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The Probability Distribution of the Number of Customers Accumulated when a Vacation Ends in the Geo/G/I Gated System

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Abstract

We present a procedure which finds the probability distribution of number of customers accumulated when the server ends a vacation in the Geo/G/1 gated queueing system, where the service for a customer and the vacation, respectively, takes one slot time. The p.g.f for the number of customers accumulated at the gate closing epoch is obtained as a recursive form. The full probabilities, then, are derived using an iterative procedure. This system finds an application in a packet transmitting telecommunications system where the server transmits(serves) packets(customers) accumulated at a gate closing epoch, and takes one slot time vacation thereafter.

1. Introduction

We consider a Geo/G/1 gated queueing system with server vacations. Time is segmented into a sequence of equal-length slots. Customers arrive according to a Bernoulli process with rate λ at the beginning of each slot and are stored in a FCFS queue. A service (also a vacation) is assumed to start only at the beginning of each slot, and to take one slot time. This special case of Geo/G/1 gated system finds an application in a packet transmitting telecommunications system where the server transmits(serves) packets(customers) accumulated at a gate closing epoch, and takes one slot time vacation thereafter[2].

The first and second moments of several performance measures of the Geo/G/1 gated system are

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well known[1]. Probability distribution functions of performance measures are, however, indirectly obtained in the form of z-transforms, which may not always result in explicit expressions of the whole probabilities. In this paper, we present a procedure which finds the probability distribution of number of customers accumulated at the time the server ends a vacation.

Let f_n denote the probability that the server finds n customers accumulated at the time a vacation ends, $n=0,1,2,\cdots$, and let $F(z)$ denote its p.g.f(probability generating function). Then by definition,

$$F(z)=\sum_{n=0}^{\infty} f_n z^n. \tag{1}$$

Let $\beta(\cdot)$ and $\omega(\cdot)$ be the p.g.f.s of the service time for a customer and the vacation time, respectively. If a vacation (also a service for a customer) takes integer multiple of a slot time, then it can be shown that[1]

$$F(z)=F\{\beta(1-\lambda+\lambda z)\}\omega(1-\lambda+\lambda z), \tag{2}$$

where λ is the mean arrival rate of customers during a slot time. In order for the system to be stable, we have $\lambda < 1$.

Since the system we consider takes one slot time in the service for a customer and also one slot time for a vacation, it is easily seen that $\beta(z)=z$ and that $\omega(z)=z$. Thus, equation (2) reduces to

$$F(z)=(1-\lambda+\lambda z)F(1-\lambda+\lambda z) . \tag{3}$$

Let $F^{(n)}(z)$ denote the n-th derivative of $F(z)$. We are interested in finding $f_n=\dfrac{F^{(n)}(0)}{n!}$.

In the following, we derive a recursive equation for $F^{(n)}(0)$and present an iterative procedure which finds f_n .

2. Development of the recursive equation for $F^{(n)}(0)$

We begin with finding $F^{(n)}(0)$ from equations (1) and (3). From equation (3) we expect that the term $F^{(n)}(0)$ will be represented in a recursive form in relation to $F^{(n-1)}(0)$. By consecutively taking the derivative in the right hand side of equation (3), we obtain

$F^{(1)}(z)=\lambda\{\ F^{(1)}(1-\lambda+\lambda z)(1-\lambda+\lambda z)+F(1-\lambda+\lambda z)\}$, and

$F^{(2)}(z) = \lambda^2\{F^{(2)}(1-\lambda+\lambda z)(1-\lambda+\lambda z)+2F^{(2)}(1-\lambda+\lambda z)\}$, and so on.

In general, it can be shown that the following recursive equation for $F^{(n)}(z)$ holds for $n \geq 1$ (See the appendix.) :

$$F^{(n)}(z) = \lambda^n\{F^{(n)}(1-\lambda+\lambda z)(1-\lambda+\lambda z)+nF^{(n-1)}(1-\lambda+\lambda z)\}, \quad n \geq 1. \quad (4)$$

Thus, $$F^{(n)}(0) = \lambda^n\{F^{(n)}(1-\lambda)(1-\lambda)+nF^{(n-1)}(1-\lambda)\}, \quad n \geq 1. \quad (5)$$

Now, equation (1) can be rewritten as

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} z^n. \quad (6)$$

It is easily seen, by taking consecutive derivatives on the right hand side of equation (6), that

$$F^{(n)}(z) = \sum_{i=n}^{\infty} \frac{F^{(i)}(0)}{(i-n)!} z^{i-n}. \quad (7)$$

Equation (7) can also be proved using mathematical induction. It immediately follows that

$$F^{(n)}(1-\lambda) = \sum_{i=n}^{\infty} \frac{F^{(i)}(0)}{(i-n)!} (1-\lambda)^{i-n}. \quad (8)$$

By substituting (8) into (5), we obtain a recursive expression for $F^{(n)}(0)$ as

$$F^{(n-1)}(0) = \frac{F^{(n)}(0)\{1-\lambda^n(1-\lambda)(n+1)\}-A(\lambda,n)}{n\lambda^n}, \quad (9)$$

where $A(\lambda,n) = \sum_{i=n+1}^{\infty} \frac{F^{(i)}(0)}{(i-n+1)!} (1-\lambda)^{i-n+1}(i+1)$.

3. Iterative procedure for finding f_n

From equation (9), we obtain f_n as follows :

Step 1: Set $F^{(N)}(0)=1$ for a sufficiently large integer N, and set $F^{(j)}(0)=0$ for $j > N$.

Step 2: Recursively evaluate $f_n = \frac{F^{(n)}(0)}{n!}$ for $n=N-1, N-2, \cdots, 1, 0$, using equation (9).

Step 3: Obtain $f_n = \frac{f_n}{\sum_{i=0}^{N} f_i}$.

4. Numerical Examples

We present some numerical examples to verify the iterative procedure for obtaining the full probabilities. Five levels of arrival rate λ are considered : λ = 0.1, 0.3, 0.5, 0.7, and 0.9. Note that we reguire $\lambda < 1$ for system stability. N is set to be 30 in Step 1 in the procedure given above.

Let X be the number of customers accumulated at a gate closing epoch. Then, it can be shown from equations (1) and (3) that the first and second factorial moment of X are

$$E(X) = F^{(1)}(1) = \frac{\lambda}{1-\lambda}, \tag{10}$$

and $E(X(X-1)) = F^{(2)}(1) = \dfrac{2\lambda^3}{(1-\lambda)(1-\lambda^2)}$, respectively.

It immediately follows that

$$Var(X) = F^{(2)}(1) + F^{(1)}(1) - E(X)^2 = \frac{\lambda}{1-\lambda^2}. \tag{11}$$

Since there exist no closed form solution for the probabilities, we can compare only the mean and

X		$f_n = P(X=n)$				
		λ = 0.1	λ = 0.3	λ = 0.5	λ = 0.7	λ = 0.9
0		0.890010	0.612648	0.288788	0.042316	0.000001
1		0.108870	0.347289	0.463994	0.201265	0.000035
2		0.001119	0.038915	0.208524	0.334466	0.000397
3		0.000000	0.001138	0.035913	0.267463	0.002592
4			0.000009	0.002687	0.117940	0.011073
5			0.000000	0.000093	0.030925	0.033344
6				0.000002	0.005055	0.074305
7				0.000000	0.000532	0.126793
8					0.000037	0.169926
9					0.000002	0.182393
10					0.000000	0.159257
11						0.114557
12						0.068601
13						0.034502
14						0.014683
15						0.005322
16						0.001652
17						0.000441
18						0.000102
19						0.000020
20						0.000004
21						0.000001
22						0.000000
$E(X)$	exact	1/9	3/7	1	7/3	9
	calculated	0.111111	0.428571	1.000000	2.333333	9.000000
$Var(X)$	exact	10/99	30/91	2/3	70/51	90/19
	calculated	0.101010	0.329670	0.666667	1.372549	4.736842

variance calculated using our procedure to those obtained in (10) and (11). The following table shows that our iterative procedure produces exactly the same values as the ones obtained in (10) and (11).

5. Conclusion

In many cases of queueing system analysis, the probability distributions of performance measures are indirectly obtained as z-transforms (or as Laplace-Stieltjes transforms in continuous cases), which may not always result in explicit expressions of the probabilities. We present a procedure which finds the probability distribution of number of customers accumulated when the server ends a vacation in the Geo/G/1 gated queueing systeme, where a vacation (also a service for a customer) takes one slot time. The p.g.f for the number of customers accumulated at a gate closing epoch is obtained as a recursive form. The full probabilities, then, are derived using an iterative procedure. Numerical examples show that the probability distribution calculated using our procedure results in exactly the same mean and variance as exact ones.

References

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Appendix : Proof of Equation (4)

We employ mathematical induction technigue for the proof. By taking the derivative on the right hand side of equation (3), we obtain

$$F^{(1)}(z) = \lambda\{ F^{(1)}(1-\lambda+\lambda z)(1-\lambda+\lambda z) + F^{(0)}(1-\lambda+\lambda z)\}. \quad \text{(A1)}$$

This satisfies equation (4) when $n=1$. Now if equation (4) is assumed to hold when $n=k$, then

$$F^{(k)}(z) = \lambda^{k}\{ F^{(k)}(1-\lambda+\lambda z)(1-\lambda+\lambda z) + k\,F^{(k-1)}(1-\lambda+\lambda z)\}. \quad \text{(A2)}$$

By taking the derivative on both sides of (A2), we can show that

$$F^{(k+1)}(z) = \lambda^{k+1}\{ F^{(k+1)}(1-\lambda+\lambda z)(1-\lambda+\lambda z) + k\,F^{(k)}(1-\lambda+\lambda z)\}. \quad \text{(A3)}$$

Equation (A3) also satisfies equation (4). Thus equation (4) holds for $n \geq 1$. ■