Process Reliability Improvement and Setup Cost Reduction in an Imperfect Production System

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Abstract

In studying an EOQ-like inventory model for a manufacturing process, a number of findings were made. The system can "go out of control" resulting in a relatively minor problem state or "break-down." When the production system is in the minor problem state it produces a number of defective items. It is assumed that each defective piece requires rework cost and related operations. Once the machine breakdown takes place, the production system produces severely defective items that are completely unusable. Each completely unusable item is immediately discarded and incurs handling cost, scrapped raw material cost and related operations. Two investment options in improving the production process are introduced: (1) reducing the probability of machine breakdowns, and (2) simultaneously reducing the probability of machine breakdowns and setup costs. By assuming specific forms of investment cost function, the optimal investment policies are obtained explicitly. Finally, to better understand the model in this paper, the sensitivity of these solutions to changes in parameter values and numerical examples are provided.

Keywords: Inventory, Production, Quality

1. Introduction

Recent studies of Just-In-Time (JIT) systems have emphasized small production lot sizes and frequent deliveries. It has been widely reported that smaller lot production not only reduces inventory holding costs but also increases process quality (see, for example, Porteus (1986)). However, one of the major impediments to the successful operation of such tightly coupled organizations is the

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breakdown in bottleneck resources or the production of defective items. Thus, the model discussed here has been designed to investigate the option of investing in a production process improvement program to reduce the defective items and impact of equipment breakdowns on the operating policy in an EOQ-like inventory system. A similar concept was mentioned by Hutchins (1992), with regard to the case of the Matsuchita Refrigerator Company. According to him, the company spent more time in preproduction stages than its western counterparts. However, this time scale is the only persistent area of Japanese inferiority, as Hutchins points out. Following the launch of production, a Japanese company needs two weeks to achieve >99 percent defect-free production, compared to its typical western counterpart, which is 40 percent defect-free two years after the launch of production.

In brief, the inventory system considered in this model is subject to two types of uncertainties: The first type is the production of defective items. This has been studied extensively by Porteus (1986, 1990), Rosenblatt and Lee (1986), Lee and Rosenblatt (1987), and Lee (1992). We refer to Yano and Lee (1995) for a review. Among these, Porteus (1986) and Rosenblatt and Lee (1986) model the stochastic process of making defective items very simply: while producing a lot, a process may go "out of control." Once the production process is in that state, it is assumed that it will produce defective items and continue to do so until the entire lot is finished. At the beginning of the next lot's production, the process is restored to the same initial in-control state. The second type of uncertainty is machine breakdowns, which has been considered by Groenevelt, Pintelon, and Seidmann (1992a,b). Their assumptions are that the production process can be interrupted by machine breakdowns, and that the times until the breakdowns are exponentially distributed. The production process improvement program to be considered in this model extends the work of Porteus (1986). The novel aspects of this model, compared to Porteus's (1986), are, first, his model addressing the benefits of improved quality control. Here we begin to consider effects of improved process reliability (fewer machine breakdowns). We show that greater reliability can both reduce the optimal target lot size and improve output quality. Second, Porteus explicitly shows a significant relationship between quality and lot size. We have extended this relationship to include a reliability issue (machine breakdown). Therefore, both the relationship between quality and lot size, and that between reliability and lot size, and reliability and quality, are investigated in this work.

Two options for investing in the improvement of the production process are introduced: (1) reduce the probability of machine breakdowns, and (2) simultaneously reduce the probability of machine breakdowns and setup costs. The mathematical formulation for the investment cost function is similar to that for the setup reduction models and their related topics. These models have been studied extensively by Porteus (1985, 1986), Spence and Porteus (1987), and Fine and Porteus (1989), to name a few, so explanatory details are omitted here.

This paper is organized as follows. In § 2, assumptions and the stochastic nature of the production system are described. In § 3, an optimal lot size that accounts for defective items and machine breakdowns is specified. In § 4, a process reliability improvement program to reduce the probability of machine breakdowns is analyzed. In § 5, a simultaneously process reliability improvement and setup reduction program is studied. Conclusions are provided in the final section.

2. Problem Definition

Let us assume a traditional EOQ world in which the basic model is modified as follows. First, while producing a single unit of product, equipment breakdown occurs with probability $\alpha\,(1-\alpha=\beta)$.

That is, the production system is assumed to follow a two-state Marcov chain during production, with a transition occurring with each unit produced. Once the breakdown takes place, the production system begins to produce severely defective items that are completely unusable. Equipment maintenance (including repair of the broken machine) is carried out at the beginning of the next lot's production. The time for repair and setup is assumed to be negligible. Each maintenance action restores the system to the same initial working conditions. Throughout the work we will use the term "unusable items" to refer to the items produced in machine breakdown state.

Second, while in production, with probability $q\,(1-q=\rho)$ the production system can "go out-of-control". The production system begins to produce defective units. The stochastic process is similar to that for equipment breakdowns that is, it follows a two-state Marcov chain, with a transition occurring with each unit produced. Once the system is out of control, it remains that way until the remainder of the target lot has been completed or equipment breakdown takes place. Each defective item incurs an extra cost for reworking and related operations. Again, each maintenance action restores the system to the same initial working conditions. We will use the term "defective items" to refer to the items produced in "out-of-control" state.

The assumption that the machine can either go "out-of-control" (producing defective items) or breakdown (producing unusable items) is based on the work of Pate-Cornell, Lee and Tagaras (1987), in which they assume that the production system can be in one of the following four states:

(1) State 1: perfect condition; (2) State 2: minor problem;
(3) State 3: major problem; or (4) State 4: machine break-down.

Our assumption of "out of control state" is analogous to that of States 2 and 3, and our assumption of machine breakdown is akin to that of State 4 in Pate-Conell et al. Pate-Conell et al.

assume that States 2 and 3 can be observed at the earliest L time units after it occurs, and the failure of the machine (state 4) is assumed to be immediately detected. We assume that both "out of control" state and "achine breakdown state" can only be detected after the whole lot has been completed. This assumption has been used by Rosenblatt and Lee (1986), Lee and Rosenblatt (1987), and Porteus (1986, 1990; for the case in which $Q < C_1/(qC_RQ_1)+Q_D+Q_1)$. Porteus (1986) furnishes an example of such operation

> "..One interpretation is that the firm uses the inspection policy suggested by Hall (1983) that inspects only the first and last pieces of the lot. (If the last piece is good, then the entire lot is judged to be good. If not, further inspection is needed to determine the defective pieces.)....

In addition, the assumption that when the machine is in an "out of control state" or "machine breakdown state" all items produced are defective or unusable is based on the work of Porteus (1986,1990). Porteus (1990) provides an example of such an operation.

> "...This Simple Modeling approach is supported by Moden (1983).... A high-speed automatic punch press, for example, where lots of 50 or 100 units are kept in a chute, only the first and the last unit in the chute are inspected. If both units are good, all units in the chute are considered good....."

Figure 1 illustrates the resulting sample path for this model. For example, the first inventory cycle represents the case in which the target lot is accomplished, and the whole produced lot consists of good items. The second cycle illustrates the case in which the target lot is not achieved due to machine failure. However, the produced lot consists of good items. The third cycle illustrates the case in which target lot is achieved. However, a portion of the produced lot is defective, and require reworks. The fourth cycle shows the case in which a target lot is not achieved due to machine failure, and a part of the produced lot consists of the defective items

We begin our analysis by briefly introducing the notations. Let P be the cost of production, Q the target lot size, r_1 the opportunity cost rate (Pr_1 holding cost), r_s the unit rework cost as a percent of production cost, r_u the unit cost for disposal of each completely unusable item, d' the deterministic, constant demand rate. The case of a finite production rate m is covered by adjusting $d=d'(m/(d'+m))$, S the setup cost, Z(Q) the expected lot size including good items and defective items, and δ_Z the expected defective items. Note that neither setups nor production time are assumed to be time-consuming in this model. The shortages are not allowed. We develop the expected lot size Z(Q) for a target lot size of Q, and the expected number of defective items δ_Z.

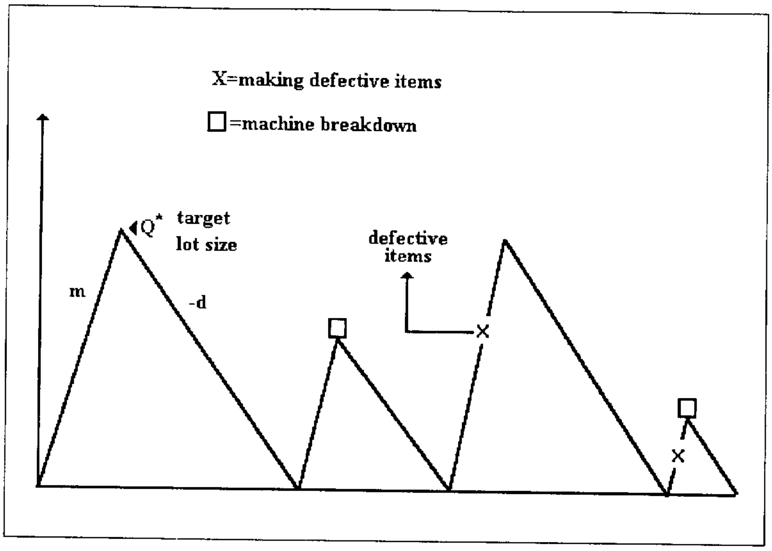

Figure 1. Inventory Sample Path

Lemma 1.

(1.1) The expected lot size (including good items and defective items) is

$$Z(Q) = \frac{\beta(1-\beta^Q)}{\alpha} \tag{1}$$

which is strictly concave and strictly increasing in Q. We see that $Z(Q) < \beta/\alpha$, *and* $\lim_{Q\to\infty} Z(Q) = \beta/\alpha$. *Let* $\zeta_Z = Q - Z(Q)$ *be the expected number of completely unusable items due to machine breakdown, which is a strictly increasing, strictly convex function of Q.*

(1.2) The expected number of defective items is

$$\delta_Z = Z(Q) - \frac{\beta\rho(1-(\beta\rho)^Q)}{1-\beta\rho} \tag{2}$$

which is strictly increasing in Q . $Y(Q) = \beta\rho(1-(\beta\rho)^Q)/(1-\beta\rho)$ *is the expected number of good items; and it is strictly concave and strictly increasing in Q.*

Proof. The proof of Lemma 1 is given in Appendix 1.

Lemma 1.1 and 1.2 give the expected lot size and the number of defective items. For example, if $q = 0.01$, $\alpha = 0.01$, and $Q = 100$; then the expected lot size $Z(Q)$ and defective items δ_Z are 62.76(62.76% of 100) and 20.11(32% of 62.76), respectively. If α is decreased to 0.001, and Q is decreased to 65, then the expected lot size and defectives are 62.96(97% of 65) and 16.84(27% of 62.96). These figures tell us two things. First, a system with relatively high reliability (small α) incurs fewer unusable items. Second, the percentage of defectives is smaller for a reliable system.

3. Optimal Lot Size

Our first objective is to determine the optimal lot size Q^*, i.e., the target lot size that minimizes the long-run average cost per unit time while accounting for the quality and reliability effects of this model. Note that the inventory process has a renewal epoch at the beginning of each inventory cycle due to setup operations. Therefore, using the well-known renewal reward theorem, the long-run average cost can be found by taking the ratio of the expected cost per renewal cycle and the expected duration of a renewal cycle. The long-run average cost function (3) consists of setup, holding, rework, and disposal costs.

$$\overline{C}(Q) = \frac{Sd}{Z(Q)} + \frac{\Pr_1 Z(Q)}{2} + \Pr_s d\left[1 - \frac{Y(Q)}{Z(Q)}\right] + \Pr_u d\left[\frac{Q}{Z(Q)} - 1\right] \tag{3}$$

An optimal target lot size Q^* minimizes cost expression (3). Proposition 1 summarizes the structural properties of the model.

Proposition 1. *When α and q are very small:*

(1.a) The total cost is approximately

$$\cong \frac{dS}{Z(Q)} + \frac{Z(Q)P(r_1 + d(r_s q + r_u \alpha)/\beta^3)}{2} \tag{4}$$

(1.b) Optimal expected production lot size minimizing (4) is

$$Z(Q^*) = \sqrt{\frac{2dS}{P(r_1 + d(r_s q + r_u \alpha)/\beta^3)}}.$$

By Lemma (1.1), the optimal target lot size is

$$Q^* = \frac{\ln(1-\alpha Z^*/\beta)}{\ln\beta}, \quad \text{if } Z^* < \frac{\beta}{\alpha}. \tag{5}$$

(1.c) If (5) exists, then the optimal cost is

$$\overline{C}(Q^*) = \overline{C}(Z^*) = \sqrt{2dSP\ [r_1 + \frac{d(r_s q + r_u \alpha)}{\beta^3}]} \tag{6}$$

(1.d) $\overline{C}(Q^*)$ *is strictly increasing and strictly concave in* q, r_1, P, r_s, r_u, S, *and d and strictly decreasing and strictly convex in* β

Proof. Proofs for Proposition 1 are given in Appendix 2.

Proposition 1.b tells us that optimal production lot size Q^* can be obtained by computing Z^* first, and then substituting it into (5). Proposition 1.b also shows us that the optimal solution approaches infinity as Z^* approaches β/α. However, the model assumes α to be very small so that the ratio β/α, is relatively large. Therefore, there is a relatively wide feasible area for the optimal solution. Propositions 1.d tells us that the parametric properties of the classical EOQ model are preserved in this model. Consider a numerical example given in Porteus (1986): $S = 100$, $d = 1000$, $r_1 = 0.15$, $r_s = 0.5$, $q = 0.0004$, $P = 50$, and $\beta = 0.999$. Here, additionally, we let $r_u = 0.03$. If the quality and reliability effects of this model are ignored that is, if the classical EOQ model is used, then, Q, $Z(Q)$, and δ_Z are 163.3, 150.6, and 4.7 respectively, and the total cost by (3) is 2,141. If

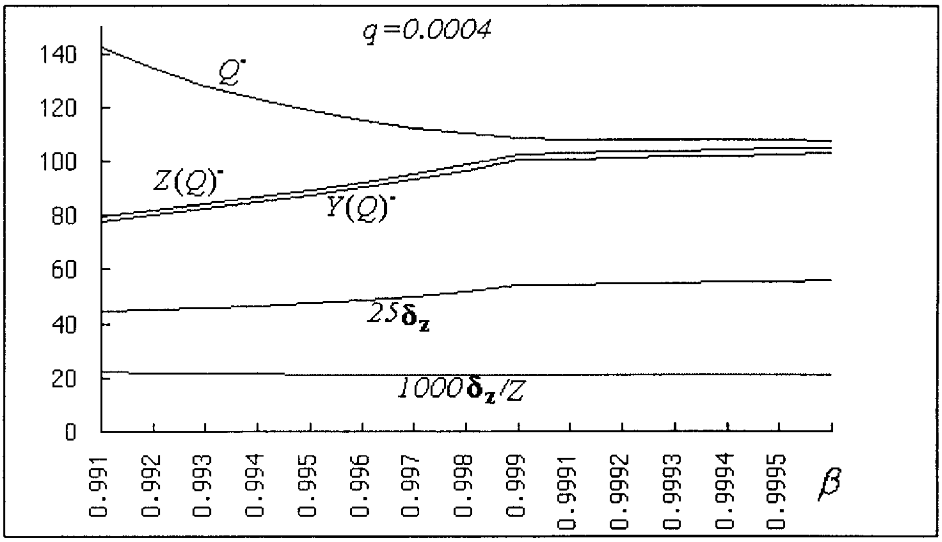

Figure 2. Z(Q), Q and δ as a Function of β

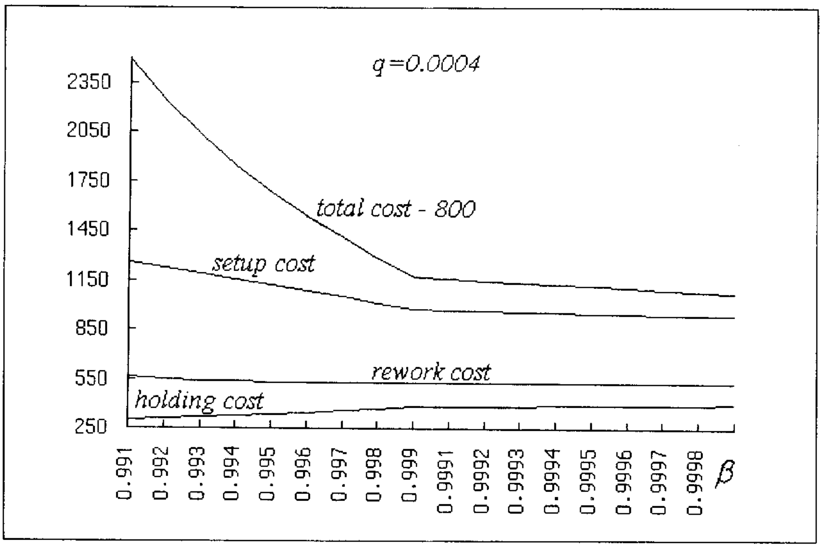

Figure 3 . Long-run Average Cost Components as a Function of β

the formula from (5) is used, Q^*, $Z(Q^*)$, and δ_Z and drop to 108.2, 102.5, and 2.2, respectively, and the total cost by (3) is 1,973, which is a 7.8% reduction.

Here, with the same set of numerical examples, the model is studied in light of changes in the underling parameter values. Figure 2 illustrates that Q^* is decreasing, convex in β, and Z^* is creasing, concave in β. We also see that $Q^* - Z^*$ and δ_Z / Z^* are decreasing in β. Therefore, improving reliability not only can reduce the unusable items, but also can reduce the percentage of defectives.

Figure 3 gives the total costs per unit time of the optimal policy and its components as a function of β. As expected, the expected rework costs decrease with β due to the percent of defectives δ_Z / Z^* decreasing in β. As β increases, the average inventory holding costs increase. This increase is the result of an increase in Z^*. Finally, setup costs decrease in β because a reliable production system requires fewer setups. We also see that the optimal total cost function is strictly convex, decreasing in β. In Figures 2 and 3, some variables are modified to fit into one graph. Their general shapes are preserved, however. Some of the properties shown in Figures 2 and 3 also are verified in Proposition 1.

Figure 4 displays the total cost as a function of Q for $\beta = 0.992, 0.994, 0.999$. For a smaller value of β the optimum value of Q shifts to the right. If we let $r_u = 0$, then the total

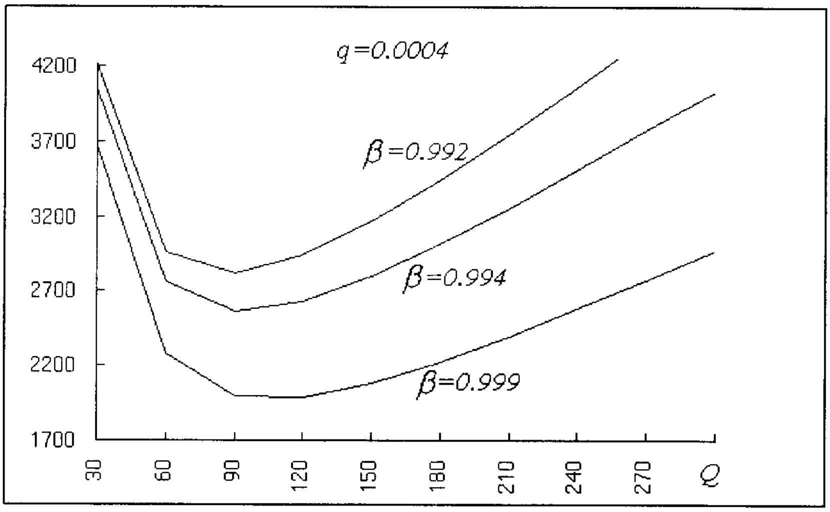

Figure 4. Long-run Average Cost as a Function of Lot Size

cost function can be illustrated as in Figure 5. Interestingly, the cost function flattens out for smaller values of β For unreliable machines, actual lot size (including good items and defective items) will almost always be determined by equipment breakdown. Therefore, when all other factors are kept unchanged, the cost function is almost indifferent to the target lot size after it passes a certain number. The two properties mentioned above also have been verified by Groenevelt, Pintelon and

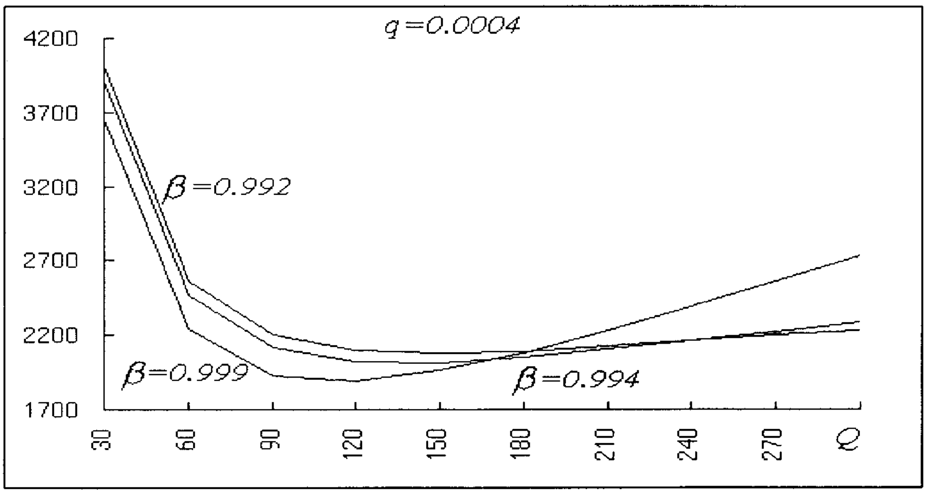

Figure 5. Long-run Average Cost as a Function of Lot Size

Seidmann (1992a) in a similar model with a different set of assumptions about the machine breakdown process.

4. Process Reliability Improvement

Here we improve the reliability of the production system by investing in the reliability improvement program. The approach used to tackle the problem is the reduction of the reciprocal of β Therefore, the lowest possible value of $1/\beta$ is 1 (when $\beta = 1$), and the highest possible value of is $1/\beta^o$ (when $\beta = \beta^o$), In this way, we choose the optimal β value between $[\beta^o, 1]$ to improve the production system's reliability. In common with other works (for example, Porteus (1985, 1986)), we assume that the investment cost function has a particular form and then derive the explicit results. Let $b_\beta r_1 \ln(\beta/\beta^o)$ be the investment cost of changing the reciprocal of β from the original to the improved level, where b_β is the cost of making an approximately 63% reduction in $1/\beta$. The particular form of the investment cost function is well documented in Porteus (1985); therefore, explanatory details are omitted here. In general, we minimize

$$\min_{\beta^o \le \beta \le 1} \omega(Q^*, \beta) = \overline{C}(Q^*, \beta) + b_\beta r_1 \ln(\beta/\beta^o) \tag{7}$$

In the following work, with the assumption that β is near 1, we let

$$Z(Q^*) \cong \sqrt{\frac{2dS}{P(r_1 + d(r_s q + r_u a)/\beta)}} \tag{5.1}$$

$$\overline{C}(Q^*) = \overline{C}(Z^*) = \sqrt{2dSP\left(r_1 + \frac{d(r_s q + r_u a)}{\beta}\right)} \tag{6.1}$$

The reasons for using the approximation are twofold. First, in so doing, the analysis in the latter portion of the paper can be greatly simplified. Second, if β is significantly large, then the accuracy of the approximation should not be reduced too much. Consider again the numerical example discussed earlier, with the addition of four levels of β = 0.992, 0.993, 0.995, and 0.999. The accuracy of the approximations is determined by testing one against the other and the exact optimal

Table 1. Comparison of the Optimal Solutions

β	0.992	0.993	0.995	0.999
Optimal Q by (5.1)	135	128	119	108
Total Cost by (3)	3045	2831	2482	1972
Optimal Q by (5)	134	127	118	108
Total Cost by (3)	3034	2824	2481	1972
Exact Optimal	86	91	95	105
Total Cost by (3)	2816	2688	2436	1972
% of Cost Difference Between (5.1) and Exact Optimal	8.1	5.2	1.8	-

solution, where the exact optimal solution is numerically searched from cost function (3). Approximations are obtained from (5) and (5.1) Total costs are obtained by substituting them into cost function (3). Table 1 shows that the total costs and target lot sizes obtained from (5) and (5.1) are almost identical when β is relatively large.

Proposition 2 summarizes the decision rules of the optimal solution.

Proposition 2: *Optimal Probability of Machine Breakdowns.*

We use $\min_{x \vee y} \overline{C}(Q^*)$ *to denote x or y, which minimizes* $\overline{C}(Q^*)$.

(2.a) The situation is divided into three mutually exclusive and collectively exhaustive cases.

Case 1. $r_1 > dr_u$ *or* $(b_\beta r_1)^2 - 2PdS(r_u d - r_1) \geq 0$ and $d(r_s q + r_u)/2(dr_u - r_1) > 1$,

$\omega(Q^*, \beta)$ *has a unique local minimum* β^**on* $[\beta^o, 1]$ *and the optimal solution is given in Proposition 2.b.*

Case 2. $(b_\beta r_1)^2 - 2PdS(r_u d - r_1) \geq 0$ and $d(r_s q + r_u)/2(dr_u - r_1) \leq 1$ *The optimal solution* satisfies $\min_{\max[\beta^1 \beta^o] \vee 1} \overline{C}(Q^*)$, *where* β^1 *is given in Proposition 2.b.*

Case 3. $(b_\beta r_1)^2 - 2PdS(r_u d - r_1) < 0$, *The optimal solution is* $\beta^* = 1$

(2. b) The optimal probability β^* satisfies $\beta^* = \min[\max[\beta^o, \beta^1], 1]$, *where* β^1 *satisfies*

$$\beta^1 = \frac{(r_s q + r_u) PSd^2}{(b_\beta r_1)^2 + b_\beta r_1 \sqrt{(b_\beta r_1)^2 - 2PdS(dr_u - r_1)}} \tag{8}$$

(2.c) β^1 *is increasing in* P, d, S, r_s, r_u *and* q, *and decreasing in* b_β, *and* r_1

Proof. The proof for Proposition 2 is given in Appendix 3.

Proposition 2.a reveals that a unique optimum solution may exist in $[\beta^o, 1]$ under certain conditions. Proposition 2.b gives the optimal solution; Proposition 2.c tells how the optimal solution varies as a function of the parameters. Here the probability of machine breakdown (q) is negatively related to the probability of making defective items (α). That is, the lower the process quality is, the greater the investment should be in improving process reliability. Consider again the numerical example discussed earlier, adding the option of investing in lowering the probability of machine breakdown, and let $b_\beta = 100{,}000$. Proposition 2.a suggests β be increased to 1. The total cost by (3) is 1,883, a 12% reduction on the classical EOQ model. $Q^*, Z(Q^*)$, and δ_Z are, respectively, 107, 107, and 2.3.

5. Optimal Reliability Improvement and Setup Reduction

In this section we will investigate the option of simultaneously reducing the probability of machine breakdown and setup costs. In general, we seek to minimize

$$\min_{\beta, S} \ \omega(Q^*, \beta, S) = \overline{C}(Q^*, \beta, S) + b_\beta r_1 \ln(\beta/\beta^o) + b_S r_1 \ln(S^o/S) \tag{9}$$

where $b_S r_1 \ln(S^o/S)$ is the investment cost of changing the setup cost from the original level S^o to the improved level S, and b_S is the cost of making an approximately 63% reduction in S. If we only consider the option of setup reduction, then, as shown in Porteus (1986), the optimal setup reduction policy is $S^* = \min[S^o, S^1]$, where S^1 satisfies

$$S^1 = \frac{2(b_s r_1)^2}{dP(r_1 + d(r_s q + r_u \alpha)/\beta)} \tag{10}$$

which is a strictly increasing function of β. That is, the higher the process reliability, the lower the investment in the setup cost reduction program. Proposition 3 summarizes the decision rules under different situations.

Proposition 3 *We divide the situation into four mutually exclusive and collectively exhaustive cases.*

Case 1. If $dr_u - r_1 > 0$ *and* $b_\beta < b_S$, *then the optimal probability of machine breakdowns is* $\beta^* = 1$

Case 2. If $dr_u - r_1 > 0$ *and* $b_\beta \geq b_S$.

If $1 - \frac{(dr_u - r_1)\beta^o}{(r_s q + r_u d)} < \frac{b_s}{b_\beta}$ *then* $\beta^* = 1$,

Else if $1 - \frac{(dr_u - r_1)\beta^o}{(r_s q + r_u d)} \geq \frac{b_s}{b_\beta}$, *then the optimal solution satisfies* $\min_{\beta^o \vee 1} \overline{C}(Q^*)$,

Case 3. If $dr_u - r_1 \leq 0$ *and* $b_\beta \leq b_S$,

If $1 - \frac{(dr_u - r_1)}{(r_s q + r_u d)} \leq \frac{b_s}{b_\beta}$ *then* $\beta^* = 1$.

Else if $1 - \frac{(dr_u - r_1)}{(r_s q + r_u d)} > \frac{b_s}{b_\beta}$ *then a unique optimal solution* β^* *exists on* $[\beta^0, 1)$.

Case 4. If $dr_u - r_1 \leq 0$ and $b_\beta > b_s$, *then the optimal probability of machine breakdowns is* $\beta^* = \beta^0$.

The optimal setup cost for $\beta^* = \beta^0$ *or 1 can be obtained by directly substituting* β^* *into (10).*

Proof. The proof for Proposition 3 is given in Appendix 4.

For the purposes of the next proposition, let

$$S1 = \sqrt{\frac{S^o dP(r_1 + d(r_s q + r_u a^o)/\beta^o)}{2r_1^2}}, \quad S2 = \frac{S^o dP(r_1 - r_u d)}{2r_1^2},$$

$$S3 = \frac{(r_1 - r_u d)\beta^o}{d(r_s q + r_u)} + 1, \qquad S4 = \frac{d^2 S^o P(r_s q + r_u)}{\beta^o r_1 \sqrt{2S^o\, dP(r_1 + d(r_s q + r_u a^o)/\beta^o)}}$$

Proposition 4 summarizes the decision rules for the optimal simultaneously process reliability improvement and setup reduction program when $dr_u - r_1 \leq 0$ and $b_\beta \leq b_s$ (Case 3 in Proposition 3.)

Proposition 4. *The following approach resembles that of Porteus (1986) in many ways. We now introduce four different cases that emerged through this process.*

Let

$$C1: = \{b_s / b_\beta > S3,\ \ b_s(b_s - b_\beta) < S2\},\quad C2: = \{b_s < S1,\ \ b_s / b_\beta \leq S3\}.$$

$$C3: = \{b_\beta < S4,\ \ b_s(b_s - b_\beta) \geq S2\} \text{ and } C4: = \{b_s \geq S1, b_\beta \geq S4\}.$$

(4.a) C1, C2, C3, and C4 are mutually exclusive and collectively exhaustive.

(4.b) If $1 + \dfrac{(r_1 - dr_u)}{(r_s q + r_u d)} \leq \dfrac{b_s}{b_\beta}$, *then* $\beta^* = 1,\ S^* = S^1(1) = \min\left[S^o,\ \dfrac{2(b_s r_1)^2}{dP(r_1 + r_s dq)} \right]$

(4.c) If $1 + \dfrac{(r_1 - dr_u)}{(r_s q + r_u d)} > \dfrac{b_s}{b_\beta}$, *then the optimal solution satisfies one of the following four cases.*

Case C1: C1 holds if and only if

$$\beta^* = \beta^1(S^1) = \frac{(b_s - b_\beta)(r_s q + r_u)d}{b_\beta(r_1 - r_u d)},\ S^* = S^1(\beta^1) = \frac{2b_s(b_s - b_\beta)r_1^2}{dP(r_1 - r_u d)}.$$

$S^1(\beta^1)$ *and* $\beta^1(S^1)$ *are obtained by simultaneously solving (10) and (8).*

Case C2: C2 holds if and only if

$$\beta^* = \beta^0,\ S^* = S^1(\beta^0) = \frac{2(b_s r_1)^2}{dP(r_1 + d\,(r_s q + r_u a^o)/\beta^0)}$$

$S^1(\beta^0)$ *is obtained by substituting* $\beta^* = \beta^0$ *into (10).*

CaseC3:C3 holds if and only if

$$S^* = S^0,\ \beta^* = \beta^1(S^0) = \frac{(r_s q + r_u)PS^0 d^2}{(b_\beta r_1)^2 + b_\beta r_1\sqrt{(b_\beta r_1)^2 - 2PdS^0(r_u d - r_1)}}$$

$\beta^1(S^0)$ *is obtained by substituting* $S^* = S^0$ *into (8).*

Case C4: C4 holds if and only if $S^* = S^0$, $\beta^* = \beta^0$,

Proof. The proof for Proposition 4 is given in Appendix 5.

The following example illustrates the optimal policy described in Proposition 3. We allow both setup reduction and process reliability improvement programs. Consider again the numerical example given above. In addition, let $b_\beta = 100,000$, $r_u = 0.15$, and $b_s = 3000$, The example shows $b_\beta > b_s$ and $1-(dr_u - r_1)\beta^0 / (r_s q + r_u d) < b_s / b_\beta$; hence, by Proposition 3 the first part of Case 2 applies. The numerical example is summarized in Table 2. For example, Classical EOQ gives the result of the classical EOQ model that is, we assume that q and β are equal to 0 and 1, respectively, even though they are not. q, β adjusted EOQ leads to the result of the EOQ model discussed in section 3; optimal adjusted β provides the result for the EOQ model discussed in section 4 (reliability improvement). The optimal adjusted setup reveals the result for the setup reduction program, adjusting for q, β. Finally, the optimal setup and β provide the results for the setup reduction and reliability improvement programs.

Table 2 reveals that the cost savings generated from the reliability improvement and setup reduction programs are 28.8% and 34.5%, respectively. A total reduction of 40.4% is achieved through the simultaneously reliability improvement and setup reduction programs.

Table 2. Results of the Numerical Example

Characteristic	Classical EOQ	q, β Adjusted EOQ	Optimal Adjusted β	Optimal Adjusted Setup	Optimal Setup β
β	0.999	0.999	1	0.999	1
S	100	100	100	16.20	23.14
Q	163.3	93.7	106.9	36.6	51.4
Z(Q)	150.6	89.4	106.9	35.9	51.4
% Shortfall	7.8	4.6	0	1.9	0
Cost	2645	2275	1883	1733	1577
% Savings	-	13.9	28.8	34.5	40.4

Table 3. Results of the Numerical Example

Characteristic	Classical EOQ	q, β Adjusted EOQ	Optimal Adjusted β	Optimal Adjusted Setup	Optimal Setup β
β	0.997	0.997	1	0.997	1
S	100	100	100	10.1	23.14
Q	163.3	79.5	106.9	23.3	51.4
Z(Q)	128.9	70.6	106.9	22.4	51.4
% Shortfall	21.0	11.2	0	3.6	0
Cost	4003	3008	1913	1962	1607
% Savings	-	24.9	52.2	50.9	59.9

We now continue to consider a less reliable system. We let $\beta = 0.997$; everything else remains the same. Table 3 summarizes the numerical example. By observing the two examples, we see that the contributions of the reliability improvement program become more and more significant as the system becomes less and less reliable.

6. Conclusion

An EOQ-like inventory system subject to machine breakdown and the production of defective items is considered. Two manufacturing process improvement programs were studied here using EOQ-like inventory models. The first introduces the option of reducing the probability of machine breakdown; the second introduces the option of simultaneously reducing the probabilities of machine breakdown and setup level. The economic production lot size, investment policy, and properties are provided. Finally, to better understand the model in this paper, the sensitivity of these solutions to changes in parameter values and numerical examples are provided. The numerical examples illustrate that the contributions of the reliability improvement program become more and more significant as the system becomes less and less reliable.

Appendix 1. Proof of Lemma 1

Lemma 1.1. The expected production lot size is

$$\alpha\sum_{k=0}^{Q-1} k\beta^k + \alpha\sum_{k=Q}^{\infty} Q\beta^k = \alpha\Big[\sum_{k=0}^{Q-1} k\beta^k + \alpha\sum_{k=0}^{\infty} Q\beta^k - \sum_{k=0}^{Q-1} Q\beta^k\Big] = Q-(Q-Z(Q)) = Z(Q).$$

The results follow since $Z'(Q) = -(\beta^Q \ln\beta)\beta/\alpha > 0$. Further differentiation reveals that the second derivative is strictly negative. Since $-\beta^Q \ln\beta \leq -\beta\ln\beta < 1-\beta$, we see that $Q-Z(Q)$ is a strictly increasing function of Q. The second differentiation reveals that the second derivative is strictly positive, which proves the result.

Lemma 1.2. The expected number of defective items is

$$\delta_z = \alpha\Big[\sum_{k=0}^{Q-1}\beta^k\Big(qk\sum_{i=0}^{k-1}\rho^i - q\sum_{i=1}^{k-1} i\rho^i\Big) + \sum_{k=Q}^{\infty}\beta^k\Big(qQ\sum_{i=0}^{Q-1}\rho^i - q\sum_{i=0}^{Q-1} i\rho^i\Big)\Big]$$

$$= \alpha\Big[\sum_{k=0}^{Q-1}\beta^k\Big(k-\frac{\rho(1-\rho^k)}{q}\Big) + \Big(Q-\frac{\rho(1-\rho^Q)}{q}\Big)\Big(\sum_{k=0}^{\infty}\beta^k - \sum_{k=0}^{Q-1}\beta^k\Big)\Big]$$

$$= \alpha\Big[\Big(\sum_{k=0}^{Q-1}\beta^k k - \sum_{k=0}^{Q-1}\beta^k Q\Big) + \frac{\rho}{q}\Big(\sum_{k=0}^{Q-1}(\beta\rho)^k - \sum_{k=0}^{Q-1}\beta^k\rho^Q\Big) + \Big(Q-\frac{\rho(1-\rho^Q)}{q}\Big)\sum_{k=0}^{\infty}\beta^k\Big]$$

$$= \Big(\frac{\beta(1-\beta^Q)}{\alpha} - Q\Big) + \Big(Q - \frac{\beta\rho(1-(\beta\rho)^Q)}{1-\beta\rho}\Big).$$

Direct differentiation reveals that

$$\partial\delta_Z/\partial Q = \frac{(\beta\rho)^Q\beta\rho\ln\beta\rho}{1-\beta\rho} - \frac{\beta\beta^Q\ln\beta}{1-\beta}$$

$$= \frac{\beta^{Q+1}}{(1-\beta\rho)(1-\beta)}\left(\rho^{Q+1}\ln\beta(1-\beta) + \rho^{Q+1}\ln\rho(1-\beta) - \ln\beta(1-\beta\rho)\right)$$

We see that $\partial\delta_Z/\partial Q > 0$, since $0 > \frac{\rho^{Q+1}\ln\rho}{(1-\beta\rho)-(1-\beta)\rho^{Q+1}} > \frac{\rho\ln\rho}{1-\rho} > -1 > \frac{\ln\beta}{1-\beta}$,

which proves the result. It easily follows, upon differentiation, that the sign of $Y'(Q) > O$ and $Y''(Q) < 0$. □

Appendix 2: Proof of Proposition 1

Proposition 1.a. When α is very small, $1-\beta^x \cong -(\ln\beta)x - [(\ln\beta)x]^2/2$. We also see that when $\beta \geq 1/2$, $\ln\beta = (-\alpha/\beta) + (-\alpha/\beta)^2/2 + (-\alpha/\beta)^3/3 + \ldots$. Assuming α is very small, we use the first term for the approximation. Thus, $Z(Q) \cong Q[1-\alpha Q/2\beta]$. Similarly, when q and α are very small $Y(Q) \cong Q[1-(1-\beta\rho)Q/2\beta\rho]$. Therefore,

$$\delta_z / Z \cong 1 - \frac{Y(Q)}{Z(Q)} \cong \frac{Q^2 q}{2\beta\rho Z(Q)} \qquad \text{(Pla.1)}$$

By linear approximation $Z(Q) = \beta(1-\beta^Q)/\alpha \cong -\beta Q \ln\beta/\alpha$. Now given that α is very small $(-\beta \ln\beta)/\alpha = -\beta[(-\alpha/\beta)+(-\alpha/\beta)^2/2+\ldots]/\alpha \cong (1-\alpha/2\beta) \cong \beta$ and so $Z(Q) \cong Q\beta$, Substituting the approximation into (Pla.1) gives $\delta_Z / Z \cong Z(Q)\, q/2\beta^3\rho$. Finally, as in Porteus (1986), the result follows upon using $\rho \cong 1$, Table 4 compares δ_Z / Z and $Zq/2\beta^3\rho$. It tells us that the approximation approaches the exact solution as the target lot size increases.

Table 4. Comparisons of δ_Z / Z and $Zq/2\beta^3\rho$ (β =0.999, ρ =0.999)

Q	10	20	30	40	50	60	70	80
δ_Z / Z	0.548	1.04	1.528	2.011	2.489	2.962	3.430	3.894
$Zq/2\beta^3\rho$	0.499	0.994	1.483	1.967	2.448	2.922	3.392	3.858
% of Difference	8.82	4.46	2.94	2.15	1.67	1.34	1.11	0.94

Similarly, we see that $\frac{Q}{Z(Q)} - 1 \cong \frac{Q\alpha/2\beta}{1-Q\alpha/2\beta} \cong \frac{\alpha Q^2}{2\beta Z} \cong \frac{\alpha Z}{2\beta^3}$, Table 5 compares ζ_Z / Z and $Z\alpha/2\beta^3$

Table 5. Comparisons of ζ_Z / Z and $Z\alpha/2\beta^3$ (β =0.999)

Q	10	20	30	40	50	60	70	80
ζ_Z / Z	0.551	1.054	1.559	2.065	2.573	3.083	3.594	4.108
$Z\alpha/2\beta^3$	0.499	0 993	1.481	1.966	2.445	2.919	3.389	3.854
% of Difference	9.54	5.87	4.98	4.84	5.00	5.32	5.72	6.18

Propositions 1.b-1.c. These results were derived after differentiation and substitution.

Proposition 1.d. Let f_x denote the differentiation of f by x.

Variable q: Direct computation reveals $Z_q^* = -Z^*\left(\frac{dr_s}{2(r_1 + d(r_sq + r_ua)/\beta^3)\beta^3}\right) < 0.$

Further analysis shows that the second derivative is strictly positive. The first differentiation reveals that $\overline{C_q}(Q^*) = Z^*dPr_s / 2\beta^3 > 0.$ Further analysis shows that the second derivative is strictly negative.

Variable β: Direct computation leads to $Z_\beta^* = Z^*\left(\frac{d(3r_sq + r_u\beta)}{2\beta^4(r_1 + d(r_sq + r_ua) / \beta^3)}\right) > 0.$

Further computation tells us that the second derivative is strictly negative.

The first differentiation shows that $\overline{C_\beta}(Q^*) = \frac{-Z^* dP}{2\beta^4}(3(r_sq + r_u) - 2\beta r_u) < 0.$

According to further analysis, the second derivative is strictly positive. Other conclusions follow from direct differentiation and substitution

Appendix 3: Proof of Proposition 2

Proposition 2.a. Let $\overline{\omega}(\beta) = \beta\partial\omega(\beta) / \partial\beta = b_\beta r_1 - Sd^2P(r_sq + r_u) / \beta\overline{C}(Q^*)$. We see that $\omega'(\beta)$ has the same sign as $\overline{\omega}(\beta)$, It easily follows, upon observing the term $B\overline{C}(Q^*)$ (see (6.1)), that $\overline{\omega}(\beta)$ is a strictly increasing function of β if $dr_u \leq r_1$, Since (i) $\lim_{\beta\to 0}\overline{\omega}(\beta) < 0$, and $\lim_{\beta\to\infty}\overline{\omega}(\beta) > 0$, (ii) $\overline{\omega}(\beta)$ is a strictly increasing function of β The first part of the Case 1 follows directly. Assuming now $dr_u > r_1$, we see that $\overline{\omega}(\beta) = 0$ implies

$$\begin{aligned}G(\beta) &= (b_\beta r_1 \beta\overline{C}(Q^*))^2 - (Sd^2P(r_sq + r_u))^2 \\ &= -(Sd^2P(r_sq + r_u))^2 - 2SdP(b_\beta r_1)^2((dr_u - r_1)\beta^2 - d(r_sq + r_u)\beta) = 0.\end{aligned}$$

The parabola $G(\beta)$ has the same sign as $\overline{\omega}(\beta)$, and has a highest vertex at $\beta^v = d(r_q + r_u)/2(dr_u - r_1)$. Direct computation reveals that $G(\beta^v) = 0 \Rightarrow 2dPS(dr_u - r_1) / (b_\beta r_1)^2 = 1$. Hence, $2dPS(dr_u - r_1)/(b_\beta r_1)^2 > \Rightarrow G(\beta) < 0 \forall\beta\in[0, 1]$. That is, $\omega(\beta)$ is a strictly decreasing function of β. Therefore, the optimal solution is $\beta^* = 1$. This proves Case 3. Assuming now $2dPS(dr_u - r_1) / (b_\beta r_1)^2 \leq 1$, $w(\beta)$ is a convex, concave function of β. Here, we partition the situation into two exclusive cases. Assume first $\beta^v > 1$; this then implies the concave part of $\omega(\beta)$ exists in [1, ∞] therefore, the second part of Case 1 follows. Assuming now $\beta^v \leq 1$, this implies that the convex and concave parts both exist in [0, 1] Therefore, the optimal solution locates either on the extreme point or on the boundary. This completes the proof of Case 2.

Propositions 2.b, 2.c. Proposition 2.b follows from solving $G(\beta) = 0$, Proposition 2.c follows from direct differentiation. □

Appendix 4 : Proof of Proposition 3

$$\text{We see that } \overline{\omega}(\beta, S^1) = b_\beta r_1 - \frac{S^1 d^2 P(r_s q + r_u)}{\beta \overline{C}(Q^*, S^1)}$$
$$= -\frac{d(r_s q + r_u) b_s r_1}{\beta(r_1 + d(r_s q + r_u a)/\beta)} + b_\beta r_1,$$

It can be easily seen that $\overline{\omega}(\beta, S^1)$ is a strictly decreasing function of β if $dr_u - r_1 > 0$, and a strictly increasing function of β if $r_1 - dr_u > 0$, We obtain from above that $1-\beta(dr_u - r_1)/d(r_s q + r_u) = d_s / b_\beta \Leftrightarrow \overline{\omega}(\beta, S^1) = 0$, Assuming now $dr_u > r_1$ and $b_s > b_\beta$. Since $1-\beta(dr_u - r_1)/d(r_s q + r_u) \le 1 < b_s / b_\beta \Rightarrow \overline{\omega}(\beta, S^1) < 0 \ \forall \beta \in [0, 1]$, Case 1 follows directly. The first part of Case 2 follows from the fact that $1 - \beta(dr_u - r_1)/d(r_s q + r_u) \le 1 - \beta^0 (dr_u - r_1)/d(r_s q + r_u) < b_s / b_\beta \Rightarrow \overline{\omega}(\beta, S^1) < 0 \ \forall \beta \in [\beta^0, 1]$ To prove the second part of Case 2, note that (i) $\overline{\omega}(\beta, S^1)$, being a strictly decreasing function of β, (ii) $\overline{\omega}(\beta^0, S^1) > 0$ implies that $\omega(\beta, S^1)$ is a strictly concave function of $\beta \ \forall \beta \in [\beta^0, 1]$. Hence, the minimum exists on the boundary points. Cases 3 and 4 follow similarly. □

Appendix 5 : Proof of Proposition 4

Proposition 4.a. Here, we show that $C3$ is disjointed from $C1$, $C2$, and $C4$ Using a similar argument, it is not difficult to show the remaining cases. It can be easily seen that the pairs $(C1, C3)$ and $(C4, C3)$ are disjointed. To see that $C2$ and $C3$ are disjointed, we show that both hold and derive a contradiction. $b_s / b_\beta \le S3$ and $b_s(b_s - b_\beta) \ge S2$ implies $b_s \ge S1$, which contradicts $b_s < S1$.

Proposition 4.b. Follows directly from Proposition 3 and (10).

Proposition 4.c. Here, we partition the result into four cases based on whether the investment is made to reduce S and $1/\beta$. The sufficient conditions of the four necessary conditions given in Proposition 4.c follow from their mutual exclusivity as verified in Proposition 4.a. For example, consider $C3$ when $\beta^* > \beta^0$ and $S^* = S^o$. That is, only the reliability improvement investment is made in this case. For β^* to be optimal, $\beta^* = \beta^1(S^0) > \beta^0$, as given in (8). This then implies $b_\beta < S4$, For $S^* = S^o$, to be optimal, $S^1(\beta^1) > S^0$ which results in $b_s(b_s - b_\beta) \geq S2$ after some algebra. The other cases follow similarly □

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