Some Properties of the Regenerative Process

Donghee Shim*

ABSTRACT

Limiting probability in the steady state of the regenerative process is one of the most useful characteristics. The formula for this limiting probability in the steady state of the regenerative process is presented in this paper. Because this formula is for the general model, it can be applied to many special systems including 2-unit redundant system. An example for this formula is also presented.

1. Introduction

Many researches[1,2,3,4,8,9] have been performed on the reliability analysis of the regenerative process. In these papers 2-unit redundant system has been analyzed because 2-unit redundant system has many applications in the real world. Most of the distributions functions for failure time, repair time, switching-over time are assumed to be arbitrary for achieving the generality of the model. Network topology between these models were different due to the various assumptions. Several important characteristics such as the expected number of visits and limiting probability to a certain state, which were expressed in terms of the distributions, were obtained in these papers. These derivations require much calculations.

These characteristics can be obtained easily without tedious calculations if the properties of the generation points of the process are used. But these properties were not used in these papers. The formula for the limiting probability in the steady state of the regenerative process is presented in this paper. System availability can be also derived using this limiting probability. The results of S.M. Sinha[9] are deduced using the formula presented in this paper.

2. Assumptions of the process

The system follows a regenerative process, i.e. has at least one regeneration point[5,6,7]. The system

* Dept. of Computer Engineering, Jeonju University.
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also has a finite state space to be a regular process[5,6,7]. The initial state epoch of the system is a regeneration point.

3. Notations

(1) $Q_{ij}(\alpha,t)$: CDF (cumulative density function) of MRP(Markov renewal process) from the regeneration point #i into #j, following the transition path α, not including any regeneration points in the path α.

(2) $q_{ij}(\alpha, s)$: Laplace transformation of the $Q_{ij}(\alpha,t)$

i.e. $= \int_0^\infty \exp\{-st\} * dQ_{ij}(\alpha, t)$

(3) $H_{ij}(t)$: first passage time CDF from the regeneration point #i into #j

(4) $h_{ij}(s)$: Laplace transformation of the $H_{ij}(t)$

i.e. $= \int_0^\infty \exp\{-st\} * dH_{ij}(t)$

(5) u_{ij} : mean time of the first passage time from the regeneration point #i into #j

i.e. $= \int_0^\infty t * dH_{ij}(t) = \lim_{s\to 0} [-\frac{d}{ds} h_{ij}(s)]$

(6) $\overline{Q}_{ij}(\alpha)$: mean time of the $Q_{ij}(\alpha, t)$

i.e. $= \int_0^\infty t * dQ_{ij}(\alpha, t) = \lim_{s\to 0} [-\frac{d}{ds} q_{ij}(\alpha, s)]$

(7) $P_{ij}(t)$: probability that the process enters #j, and is still in the state of #j at time t, starting from the regeneration point #i

(8) $p_{ij}(s)$: Laplace transformation of $P_{ij}(t)$

i.e. $= \int_0^\infty \exp\{-st\} * dP_{ij}(t)$

(9) P_j : limiting probability of the state #j

i.e. $= \lim_{t\to\infty} P_{ij}(t) = \lim_{s\to 0} p_{ij}(s)$

(10) SR_j : the set of the regeneration points through which #j may be reached without passing any other regeneration points

(11) L_j : the set of state epoches into which the process may enter after the transition from #j

(12) $Q_{ijk}(\alpha, t)$: CDF of MRP starting from the regeneration point #i into #j through the transition path α , and leaving #j for #k

(13) $q_{ijk}(\alpha, s)$: Laplace transformation for $Q_{ijk}(\alpha, t)$

(14) $\overline{Q_{ijk}}(\alpha)$: mean time of the $d(Q_{ijk}(\alpha, t))$

$$\text{i.e.} = \int_0^\infty t^* \, dQ_{ijk}(\alpha, t) = \lim_{s\to 0} \left[-\frac{d}{ds} q_{ijk}(\alpha, s) \right]$$

(15) α_{ij} : the set of transition path starting from the regeneration point #i, reaching to #j, without passing any regeneration points

4. Some Properties

We can easily see the following two properties.

(1) $\lim_{s\to 0} h_{ij}(s) = 1$ for any regeneration point #i and #j

(2) $\lim_{s\to 0} \left[\sum_{\alpha} q_{ij}(\alpha, s) - \sum_{\alpha, k} q_{ijk}(\alpha, s) \right] = 0$ for regeneration point #i

such that $i \in SR_j$, $\alpha \in \alpha_{ij}$, $k \in L_j$ and

$\lim_{s\to 0} q_{jk}(\alpha, s) = 1$ for any regeneration point #j such that $\alpha \in \alpha_{jk}$

(3) Limiting probability formula in the steady state

$$P_j = \sum_i \left(\sum_{\alpha, k} \overline{Q}_{ijk}(\alpha) - \sum_{\alpha} \overline{Q}_{ij}(\alpha) \right) / u_{ii} \text{ for the nonregeneration point \#j}$$

$\sum$ for i, α, k means $i \in SR_j$, $\alpha \in \alpha_{ij}$, $k \in L_j$

$$P_j = \sum_k \overline{Q}_{jk}(\alpha) / u_{jj} \text{ for the regeneration point \#j}$$

$\sum$ for k means $\alpha \in \alpha_{jk}$, $k \in L_j$

from now, we will omit the summation index set SR_j, α_{ij}, L_j for convenience.

pf) Using the renewal theoretic arguments[5,6,7], we have for the nonregeneration point #j

$$P_{ij}(t) = \sum_i \{1 + H_{ii}(t) + H_{ii}(t)^* H_{ii}(t) + \ldots\}^* \left\{ \sum_{\alpha} Q_{ij}(\alpha, t) - \sum_{\alpha, k} Q_{ijk}(\alpha, t) \right\}$$

$$= \sum_i \{1 - H_{ii}(t)\}^{-1} * \left\{ \sum_{\alpha} Q_{ij}(\alpha, t) - \sum_{\alpha, k} Q_{ijk}(\alpha, t) \right\}$$

The Laplace transformation of the $P_{ij}(t)$

$$p_{ij}(s) = \sum_i \{1 - h_{ii}(s)\}^{-1} * \left\{ \sum_{\alpha} q_{ij}(\alpha, s) - \sum_{\alpha, k} q_{ijk}(\alpha, s) \right\}$$

Using the above properties of this section

$$P_j = \lim_{t\to\infty} P_{ij}(t) = \lim_{s\to 0} p_{ij}(s)$$

$$= \sum_{i} \left(\sum_{\alpha, k} \overline{Q}_{ijk}(\alpha) - \sum_{\alpha} \overline{Q}_{ij}(\alpha) \right) / u_{ii}$$

For the regeneration point #j

$$P_{jj}(t) = [\ 1 + H_{jj}(t) + H_{jj}(t) * H_{jj}(t) + ...] * [\ 1 - \sum_{\alpha} Q_{jk}(\alpha, t)]$$

$$= [\ 1 - H_{jj}(t)]^{(-1)} * [\ 1 - \sum_{\alpha} Q_{jk}(\alpha, t)]$$

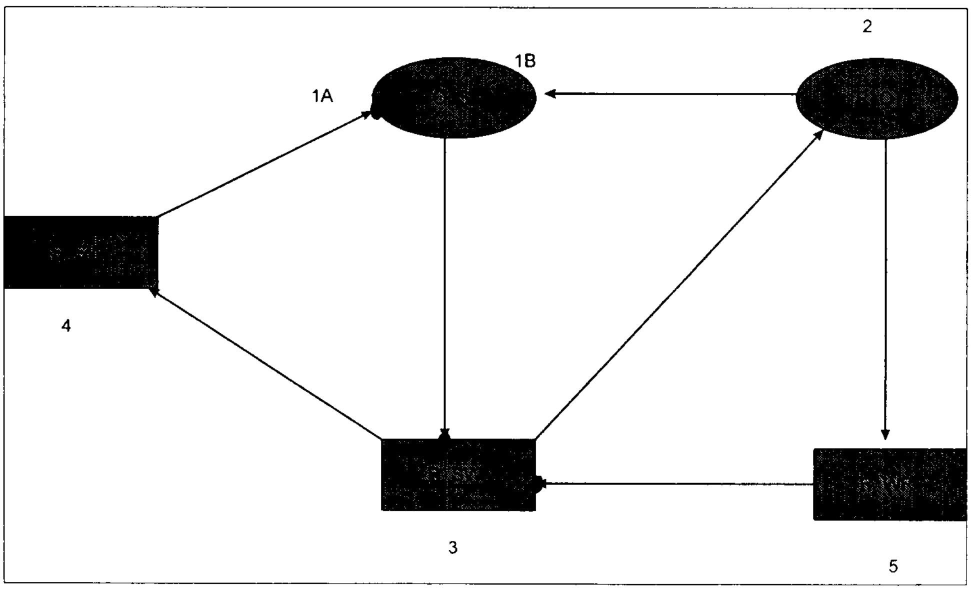
By the same procedure as for the nonregeneration point #j

$$P_j = \sum_{k} Q_{ik}(\alpha) / u_{jj}$$

5. Example

We adopt the model of S.M.Sinha[9] for example. This model is the 2-unit cold-standby system, and the time distributions for failure, repair, switch are all arbitrary. The following diagram shows the state transition.

In the above diagram, . means the regeneration point. The circular nodes mean the operating state and the rectangular nodes mean the down state. State space is {1,2,3,4,5}, and state epoch space is {#1A, #1B, #2, #3, #4, #5}. We can derive the limiting probability using the formula presented in



O:Operating S:Standby R:Repair WR:Waiting for repair
SW:Switched from standby to operating

⟨Fig. 1⟩ Diagram of 2-unit Redundant System with Slow Switch this paper without tedious calculations as in Sinha[9].

For #4

SR_4 = {#3}

L_4 = {#1A}

α_{34} = { (3 - 4) }

$P_4 = (\overline{Q}_{3\ 4\ 1A}((3\text{-}4)) - \overline{Q}_{3\ 4}((3\text{-}4))) / u_{33}$

For #2

SR_2 = {#3}

L_2 = {#1B, #5}

α_{32} = { (3 - 2) }

$P_2 = (\overline{Q}_{3\ 2\ 1B}((3\text{-}2)) + \overline{Q}_{3\ 2\ 5}((3\text{-}2)) - \overline{Q}_{3\ 2}((3\text{-}2))) / u_{33}$

For #5

SR_5 = {#3}

L_5 = {#3}

α_{35} = { (3 - 2 - 5) }

$P_5 = (\overline{Q}_{3\ 5\ 3}((3\text{-}2\text{-}5)) - \overline{Q}_{3\ 5}((3\text{-}2\text{-}5))) / u_{33}$

For #1B

SR_{1B} = {#3}

L_{1B} = {#3}

$\alpha_{3\ 1B}$ = { (3 - 2 - 1B) }

$P_{1B} = (\overline{Q}_{3\ 1B\ 3}((3\text{-}2\text{-}1B)) - \overline{Q}_{3\ 1B}((3\text{-}2\text{-}1B))) / u_{33}$

For #1A

SR_{1A} = {#3}

L_{1A} = {#3}

$\alpha_{1A\ 3}$ = { (1A - 3) }

$P_{1A} = \overline{Q}_{1A\ 3}((1A\text{-}3)) / u_{1A\ 1A}$

For #3

SR_3 = {#1A}

L_3 = {#2, #4}

$\alpha_{3\ 4}$ = { (3 - 4) }, $\alpha_{3\ 2}$ = { (3 - 2) }

$P_3 = (\overline{Q}_{34}((3\text{-}4)) + \overline{Q}_{32}((3\text{-}2))) / u_{33}$