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Design of Survivable Communication Networks with High-connectivity Constraints

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Abstract

Designing highly survivable interoffice telecommunication networks is considered. The problem is formulated as a minimum-cost network design problem with three node connectivity constraints. Three valid and facet-defining inequalities for the convex hull of the solutions are presented. A branch and cut algorithm is proposed based on the inequalities to obtain the optimal solution. With the lower bound by the cutting plane algorithm, a delete-link heuristic is proposed to obtain a good upper bound in the branch and bound procedure. The effectiveness of the branch and cut algorithm is demonstrated with computational results for a variety of problem sets; different lower bounds, two types of link costs and large number of links. The cutting plane procedure based on the three inequalities provides excellent lower bounds to the optimal solutions.

1. Introduction

Survivability becomes an important issue in the design of fiber optic communication networks. To restore services from catastrophic failures, such as the complete loss of a transmission link or the failure of switching facilities, additional connectivity is necessary in the network. Services could be restored by routing traffics through other links and nodes of the network. Clearly, a high level of redundant connectivity for improved network survivability requires the increase of overall network cost. This leads to the problem of designing a minimum-cost network that meets certain required connectivity constraints.

As an example, an interoffice telecommunication network consists of hub and central offices. A

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hub is a switching office through which all the demand from each central office are sent and routed for connection to other hub offices. Each central office is connected to a hub via a fiber optic system. They constitute clusters [16] such that a cluster contains one or two hubs and several central offices. Two connectivity is usually considered between two different central nodes in a cluster to guarantee the network survivability. It is supported by employing a self-healing ring or diverse protection architecture. On the other hand, the network topology with high connectivity is necessary among important hub nodes to provide reliable network services. Central offices in the network employ the add-drop multiplexers, while hubs are equipped with switching facilities such as digital cross-connect system.

Recent trends on designing survivable communication networks have mainly focused on network models with two connectivity constraints. The cutting plane approach based on integer programming model [5, 6, 7, 10] and various heuristics [2, 11] have been proposed for the survivable network design. However, the increased traffics and the development of new equipments such as add-drop multiplexers and digital cross-connect systems require more than two connectivity in the design of interoffice communication networks [16]. The requirement of high connectivity has resulted in the research on the related polyhedral theory and integer linear programing [1, 5] for the network design problem. However, theoretical results and algorithms are still in an early stage of development for the high node connectivity problems.

In this paper the network design problem with high connectivity, in particular three node connectivity constraint, is considered. Differently from the study in [5], the solution procedure to design the three connected network is presented with stronger polyhedral results. The underlying network consists of switching offices each of which is either hub or central office. Potential links between offices are assumed to be established with fiber facilities. Each link in the network has fixed cost of establishing fiber facilities. For the survivability of the network, at least three node disjoint paths are required between each pair of hub offices. However, such a requirement is not necessary for the central offices. The goal is to design a network that satisfies the survivability with minimum total link cost.

The design problem with three connectivity can be classified into two cases: unweighted link cost and weighted cost. It is known in [13, 14, 15] that the problem with unweighted link cost can be solved in linear time. However, the problem with weighted link cost is NP-complete even if a given initial graph is two-connected. In this paper, the design problem with weighted link cost is considered.

Section 2 presents the integer linear programing model of the survivable network design with three node connectivity constraints. In Section 3 we present three valid and facet-defining inequalities for the convex hull of the solutions to the problem. The branch and cut procedure which combines a cutting plane approach and a branch and bound technique is presented in Section 4. The effectiveness of the proposed algorithm is discussed with computational results in Section 5, and Section 6 concludes the paper.

2. Problem Description

In this section, we formalize the network design problem that is being considered in this paper. A set V of nodes is given which represents locations of the switching offices that must be interconnected into a network in order to provide the desired services. A collection E of possible links is also specified which represents the possible pair of nodes between which a direct transmission link connection can be placed. Let $G(V,E)$ be the graph of possible link connections. Each link $e \in E$ has a nonnegative fixed cost c_e of establishing the link connection. The cost of establishing a network $N(V,F)$ which consists of a subset $F \subseteq E$ of links is the sum of the costs of the individual link contained in F. The goal is to build a minimum-cost network so that the required survivability conditions, which we described below, are satisfied.

The survivability conditions require that the network satisfy node connectivity requirement. Groetschel and Monma [5] introduced a nonnegative integer r_v for each node $v \in V$ which represents its connectivity type. They studied basic polyhedral properties for three connected polytope. However, no computational analysis is appearing for the effectiveness of the theoretical study. This paper expands the polyhedral properties which is necessary to design an efficient solution procedure. We also develop a branch and cut algorithm for the optimal three node connected network based on the polyhedral results.

We consider the case that the connectivity requirement satisfies $r_v \in \{1, 3\}$ for all $v \in V$. The node set V is divided into a set of hub nodes V_H and a set of central nodes V_C. Each hub node $v \in V_H$ with $r_v = 3$ requires three node connectivity. In other words for any origin and destination pair of hub nodes at least three internally disjoint paths must exist. The central node $v \in V_C$ with $r_v = 1$ on the other hand needs one connectivity. That is, it is enough for a central node to have a connection with any other nodes in the network. We consider the three node connectivity (3NCON) problem as follows.

Given a graph $G(V,E)$ and $W, Z \subseteq V$, we use the following notations.

$\delta(W) := \{ ij \in E \mid i \in W, j \in V \setminus W \}$: the cut induced by W;

$E(W) := \{ ij \in E \mid i, j \in W \}$: the set of links induced by W;

$G[W] := (W, E(W))$: the subgraph induced by W;
G/W : the graph where $W \subseteq V$ is shrunk to a node;
$G-Z$: the graph obtained by removing each node $z \in Z$ and its incident links from G;
$\delta_{G-Z}(W) := \{ ij \in E \mid i \in W\backslash Z,\ j \in V\backslash(W \cup Z) \}$;
$r(W) := \max\{ r_s \mid s \in W \}$: the connectivity type of W.

Let us now introduce, for each link $e \in E$, a variable x_e and the vector space R^E such that every subset $F \subseteq E$ induces an incidence vector $X^F \in R^E$ by setting $X_e^F := 1$ if $e \in F$, and $X_e^F := 0$ otherwise. For any subset of links $F \subseteq E$, we define $x(F) := \sum_{e \in F} x_e$. We now formulate the 3NCON problem as the following integer linear program.

(1) Minimize $\sum_{e \in E} c_e x_e$ subject to

(i) $x(\delta(W)) \geq 3$	for all $W \subseteq V$ with $r(W) = r(V\backslash W) = 3$;
(ii) $x(\delta_{G-Z}(W)) \geq 1$	for all $Z \subset V$ with $\lvert Z\rvert = 2$ and
	for all $W \subseteq V\backslash Z$ with $r(W) = r(V\backslash(W \cup Z)) = 3$;
(iii) $x(\delta(W)) \geq 1$	for all $W \subseteq V$ with $r(W) = 1$ or $r(V\backslash W) = 1$;
(iv) $x_e \in \{0, 1\}$	for all $e \in E$.

In the model, each cut inequality of type (i) requires three link disjoint paths for each origin-destination hub pair. The node cut inequality of type (ii), on the other hand, requires three node disjoint paths. The inequalities of type (iii) require the network to be connected. The solution space of the problem (1) can be described with the following connected polytope

3NCON(G) $:= \text{conv}\{ x \in R^E \mid x$ satisfies (i), (ii), (iii) and $(iv)\}$

where conv denotes the convex hull operator. By deleting inequalities (iii), we obtain a special 3NCON problem with a node type $r_v = 3$ for all $v \in V$.

3. Valid Inequalities for 3NCON Polytope

In this section, we assume the survivable network design problem with a node type $r_v = 3$ for all $v \in V$. Thus the following connected polytope is considered.

3NCON(G) $:= \text{conv}\{ x \in R^E \mid x$ satisfies (i), (ii) and $(iv)\}$.

Three classes of valid inequalities for 3NCON(G) and conditions under which they define a facet of 3NCON(G) are considered. The inequalities will be employed to implement a branch and cut procedure for the problem with a node type $r \in \{1, 3\}^V$.

An inequality $a^T x \le b$ is valid with respect to a polyhedron P if P $\subseteq \{ x \mid a^T x \le b \}$ and the set $F_a := \{x \in P \mid a^T x = b\}$ is called the face of P defined by $a^T x \le b$. If dim(F_a) = dim(P) - 1 and $F_a \ne \emptyset$, then F_a is a facet of P and $a^T x \le b$ is called facet-defining or facet-inducing. Given a graph G(V,E), the inequalities (1)-(i) and (ii) are known to be valid and define a facet of 3NCON(G) [5].

3-1. Partition Inequalities

Let us introduce the so-called partition inequalities that generalize a class of cut inequalities (1)-(i). Consider a graph $G(V,E)$ and a partition of V into node disjoint nonempty sets, $W_1, W_2, \ldots, W_p$, $p \ge 2$. A collection $W_1, W_2, \ldots, W_p$ of subsets of V is called a partition of V. Then the partition inequality induced by $W_1, W_2, \ldots,$ Wp is given by

$$(2) \qquad 1/2 \sum_{i=1}^{p} x(\delta(W_i)) \ge \lceil 3/2 * p \rceil.$$

where $\lceil x \rceil$ denotes the smallest integer not smaller than x.

Suppose that each W_i be composed of a node in V for $i = 1, \ldots, |V|$. Then the partition inequality (2) can be expressed as

$$(3) \qquad x(E) \ge \lceil 3/2 * |V| \rceil.$$

The following theorem shows the inequality (3) defines a facet of 3NCON(G).

Theorem 1. Given a graph $G(V,E)$ where $|V|$ is odd and a node type $r \in \{3\}^V$, if G contains two node-disjoint spanning cycles $C_1, C_2 \subseteq E$ (See Figure 1(a)), then the partition inequality (3) defines a facet of 3NCON(G).

(Proof) Let $F_a := \{x \in 3NCON(G) \mid x(E) = \lceil 3/2 * |V| \rceil\}$. Clearly, F_a is a face of $3NCON(G)$ since the partition inequality (3) is valid for $3NCON(G)$. Also, $F_a \ne \emptyset$, since G satisfies four node connectivity. That is, a 3NCON network with $3/2|V|$ links is always constructed on the given network G as described below. Now, to prove that F_a is a facet of 3NCON(G), we construct $|E|$ affinely independent vectors in F_a as follows. This implies that dim(F_a) = $|E|$ - 1 = dim($3NCON(G)$) - 1.

First, take a cycle C_1 of $G(V,E)$, and then for a node $u \in V$ a set of links $T \subset C_2$, we call T a matching forest, may be added to C_1 as follows;

(i) two incident links to the node u such as (u, w_2), (u, w_5) are added to C_1.

(ii) then a subset of links in C_2 that no two are adjacent in C_2 are added (each link such as (w_1, w_3), (w_4, w_6) is called a matching link).

Note that such a matching forest T where $|T| = \lceil |V|/2 \rceil$ always exists in C_2 and the incidence vector X^D where $D := C_1 \bigcup T$ is in F_a. By applying this argument for each node in V successively we can construct $|C_2|$ affinely independent vectors in F_a (affinely independence can be shown easily).

Now, take a cycle C_2 of $G(V, E)$ and apply above arguments for each node in V. Then we can construct $|C_1|$ affinely independent vectors in F_a, which are also affinely independent of above $|C_2|$ vectors. Until now, we have shown that $|C_1| + |C_2|$ affinely independent vectors exist in F_a.

For any other link $e = uv \in E\backslash(C_1 \bigcup C_2)$ we want to construct a cycle C and its matching forest T using e but no other link of $E\backslash(C_1 \bigcup C_2)$, so that the corresponding incidence vector X^D, $D := C \bigcup T$, exists in F_a. This can be easily done by finding a cycle including e and constructing its matching tree in the remaining components of $C_1 \bigcup C_2$ (See Figure 1(b)). This vector is affinely independent of all the others exhibited so far because all of those satisfied $x_e = 0$. So we have $|E|$ affinely independent vectors in F_a. This completes the proof. □

Given a partition $W_1, W_2, \ldots, W_p$ of V, let G^* be $G/W_1/W_2/\ldots/W_p$ where each node set W_i is shrunk to a node w_i of type $r(W_i)$ for $i = 1, \ldots, p$. G^* is called the shrinking graph of V. Grotschel and Monma [5] presented conditions under which valid inequalities for the NCON (G^*) on a graph G^* can be lifted to valid inequalities for high-dimensional NCON(G) on a graph G that contains G^* as a subgraph. Such a lifting theorem helps to simplify the proofs for facet-defining inequalities. With the help of the lifting theorem, we claim the following result without proofs.

Corollay 1. Given a graph $G(V, E)$, let G^* be a shrinking graph $G/W_1/W_2/\ldots/W_p$ of G where p is odd. Then the partition inequality (2) defines a facet of 3NCON(G) if the following conditions hold

(a) G^* contains two disjoint cycles spanning every W_i for $i = 1, \ldots, p$ and

(b) Each $G[W_i]$ is three node connected for $i = 1, \ldots, p$.

We note that a partition inequality induced by a partition with $p = 2$ is exactly the cut inequality $x(\delta(W)) \geq 3$. However, such a inequality does not define a facet of 3NCON(G) if p is even.

3-2. Node Partition Inequalities

The node cut inequalities (1)-(ii) can also be generalized to node partition inequalities. Given a graph $G(V,E)$ and $z_1, z_2 \in V$, let $W_1, W_2, \ldots, W_p$ be a partition of $V\backslash\{z_1,z_2\}$. Then the node partition inequality induced by z_1, z_2 and $W_1, W_2, \ldots, W_p$ is given by

(4) $$1/2 \sum_{i=1}^{p} x(\delta_{G-Z}(W_i)) \geq p - 1 \text{ where } Z = \{z_1, z_2\}.$$

Given a graph $G(V,E)$ and $r \in \{3\}^V$, Grotschel and Monma characterized the conditions under which the node partition inequality (4) defines a facet of 3NCON(G) [5]. However, since some of the conditions are too strict and have redundancies, we propose simple and reasonable conditions for the node partition inequalities to define a facet of 3NCON(G) as follows.

Theorem 2. Given a graph $G(V,E)$, a node type $r \in \{3\}^V$ and $z_1, z_2 \in V$, let $W_1, W_2, \ldots, W_p$, $p \geq 4$, be a proper partition of $V\backslash\{z_1,z_2\}$. Let $G^*(V^*,E^*)$ be the shrinking graph of G. Then the node partition inequality (4) defines a facet of 3NCON(G) if the following conditions hold

(a) z_1 and z_2 are adjacent to all W_i for $i = 1, \ldots, p$ and

(b) $G^*(V^*,E^*)$ is two node connected and

(c) Each $G[W_i] - e$ is three node connected for all $e \in E[W_i]$, $i = 1, \ldots, p$.

(Proof) The node partition inequality can be written as $x(E^*) \geq p - 1$ in the graph G^*. Note that condition (c) is necessary only for lifting the inequality to the inequality (4). Thus we need to prove that $x(E^*) \geq p - 1$ defines a facet of 3NCON(G^*) if conditions (a) and (b) hold.

Suppose that $G^*(V^*,E^*)$ satisfies conditions (a) and (b) (See Figure 2(a)). Set $F_a := \{x \in 3NCON(G^*) \mid x(E^*) = p - 1\}$. Let $\hat{E} = E^* \cup E'$ where $E' = \delta(z_1) \cup \delta(z_2)$. Clearly, F_a is a face of $3NCON(G^*)$ since the inequality $x(E^*) \geq p - 1$ is valid for $3NCON(G^*)$. Now, to prove that F_a is a facet of 3NCON(G^*), we construct $|\hat{E}|$ affinely independent vectors in F_a. This implies that $\dim(F_a) = |\hat{E}| - 1 = \dim(3NCON(G^*)) - 1$.

We first show that for every spanning tree T of G^* the link set $D := T \cup E'$ is three node connected such that $X^D \in F_a$. Let T be a link set in E^*. Then

- For z_1 and z_2, clearly three node disjoint paths exist not using T.

- For z_1 (or z_2) and $u \in V^*$, take two node disjoint paths not using T (path $z_1 \rightarrow u$ and path $z_1 \rightarrow v \rightarrow z_2 \rightarrow u$ where $v \neq u$ in V^*) and another disjoint path using T but not using v (path $z_1 \rightarrow T \rightarrow u$).

- For u, $v \in V^*$, take two node-disjoint paths not using T (path $u \rightarrow z_1 \rightarrow v$ and path $u \rightarrow z_2 \rightarrow v$) and another disjoint path using T (path $u \rightarrow T \rightarrow v$).

Since G^* is two node connected, we can find $|E^*|$ affinely independent vectors of the form $X^D \in F_a$ in G^* [5].

Now let $D := T \bigcup E' \backslash \{e\}$ for each link $e = z_1 u \in E'$. We want to show that $|E'|$ affinely independent vectors X^D in F_a exist. To do so, for each link $e = z_1 u$ we construct a spanning tree T in E^* such that u is not a leaf of T (See Figure 2(b)). Consider the three connectivity between z_1 and u. Suppose nodes v, w and y are on T. Then we can find three node disjoint paths in D (path $z_1 \rightarrow w \rightarrow u$, path $z_1 \rightarrow v \rightarrow u$ and path $z_1 \rightarrow y \rightarrow z_2 \rightarrow u$). By applying this argument for each e in E' successively, we can construct $|E'|$ vectors of the form $X^D \in F_a$ in G. These vectors are affinely independent of all the others exhibited so far because all of those satisfied $x_e = 1$. So we have $|E| = |E^*| + |E'|$ affinely independent vectors in F_a. □

In [5] instead of condition (a) and (c), a condition that each $G[W_i \bigcup Z] - e$ is three node connected for all $e \in E[W_i]$, $i = 1, \ldots, p$ was used to prove that F_a is a facet of 3NCON(G). Note that conditions (a) and (c) are more simple and reasonable than conditions in [5] on a graph $G(V, E)$. The node partition inequality induced by a partition with $p = 2$ is exactly the node cut inequality $x(\delta_{G-Z}(W)) \geq 1$.

3-3. Cycle Partition Inequalities

In this section, we propose two new classes of valid inequalities on low dimensional 3NCON(G) that can be lifted to high dimensional 3NCON(G) by using lifting theorem. Such inequalities as variants of the partition inequality (3) can be employed for an effective implementation of the branch and cut procedure.

Given a graph $G(V, E)$, let $C \subseteq E$ be a cycle (C may not span) V. Then we define two inequalities as follows. On the cycle C which spans a set of nodes V, we define the crown inequality (See Figure 3(a)) as

$$(5) \qquad x(E \backslash C) \geq \lceil |C| / 2 \rceil.$$

When the cycle C does not span a node set V, let $Z \subset V$ be a set of nodes spanned by the cycle C. Then on a set of nodes Z and a set of links C we define the wheel inequality (See Figure 3(b)) as

(6) $$x(\delta(V\backslash Z)) + x(E(Z)\backslash C) \geq 1 + \lceil |C|+1 / 2 \rceil .$$

Given a graph $G(V, E)$ with a type vector $r \in \{3\}^V$, the following lemmas show that the two classes of inequalities are valid for 3NCON(G).

Lemma 1. The crown inequality (5) is valid for 3NCON(G).

(Proof) We know that the partition inequalities (3) is valid for 3NCON(G). Thus we obtain $x(E) = x(E\backslash C) + x(C) \geq |C| + \lceil |C|/2 \rceil$. By setting $x_e = 1$ for all $e \in C$, the result follows. □

Lemma 2. The wheel inequality (6) is valid for 3NCON(G).

(Proof) Let $Z \subset V$ be the node set spanned by the cycle C. From the partition inequalities (3) we obtain the following inequality

$$x(E) \geq x(\delta(VZ)) + x(E(Z)\backslash C) + x(C) \geq |C| + 1 + \lceil (|C|+1)/2 \rceil.$$

By setting $x_e = 1$ for all $e \in C$, the result follows. □

4. Branch and Cut Procedure

In the previous section we discussed the polyheral properties of the 3NCON(G) with a type vector $r \in \{3\}^V$ and showed three strong valid inequalities to define the facet of 3NCON(G). This implies that the partition, node partition and cycle inequalities may give a good effect on solving the problem. Note that the identification problem of these facet-defining inequalities is very difficult since the problem is NP-hard. However, those inequalities are very helpful to solve the problem efficiently. Now, to solve a survivable network design problem (1) with $r \in \{1, 3\}^V$, we need the following condensing operation.

Given a connected graph $G(V, E)$ where $V = V_H \cup V_C$, condensing of a set V_C of nodes to a set V_H consists of shrinking every node $u \in V_C$ into a node $v \in V_H$ connected to the node u. Note that each node $u \in V_C$ satisfies its connectivity requirement since G is connected. All links that were incident to either u or v in G are now incident to the node v. The resulting graph consists of nodes only in V_H not in V_C.

Given a condensed graph, the branch and cut procedure is employed based on the cutting plane method proposed in [7] and a simple branch and bound procedure. Thus the solution procedure is divided into the cutting plane stage and the branch and bound stage.

4.1 An Outline of the Branch and Cut Procedure

The inputs of the survivable network design problem are an underlying network $G(V, E)$ with link costs $c \in R^E$ and node types $r \in \{1, 3\}^V$. To solve the integer linear program (1), we consider a sequence of the following linear programming relaxations;

$$
(7) \qquad \begin{array}{ll}
\min\ c^T x & \text{subject to} \\
x(\delta(v)) \geq 3 & \text{for all } v \in V_H; \\
x(\delta(v)) \geq 1 & \text{for all } v \in V_C; \\
0 \leq x_e \leq 1 & \text{for all } e \in E.
\end{array}
$$

Note that the problem (7) contains the degree constraint of each node in the network, that is, a subset W of nodes is equal to a single node in (1)-(i) and (iii). Suppose that the current LP-relaxtion is solved and an optimal solution y is obtained. If y satisfies the survivability conditions (1)-(i), (ii), (iii) and (iv), then the problem is solved. If y does not satisfy any of the four conditions, then we try to generate either partition, node partition or cycle partition inequality that is violated by y (we call this a separation routine). If we can produce such inequalities, we add the inequalities to the current LP, solve the new LP, and repeat these steps.

It may happen that y is not an integer solution and that we are unable to find a valid inequality violated by y. In this case with the lower bound of the 3NCON problem (1) the branch and bound stage is started. With inequalities obtained from the cutting plane stage and integer constraints, we solve the integer programming problem by a branch and bound technique and obtain an integer solution y^*. Clearly, the solution y^* satisfies the integer constraint (1)-(iv), but may not satisfy the other constraints (1)-(i), (ii) and (iii). Thus the inequalities that are violated by y^* need to be generated as in the cutting plane stage. If no valid inequality violated by y^* can be found, then the current solution y^* is an optimum of 3NCON problem (1) since y is an integer solution satisfying the survivability constraints (1)-(i), (ii) and (iii).

Branch and Cut Procedure for the 3NCON Problem

A. *Cutting Plane Stage*

1. Solve the LP (7). Let y be an optimal solution to this LP.
2. Do the following steps if y is not feasible for the 3NCON(G).
   2-1. Find the partition inequalities violated by y and add them to the LP.

2-2. Take $E' = \{e \in E \mid y_e \geq 1/2\}$ from $G(V, E)$ and y.
If $G(V, E')$ is not connected, go to Step 1.
Else condense V_C to V_H and let G^* be the condensed graph.

2-3. Try to find node partition and cycle partition inequalities violated by y in G^* and add them to the LP.

At this point, the graph of $G(V, E')$ satisfies 3NCON(G).
Go to the branch and bound stage (B). Else go to Step 1.

B. *Branch and Bound Stage*

3. Consider an integer program with inequalities obtained from (A) and integer constraints (note that, fortunately, if the solution y obtained from (A) is an integer, such an integer solution y is an optimum).

3-1. Perform the branch and bound procedure.
Then let y^* be an optimal solution to this integer program.

3-2. Try to find the inequalities violated by y^* as in Step 2 of (A).
If no inequality can be found (this means that the graph induced by y^* satisfies 3NCON(G)), then stop.
Else add them to the integer program and go to Step 3-1.

4.2 Separation Routines

By assuming that y is a feasible solution obtained from the current LP (or integer program in the branch and bound stage), we need to check whether y satisfies all inequalities described in Section 3. Since each class of inequalities contains a number of inequalities that is exponential in $|V|$, it is impractical to consider all these inequalities explicitly.

For the cut inequalities (1)-(i) and (iii), an efficient exact separation procedure exists. It works by using Gomory-Hu algorithm [4, 12]. The separation problems for the node cut inequalities (1)-(ii) can be solved in a similar manner by first deleting two nodes so that the resulting graph becomes disconnected (these two nodes are called a separation pair [9]) and applying Gomory-Hu algorithm to the resulting graph, and repeating this for all separation pairs in V. Note that these operations are done on the condensed graph.

In theorem 1 and 2 of Section 3, we showed the partition and node partition inequalities are valid and facets of 3NCON polytope. However, the separation problems associated with partition and node partition inequalities are known to be NP-hard [7]. We employ efficient heuristics [7] to generate violated inequalities in the branch and cut algorithm. The inequalities generated in the algorithm are valid for 3NCON problem, and they may be a facet of 3NCON polytope. However, it is difficult to check whether a valid inequality is a facet or not, since the related separation problem is NP-hard.

The separation problem for the cycle partition inequalities is also NP-hard since those inequalities are derived from the partition inequalities. Thus a separation heuristic for cycle partition inequalities is designed as follows. Given a condensed graph $G(V_H, E)$, let z_1, z_2 be a separation pair of $G(V_H, E)$. We first identify a cycle spanning nodes z_1 and z_2. This is done by combining two node disjoint paths: a path from z_1 to z_2 and a path from z_2 to z_1. Let $C \subseteq E$ be a cycle found by such a procedure and let $Z \subset V_H$ be the set of nodes that are spanned by the cycle C.

To generate the crown inequality (5), each node u in $V_H \backslash Z$ is shrunk to a node v in Z that is connected to the node u. Then the graph $G(V, E)$ is partitioned into a set of nodes in Z. For the wheel inequality (6), each node $u \in V_H \backslash Z$ is successively shrunk to a node $w \in V_H \backslash Z$ that is connected to the node u. Then the Gomory-Hu algorithm is applied to generate the cycle partition inequalities violated by the current solution y.

4.3 Delete-Link Heuristic for Obtaining Upper Bound

The cutting plane routine gives a lower bound to the subsequent branch and bound stage for 3NCON problem. To design more efficient branch and bound procedure, it is reasonable to use a good feasible solution as an upper bound. To get an upper bound, the delete-link heuristic is presented.

Let the solution obtained from the cutting plane stage be y. Usually the solution is not an integer. Thus by taking links in $E' = \{e \in E \mid y_e \geq 1/2\}$, the solution becomes feasible for 3NCON(G). Then this solution has redundant links which are not necessary for the three node connectivity constraint. From this solution, the delete-link heuristic improves the network cost by deleting expensive links in the current solution until no further cost improvement is made. The procedure of the delete-link heuristic is presented as follows:

Delete-Link Upper Bound Heuristic

1. Perform the cutting plane stage. Then let y be a lower bound solution.
2. Take $E' = \{e \in E \mid y_e \geq 1/2\}$ from the solution y and $G(V,E)$.
3. From the network $G(V,E')$, sort all links in descending order of link costs.
   Do the following steps for all links in E'.
   3-1. Let e be the most expensive link in $G(V,E)$ that is not considered.
   3-2. If $G(V,E' \backslash \{e\})$ is feasible for the 3NCON(G).
   Then delete the link e from $G(V,E')$.

Note that the above heuristic improves the network cost such that the feasibility is maintained and gives an upper bound to the subsequent branch and bound procedure. The delete-link procedure is applied for all links in E' and requires to check if the network satisfies the three node connectivity.

5. Experimental Results

Computational experiments are performed to test the effectiveness of the branch and cut procedure for the survivable network design with three connectivity. We examine the performance of the cutting plane procedure and the gap of lower and upper bound from the optimal solution. We also compare the branch and bound with two different lower bounds: a linear programming relaxation and a cutting plane method.

All the computational results reported in this section are performed on HP-UX 9000/715 Workstation. The branch and cut procedure is coded in C programming language and uses the LP-solver CPLEX and IP-solver CPLEX-MIP. All separation routines outlined in Section 4 are also coded in C.

Figure 4, as an example, shows a problem with 50 nodes where squares and circles represent hubs and central offices repectively. In the figure 100 candidate links are located in 400 by 400 Euclidean plane and each link cost is given as the distance between two nodes. Figure 5 demonstrates a three node connectivity solution obtained by the branch and cut algorithm.

Five instances are generated and averaged for each problem with 50, 100, 150 and 200 nodes. In each case it is assumed that $|V_H| = |V_C| = |V|/2$. The computational results of the twenty problems are summarized in Table 1 - 5. In Table 1, 2 and 3, the underlying network of each problem is generated with $2|V|$ candidate links where each link cost is given as Euclidean distance between two nodes. In Table 4 and 5, the experiments are performed with various link costs and different number of candidate links.

Table 1 shows the performance of cutting plane procedure for 3NCON problem. In the table, valid inequalities generated during the cutting plane procedure are shown for each problem. Partition and node partition inequalities generated in the cutting plane procedure give good effects in obtaining the lower bound for 3NCON problem. On the average, partition inequalities are generated approximately twice of the number of nodes while node partition inequalities about 1/2 of the number of nodes. The use of crown and wheel inequalities are relatively small and indifferent to the number of nodes. This implies that the partition and node partition inequalities give better performance than the crown and wheel inequalities.

Table 2 shows gaps of lower and upper bounds from the optimal solution. Linear programming relaxation and cutting plane lower bound are compared with optimal solution. Each gap represents the percent relative error from the optimal solution, i.e., 100(bound-optimum)/ optimum. The table shows that the cutting plane procedure provides excellent lower bounds compared to the LP relaxation. The gap is approximately 0.8 % in problems with 50 nodes and 5.1 % with 200 nodes. The delete-link heuristic gives a good upper bound with the gap of approximately 6 - 11 % for all problems.

Table 3 shows the performance of the branch and bound with different lower and upper bounds. Two lower bounds, LP-relaxation and cutting plane, are employed for branch and bound procedure. Each case is then compared with the procedure with upper bound. The table shows the computational time of each procedure for optimal solution. The cutting plane method is much more efficient than the LP-relaxation for all problems. In problems with 200 nodes, two methods with LP relaxation fail to obtain an optimal solution within 3 hours. The cutting plane method with the delete-link upper bound is the most efficient among the four procedures.

Table 4 shows the results of the branch and bound with two different link costs. The cutting plane with upper bound procedure, which is the most efficient, is employed. The random links are generated by assuming uniform cost distribution over the integers 1, 2, ..., 400. In the table, the gap of lower and upper bound from the optimal solution is shown with the CPU time of the branch and cut procedure. It is clear from the table that the problem with random link cost is more difficult than that with Euclidean link cost.

Table 5 illustrates the branch and bound with three different sizes of candidate link set. The number of candidate links in the three sets are $2|V|$, $3|V|$ and $4|V|$ respectively.

Link costs are assumed Euclidean. As shown in the table, when $|E| = 3|V|$ problems with 150 and 200 nodes are not solved within 3 hours. When the problem size becomes $|E| = 4|V|$, optimal solutions could be obtained only in problems with 50 nodes within 20 minites.

6. Conclusions

The problem of designing survivable telecommunication networks is considered. Valid and facet-defining inequalities for the three node connectivity solutions are proposed. Partition, node partition, crown and wheel inequalities are developed for the cutting plane procedure to obtain a good lower bound. It is demonstrated that the partition and node partition inequalities give better performance than the crown and wheel inequalities.

The computational results show that the cutting plane procedure provides excellent lower bounds compared to the LP relaxation. The gap from the optimal solution by the cutting plane is approximately 0.8 % in problems with 50 nodes and 5.1 % with 200 nodes. However, the LP relaxation method failed to obtain an optimal solution within 3 hours in problems with 200 nodes. The use of delete-link heuristic is promising for a good upper bound. The cutting plane procedure with the delete-link upper bound is proved to be the most efficient among the procedures experimented. It is also demonstrated that the largest size of candidate links to obtain an optimal solution within a reasonable computing time is $|E| = 3|V|$ for problems with 100 nodes and $|E| = 2|V|$ with more than 150 nodes.

References

[1] Bienstock, D., E. F. Brickell and C. L. Monma, On the Structure of Minimum-Weight k-connected Spanning Networks, ***SIAM Journal on Discrete Mathematics*** 3 (1990) 320-329.

[2] Cardwell, R. H., C. L. Monma and T. H. Wu, Computer-Aided Design Procedures for Survivable Fiber Optic Networks, ***IEEE Journal on Selected Areas in Communications*** 7 (1989) 1188-1197.

[3] S. Chopra, Polyhedra of the Equivalent Subgraph Problem and Some Edge Connectivity Problems, ***SIAM Journal on Discrete Mathematics*** 5 (1992) 321-337.

[4] R. E. Gomory and T. C. Hu, Multi-Terminal Network Flows, ***SIAM Journal on Applied Mathematics*** 9 (1961) 551-570.

[5] M. Groetschel and C. L. Monma, Integer Polyhedra Associated with Certain Network Design Problems with Connectivity Constraints, ***SIAM Journal on Discrete Mathematics*** 3 (1990) 502-523.

[6] M. Groetschel, C. L. Monma, and M. Stoer, Facets for polyhedra arising in the design of communication networks with low-connectivity constraints, ***SIAM Journal on Optimization*** 2 (1992) 474-504.

[7] M. Groetschel, C. L. Monma, and M. Stoer, Computational Results with a Cutting Plane Algorithm for Designing Communication Networks with Low Connectivity Constraints, ***Operations Research*** 40 (1992) 309-330.

[8] J. Hopcropt and R. E. Tarjan, Algorithm 447: Efficient Algorithms for Graph Manipulation, ***Communications of ACM*** 16 (1973) 372-378.

[9] J. Hopcropt and R. E. Tarjan, Dividing a Graph into Triconnected Components, ***SIAM Journal***

on Computing 2 (1973) 135-158.

[10] C. L. Monma, B. S. Munson, and W. R. Pulleyblank, Minimum-weight Two-connected Spanning Networks, ***Mathematical Programming*** 46 (1990) 153-171.

[11] C. L. Monma and D. F. Shallcross, Methods for Designing Communications Networks with Certain Two-connected Survivability Constraints, ***Operations Research*** 37 (1989) 531-541.

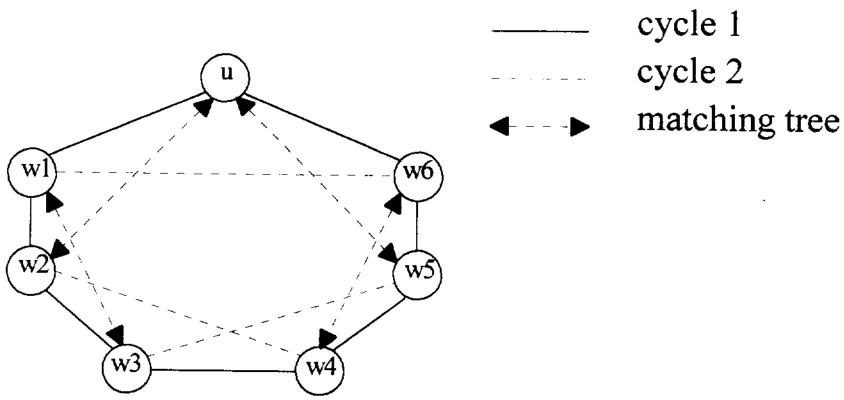
[12] M. Padberg and G. Rinaldi, An efficient Algorithm for the Minimum Capacity Cut Problem, ***Mathematical Programming*** 47 (1990) 19-36.

[13] T. Watanabe and A. Nakamura, 3-Connectivity Augmentation Problems, ***Proceeding of IEEE International Symposium on Circuits and Systems*** (1988) 1847-1850.

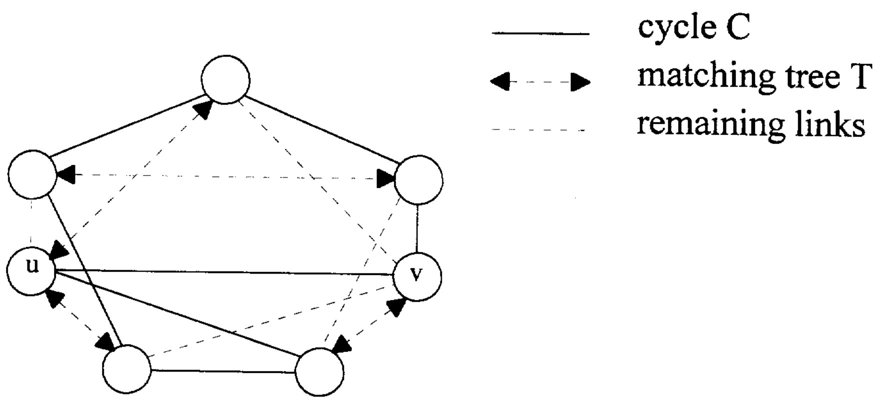
[14] T. Watanabe and A. Nakamura, A Smallest 3-Connectivity Augmentation of a Graph, ***Discrete Applied Mathematics*** 28 (1990) 183-186.

[15] T. Watanabe and A. Nakamura, A Minimum 3-Connectivity Augmentation of a Graph, ***Journal of Computer and System Sciences*** 46 (1993) 91-128.

[16] T. H. Wu, Fiber Network Service Survivability, Artech House (1992).

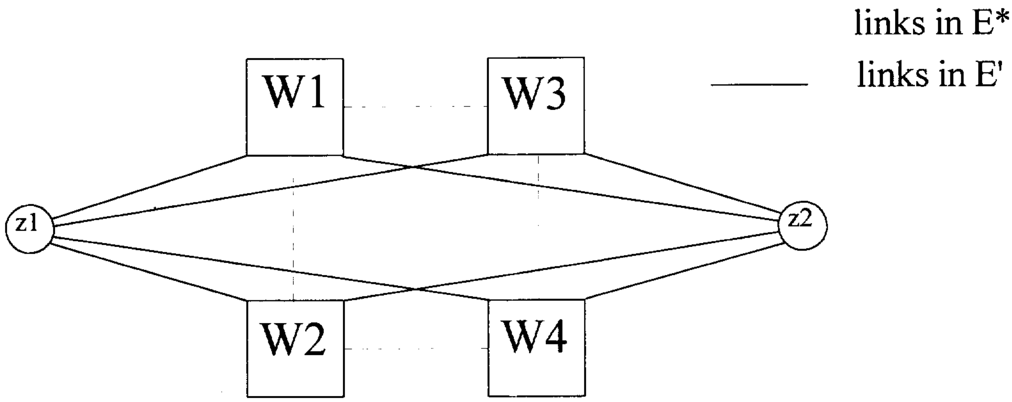


Case (a)

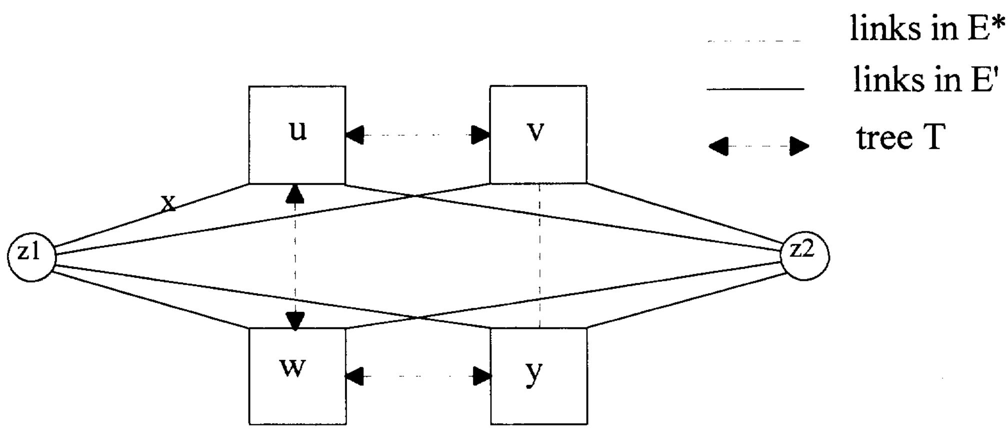


Case (b)

Figure 1. Partition Inequalities

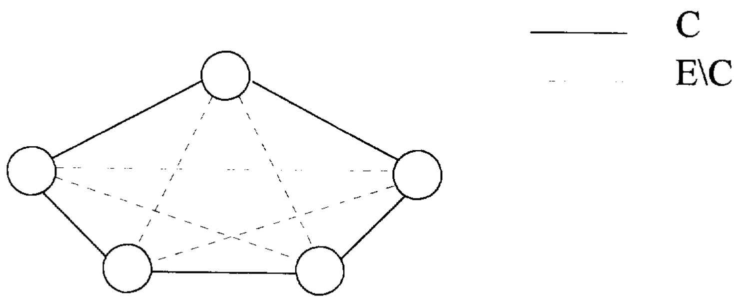


Case (a)

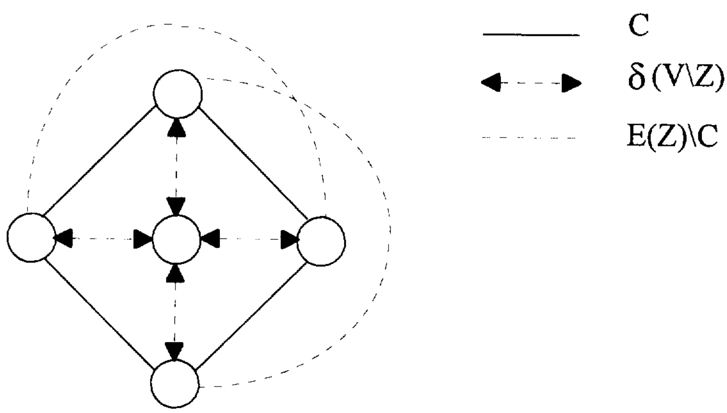


Case (b)

Figure 2. Node Partition Inequalities



(a) Crown Inequalities



(b) Wheel Inequalities

Figure 3. Cycle Partition Inequalities

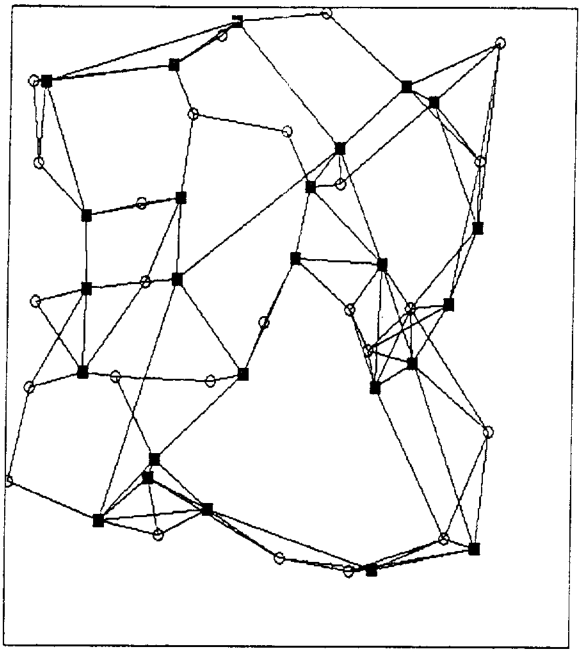


Figure 4. The Example Network with 50 Nodes

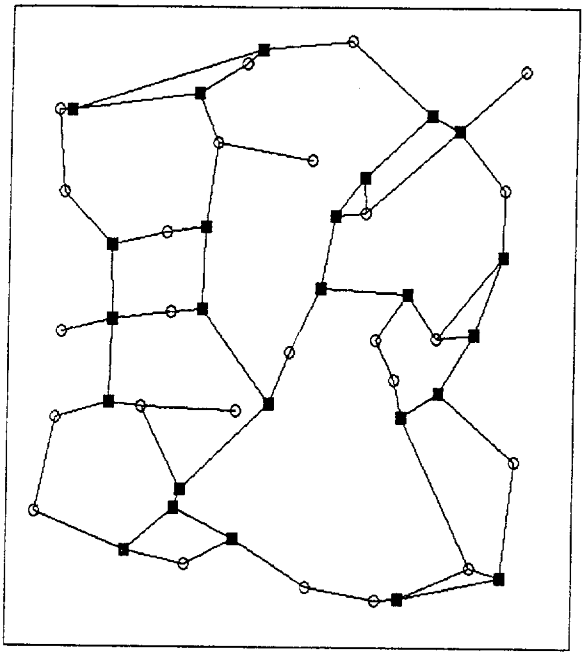


Figure 5. The Optimal Network with 50 Nodes

Table 1. Performance of Cutting Plane Procedure for 3NCON Problem

No. of Nodes	50	100	150	200
Partition	114	202	256	287
Node-Partition	23	44	68	76
Crown	7	9	14	18
Wheel	6	8	12	17

Each value is averaged over five instances.

Table 2. Comparision of Lower and Upper Bounds from the Optimal Solution

No. of Nodes	50	100	150	200
LP-relaxation(LB)	2807.8	3773.8	4584.5	5215.4.
	(-25.80%)	(-26.14%)	(-27.45%)	(-27.71%)
Cutting Plane(LB)	3759.3	4986.2	6114.8	6845.5
	(-0.76%)	(-2.41%)	(-3.21%)	(-5.11%)
Delete-Link(UB)	4048	5547	6904	8024
	(6.86%)	(8.57%)	(9.27%)	(11.23%)
Branch and Bound	3,788	5,109	6,318	7,214

Each percent in the parenthesis represents the gap from the optimal solution.

Table 3. Branch and Bound with Different Lower and Upper Bounds

No. of Nodes	50	100	150	200
LP-relaxation	30.24	405.85	9,487.23	〉3 Hours
LP-relaxation with UB	19.25	262.54	6,824.09	〉3 Hours
Cutting Plane	7.32	48.21	654.14	7,415.85
Cutting Plane with UB	6.43	32.14	513.67	5,426.15

Each value represents CPU seconds

Table 4. Branch and Bound with Two Different Link Costs

No. of Nodes	50	100	150	200
Euclidean Cost				
Cutting Plane(LB)	-0.76%	-2.41%	-3.21%	-5.11%
Delete-Link(UB)	6.86%	8.57%	9.27%	11.23%
CPU seconds	6.43	32.14	513.67	5,426.15
Random Cost				
Cutting Plane(LB)	-1.04%	2.26%	-4.25%	-6.87%
Delete-Link(UB)	7.52%	19.21%	13.45%	15.88%
CPU seconds	7.42%	103.45	897.89%	7,457.82

Table 5. Branch and Bound with Large Candidate Link Sets

No. of Nodes	50	100	150	200
|E|=2|V|				
Cutting Plane(LB)	-0.76%	-2.41%	-3.21%	-5.11%
Delete-Link(UB)	6.86%	8.57%	9.27%	11.23%
CPU seconds	6.43	32.14	513.67	5,426.15
|E|=3|V|				
Cutting Plane(LB)	-2.34%	-5.78%	‡	‡
Delete-Link(UB)	8.34%	12.12%	‡	‡
CPU seconds	78.94	1,076.34	〉3 Hours	〉3 Hours
|E|=4|V|				
Cutting Plane(LB)	-5.67%	‡	‡	‡
Delete-Link(UB)	11.34%	‡	‡	‡
CPU seconds	1,197.23	〉3 Hours	〉3 Hours	〉3 Hours

‡The branch and bound fails to obtain an optimal solution in 3 hours.