e^{-\int_r^{r_0} p(s)\tau^0(s)ds}\, d\sigma^0(r) + \lambda^0$$

$$= -p(x)\, F(r_0)\, \sigma^0(r_0) + \lambda_0$$

$$= -F(r_0)\; p(r_0) + F(r_0)\; p(r_0) \geq 0,$$

Since $C(x)$ is greater than or equal to zero, the right-hand side of Equation 2.7 is greater than or equal to zero. Therefore, the helicopter's optimal strategy σ^0 is found such that

$$M(\sigma^0, \tau) - M(\sigma^0, \tau^0) \geq 0. \tag{2.8}$$

Verification of Candidate Tank Strategy

It remains to verify that the extended tank strategy τ^0 does in fact satisfy.

$$M(\sigma^0, \tau^0) \geq M(\sigma^0, \tau^0)$$

To this end note that

$$M(\sigma^0, \tau^0) = \int_c^R F(r)\, e^{-\int_r^R \tau^0(s)p(s)ds}\, d\sigma^0(r)$$

$$= \int_c^{r_0} F(r)\, e^{-\int_r^{r_0} \tau^0(s)p(s)\,ds}\, d\sigma^0(r)$$

$$= \int_c^{r_0} F(r)\, e^{\int_r^{r_0} \frac{F'(s)}{F(s)}\,ds}\, d\sigma^0(r)$$

$$= \int_c^{r_0} F(r) \left[\frac{F(r_0)}{F(r)} \right] d\sigma^0(r)$$

$$= F(r_0) \int_c^{r_0} d\sigma^0(r)$$

$$= F(r_0)\,,$$

where $\int_c^{r_0} d\sigma^0(r) = 1.$

$$M(\sigma, \tau^0) = \int_c^{R} F(r)\, e^{-\int_r^{R} \tau^0(s)p(s)\,ds}\, d\sigma(r)$$

$$= \int_c^{r_0} F(r)\, e^{-\int_r^{R} \tau^0(s)p(s)\,ds}\, d\sigma(r) + \int_{r_0}^{R} F(r)\, e^{-\int_r^{R} \tau^0(s)p(s)\,ds}\, d\sigma(r)$$

$$= \int_c^{r_0} F(r) \left[\frac{F(r_0)}{F(r)} \right] d\sigma(r) + \int_{r_0}^{R} F(r)\, d\sigma(r)$$

$$\leq F(r_0) \int_c^{r_0} d\sigma(r) + F(r_0) \int_{r_0}^{R} d\sigma(r)$$

$$= F(r_0)\, \sigma(R) \leq F(r_0) = M(\sigma^0, \tau^0),$$

where the last eqality follows from $\sigma^0(r_0) = 1$.

Therefore, $M(\sigma^0, \tau^0) \geq M(\sigma, \tau^0)$, which, together with relation of Equation 2.8, verifies that σ^0 and τ^0 are the saddle point coordinate strategies for the helicopter and tank respectively.

3. Early Duel Start, whit Altered Restriction on the Tank.

This chapter suggests a saddle point coordinate strategy for the attack helicopter with altered restriction on the tank.

Consider any s with $c \leq s < r_0$. Suppose that the set of allowed strategies for the tank satisfy only the condition,

$$\int_s^R \tau(r)dr \leq \int_s^R \tau^0(r)dr \ ;$$

i.e., that the tank spends no more than the optimal amount in some early stage of the duel. Then a saddle point of the duel is given by the pair (σ_s^0, τ^0) ,

where $\sigma_s^0(r)$: 0 for $c \leq r < s$,

$p(r_0)/p(r)$ for $s \leq r < r_0$,

1 for $r \geq r_0$,

and $\tau^0(r)$: $-F'(r) / F(r)\, p(r)$ for $c \leq r < r_0$,

0 for $r \geq r_0$.

To verify that $M(\sigma, \tau^0) \leq M(\sigma_s^0, \tau^0)$, we can proceed as in the case of the classical unfair silent duel.

As to verifying that $M(\sigma_s^0, \tau) \geq M(\sigma_s^0, \tau^0)$, we proceed as follows:

From Equation 2.6,

$$M(\sigma_s^0, \tau) - M(\sigma_s^0, \tau^0),$$

$$= - \int_c^R (a(x) - a_0(x)) \, [\int_c^x F(r) \, e^{-\int_r^R a_0(s)ds} \, d\sigma_s^0(r)] dx, \qquad (3.1)$$

where $a_0(x) = p(x)\,\tau^0(x)$ and $a(x) = p(x)\,\tau(x)$, with $\tau(x)$ an arbitrary firing schedule for the tank over [c, R].

To begin with, for $x \geq r_0$,

$$\int_c^x [F(r) \, e^{-\int_r^R p(s)\tau^0(s)ds}] \, d\sigma_s^0(r)$$

$$= \int_c^{r_0} F(r) \, e^{-\int_r^R p(s)\tau^0(s)ds} \, d\sigma_s^0(r)$$

$$= \int_c^{r_0} F(r) \, e^{-\int_r^{r_0} p(s)\,\tau^0(s)ds} \, d\sigma_s^0(r)$$

$$= \int_c^{r_0} F(r) \, e^{\int_r^{r_0} \frac{F'(s)}{F(s)} ds} \, d\sigma_s^0(r)$$

$$= \int_c^{r_0} F(r) \, [\frac{F(r_0)}{F(r)}] \, d\sigma_s^0(r)$$

$$= F(r_0) \int_c^{r_0} d\sigma_s^0(r) = F(r_0).$$

and also, for $c \leq s \leq r_0$,

$$\int_c^x [F(r) \, e^{-\int_r^R p(s)\tau^0(s)ds}] \, d\sigma_s^0(r)$$

$$= \int_c^s 0 \, d\sigma_s^0(r) + \int_s^x F(r) \, e^{-\int_r^{r_0} p(s)\,\tau^0(s)ds} \, d\sigma_s^0(\mathrm{r})$$

$$= \int_s^x F(r) \, e^{\int_r^{r_0} \frac{F'(s)}{F(s)} ds} \, d\sigma_s^0(r)$$

$$= \int_s^x F(r)[\frac{F(r_0)}{F(r)}]d\sigma_s^0(r)$$

$$= F(r_0)\int_s^x d\sigma_s^0(r) = F(r_0)\sigma_s^0(x)$$

And so, $\displaystyle\int_c^R [a(x) - a_0(x)][\int_c^x F(r)\, e^{-\int_r^R a_0(s)ds} d\sigma_s^0(r)]dx$

$$= \int_c^{r_0} [a(x) - a_0(x)][\int_c^x F(r)\, e^{-\int_r^R a_0(s)ds} d\sigma_s^0(r)]dx$$

$$+ \int_{r_0}^R [a(x) - a_0(x)][\int_c^x F(r)\, e^{-\int_r^R a_0(s)ds} d\sigma_s^0(r)]dx$$

$$= \int_c^{r_0} [a(x) - a_0(x)][F(r_0)\sigma_s^0(x)]dx$$

$$+ \int_{r_0}^R [a(x) - a_0(x)][F(r_0)]dx \tag{3.2}$$

$$= \int_c^s 0dx + \int_s^{r_0} [a(x) - a_0(x)][F(r_0)\sigma_s^0(x)]dx$$

$$+ \int_{r_0}^R [a(x) - a_0(x)][F(r_0)]dx$$

$$\leq \int_s^{r_0} [a(x) - a_0(x)][F(r_0)\frac{p(r_0)}{p(x)}]dx$$

$$+ \int_{r_0}^R [a(x) - a_0(x)][F(r_0)\frac{p(r_0)}{p(x)}]dx$$

where this last inequlity is due to the fact that $[a(x) - a_0(x)]$ is non-negative on $[r_0, R]$,

$$= \int_s^R [\tau(x) - \tau^0(x)][F(r_0)p(r_o)]dx$$

$$= [F(r_0)p(r_o)][\int_s^R \tau(x)dx - \int_s^R \tau^0(x)dx] \leq 0$$

Hence, the right hand side of Equation 3.1 is greater than or equla to zero.

That in fact shows that $M(\sigma_s^0, \tau) - M(\sigma_s^0, \tau^0) \geq 0$

Therefore, σ_s^0 and τ^0 are the saddle point coordinate strategies for the helicopter and tank respectively for our modified model.

4. Conclusion

This reseach analyzed saddle point coordinate strategies for the attack helicopter-tank duel in several different situations. At first, the Weiss-Gillman model of the classical unfair silent duel is examined, with respect to the possible extension of the class of saddle point coordinate strategies for the helicopter. It is noted that the solutions for the Weiss-Gillman can be obtained when the duel starts before a certain natural range r_0 and after r_0.

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