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Closed Queueing Networks and Zeros of Successive Derivatives*

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Abstract

Consider a Jackson type closed queueing network in which each queue has a single exponential server. Assume that N customers are moving among k queues. We propose a candidate procedure which yields a lower bound of the network throughput which is sharper than those which are currently available : Let $(\rho_1, \cdots, \rho_k)$ be the loading vector, let x be a real number with $0 \leq x \leq N$, and let $y(x)$ denote that y is a function of x and be the unique positive solution of the equation . $\sum_{i=1}^{k} y(x)\rho_i/(N-y(x)x\rho_i)=1$ Whitt [17] has shown that $y(N)$ is a lower bound for the throughput. In this paper, we present evidence that $y(N-1)$ is also a lower bound. In doing so, we are led to formulate a rather general conjecture on "Migrating Critical Points" (MCP). The MCP conjecture asserts that zeros of successive derivatives of certain rational functions migrate at an accelerating rate. We provide a proof of MCP in the polynomial case and some other special cases, including that in which the rational function has exactly two real poles and fewer than three real zeros.

1. Introduction.

Consider a Jackson type closed queueing network where N customers are moving among k queues. Each queue has a single exponential server with a fixed service rate μ_i. When a customer completes his service at queue i, he then joins queue j with probability P_{ij} where the P_{ij} satisfy the usual condition $\sum_{j=1}^{k} P_{ij}=1$ for all i. Assume that the Markov transition matrix

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P is irreducible and let $\vec{x}=(x_1,\cdots x_k)$ be the unique positive solution of the system $x_i=\sum_{j=1}^{k} x_j P_{ji}$, $i=1,\cdots,k$ for which $\sum_{i=1}^{k} x_i=1$. Let $\vec{\rho}=(\rho_1,\cdots,\rho_k)=(x_1/\mu_1,\cdots, x_k/\mu_k)$ denote the "loading vector" (ρ_i is known as the utilization factor of queue i). The limiting probability that n_i customers are in the ith queue for $i=1,\cdots,k$, is given by

$$P(n_1,\cdots,n_k)=G(N,k,\vec{\rho})^{-1}\prod_{i=1}^{k}\rho_i^{n_i},$$

where $G(N,k,\vec{\rho})$, the normalization constant, is defined by

$$G(N)=G(N,k,\vec{\rho})=\sum_{n_1+\cdots+n_k=N}[\prod_{i=1}^{k}\rho_i^{n_i}].$$

The function $G(N,k,\vec{\rho})$ is the well known "complete symmetric function" whose properties have been the object of extensive study (cf. Macdonald [9]). The throughput, which is, by definition, the mean number of service completions per unit time, is given by the formula:

$TH=TH(N,k,\vec{\rho})=G(N-1,k,\vec{\rho})/G(N,k,\vec{\rho})$. In the sequel, whenever no confusion may arise, we shall suppress some of the arguments, N, k, and $\vec{\rho}$, from $G(N,k,\vec{\rho})$ and $TH(N,k,\vec{\rho})$ and simply write $G(N)=G(N,k,\vec{\rho})$, etc.

In practice, when one faces problems of design and control of closed queueing networks, the knowledge of various mean performance measures (such as throughput) is essential. Unfortunately, an exact calculation of TH requires summing over a huge index set, namely that which is parameterized by vectors $(n_1,\cdots,n_k)$ satisfying the condition $n_1+\cdots+n_k=N$. This is prohibitively expensive for large values of N. Buzen [3] proposed a recursive algorithm to evaluate $G(N)$ of complexity $O(N_k)$. Harrison [6] and Gordon [5] obtained closed form solutions of computational complexity $O(k^2)$. In addition to the high level of computational complexity, the implementation of the calculation of closed form solutions becomes unstable when ρ_i is close to ρ_j.

The techniques of Mean Value Analysis allow one to recursively calculate mean performance measures such as the number of customers, the waiting times, the throughputs, etc. These recursive methods have been widely used by many authors: see for example, Reiser & Lavenberg [13] and Sauer [14]. However, these methods also require a significant computational effort, similar to that involved in the direct calculation of $G(N)$.

To avoid numerical instability and to reduce computational complexity several simple bounds on the throughput have been proposed. Such bounds have practical value, since they are easy to compute, and are of theoretical interest as well. Zahorjan [19] et. al proposed the following simple lower bound for the network throughput:

$$TH_z=\frac{N}{\sum_{i=1}^{k}\rho_i+(N-1)\rho_{\max}}\leq TH(N), \tag{1.1}$$

where $\rho_{\max}=\max(\rho_1,\cdots,\rho_k)$. Whitt [17] introduced a method known as the Fixed Population Mean approach, which approximates a closed network by the corresponding open network with specified expected equilibrium population size. His lower bound, TH_w, is the unique positive solution of the following equation:

$$\sum_{i=1}^{k}\frac{TH_w\rho_i}{N-NTH_w\rho_i}=1. \tag{1.2}$$

Also note that Knessl et. al [8] derived the same procedure as (1.2) using aymptotic expansion method. In this paper, we propose a candidate, TH_n, which is sharper than TH_z and TH_w: Define TH_n as the unique positive solution to the following equation:

$$\sum_{i=1}^{k}\frac{TH_n\rho_i}{N-(N-1)\,TH_n\rho_i}=1. \tag{1.3}$$

We conjecture that TH_n is a lower bound for the throughput of the network. Since $TH_n\geq TH_w$, this is a stronger lower bound (see Proposition 3.1). Numerous examples were tested numerically, in which the two bounds were compared. In each case the mean error of TH_n is about an order of magnitude smaller than the mean error of TH_w. A sample of comparisons is provided in Table 1.1, where the mean error is defined as:

$\int_0^1[\,TH(k=2,\rho_1=1,\rho_2)-TH_b(k=2,\rho_1=1,\rho_2)]d\rho_2$ and TH_b refers to one of the bounds, TH_w, TH_z, or TH_n. For numerical integration we used a symbolic manipulation language, MAPLE.

Table 1.1: Numerical Comparison of Mean Errors for k=2

N	TH_n	TH_z	TH_w	N	TH_n	TH_z	TH_w
2	0.0105	0.0407	0.2514	15	0.0039	0.0258	0.0649
3	0.0122	0.0495	0.2145	16	0.0036	0.0245	0.0611
4	0.0116	0.0498	0.1835	17	0.0034	0.0234	0.0578
5	0.0106	0.0475	0.1591	18	0.0031	0.0224	0.0548
6	0.0094	0.0446	0.1398	19	0.0029	0.0215	0.0521
7	0.0084	0.0417	0.1244	20	0.0027	0.0206	0.0496
8	0.0075	0.0389	0.1120	25	0.0020	0.0186	0.0401
9	0.0068	0.0364	0.1016	30	0.0016	0.0146	0.0336
10	0.0061	0.0341	0.0930	40	0.0010	0.0113	0.0254
11	0.0055	0.0331	0.0856	50	0.0007	0.0092	0.0203
12	0.0050	0.0326	0.0793	60	0.0006	0.0078	0.0169
13	0.0046	0.0286	0.0739	70	0.0004	0.0068	0.0145
14	0.0042	0.0271	0.0691	80	0.0003	0.0059	0.0127

2. A New Lower Bound.

In this section we formulate our conjecture on the throughput inequality (THI) and we prove it for the case $k=2$. We also show that the conjectured inequality is sharp.

Conjecture THI: $TH(N, \vec{\rho}) \geq TH_n$.

Note that the evaluation of TH_n in (1.3) is fast since the function $\rho_i x/[N-(N-1)\rho_i x]$, for $x \in [0, 1/\rho_i]$ is increasing and convex in x. Let

$$F = F(N, k, \vec{\rho}) = \sum_{i=1}^{k} \frac{\rho_i TH(N)}{N-(N-1)\rho_i TH(N)}. \tag{2.1}$$

Then Conjecture THI is equivalent to showing that $F(N, k, \vec{\rho}) \geq 1$. Observe that the conjectured inequality is sharp: When all ρ_i are equal to a fixed number ρ (the "balanced case"), then $TH(N) = \binom{N+k-2}{k-1} / \rho \binom{N+k-1}{k-1}$, which is equal to TH_n. It is also easy to prove the conjecture for the case $k=2$. Without loss of generality, we may assume $\max(\rho_1, \rho_2) = 1$.

Let $x=\min(\rho_1, \rho_2)$. The statement $F(N, k, \vec{\rho}) \geq 1$ is equivalent to the inequality: $\sum_{i=1}^{N-1} x^i \geq Nx^{(N-1)/2}$. Note that this inequality is obtained by substituting TH_n for $TH(N)$ in (1.3) and the inequality holds by the arithmetic-geometric mean inequality.

3. Comparison with Other Lower Bounds.

In this section we show that TH_n yields a sharper lower bound than that provided by TH_w or TH_z.

Proposition 3.1.

(a) $TH_n \geq TH_w$.

(b) $TH_n \geq TH_z$.

Proof (a): trivial.

(b): Without loss of generality, we may assume $\sum_{i=1}^{k} \rho_i = 1$. Then we need to show that $\sum_{i=1}^{k} TH_z\rho_i/[N-(N-1)TH_z\rho_i] \leq 1$, which is reduced to $\sum_{i=1}^{k} \rho_i/[(N-1)(\rho_{\max}-\rho_i)+1] \leq 1$. But, since $\sum_{i=1}^{k} \rho_i = 1$, it suffices to prove $\rho_i/[(N-1)(\rho_{\max}-\rho_i)+1] \leq \rho_i$. This is equivalent to showing that $(N-1)(\rho_{\max}-\rho_i) \geq 0$, which is clear. ■

4. Migrating Critical Points (MCP).

Statement of Results

The rest of this paper will be devoted to a study of Conjecture THI. Our approach is as follows:

We first construct the generating function $g(t)$ whose Taylor coefficients are given by the sequence $G(N-1)-G(N)$. Conjecture THI has a surprisingly natural reformulation in terms of the analytic properties of the function $g(t)$ (see Theorem 4.1). We are led to a rather general conjecture which predicts the migration properties of successive derivatives of rational functions (see Main Conjecture). The Main Conjecture may be viewed as a contribution to the research area initiated by G. Polya. The main focus of Polya's school can be summarized by the following question: "How do the zeros of the nth derivative $f^{(n)}(x)$ behave as n increases?" (see Polya [11]). Much work has been done in this area: See, for example, Boas et. al [1], Craven et. al [4], Reddy [12], and Sheil-Small [16]. But all the previous authors have restricted their attention to migration properties only when n tends to infinity. In our setting, it becomes important to consider values of n, both large and small: Our theorems and conjectures predict the migrating properties of zeros of $f^{(n)}(x)$ for all $n \geq 0$. These results appear to be new, even in the case where f is a polynomial.

Let $g(x)$ be a rational function (i.e., a quotient of two polynomials). Assume that g has real coefficients, and that all the zeros (roots of the numerator) are positive real numbers, and the poles (roots of the denominator) of g are negative real numbers. Let k be the degree of the denominator of g and n the degree of the numerator.

Define the sequence of critical points: $r_j = \min\{x>0 : g^{(j)}(x)=0\}$, $j \geq 0$. (Note that without loss of generality we can restate $r_j = \max\{x>0 : g^{(j)}(x)=0\}$, $j \geq 0$ by taking $g(-x)$ instead of $g(x)$. In this section, we shall examine the migration properties of this sequence. In particular, we are interested in the "acceleration" of the sequence. That is, $a_j = v_{j+1} - v_j$, where $v_j = r_{j+1} - r_j$, $j \geq 0$. In the following we list the conjecture and theorems to be discussed in the subsequent sections.

Main conjecture: The acceleration is non-negative, that is $a_j \geq 0$ for all j.

Theorem 4.1. The **main conjecture** implies THI.

Theorem 4.2. The **main conjecture** is true in the following cases: Let k, n in (k, n) denote the degree of denominator and numerator of g, respectively.

1. $(k,n) = (0,n)$ with n arbitrary (i.e., g is a polynomial).

2. $(k,n) = (2,n)$, n = 1, 2.

Theorem 4.3. Conjecture THI is true for $N = 2, 3$, and 4.

Remark: For simplicity, we have assumed that all the poles of g are negative and all the zeros of g are positive. But with minor adjustments in formulation, we can relax the hypothesis as follows: We assume that all the zeros and poles of g are real numbers, and we further assume that all the poles are to the left (respectively to the right) of all the zeros. Then we conjecture that the acceleration is non-negative (respectively non-positive).

Proof of Theorem 4.1.

We shall show that if we assume the **main conjecture** holds in the special case where g has exactly one zero, then THI follows. To prove **THI**, without loss of generality we scale $\vec{\rho}$ such that $TH_n(N, \vec{\rho}) = 1$ and based on (1.3) assume that

$$\sum_{i=1}^{k} \frac{\rho_i}{N-(N-1)\rho_i} = 1. \tag{4.1}$$

We must show that (4.1) implies

$$G(N-1, \vec{\rho}) \geq G(N, \vec{\rho}). \tag{4.2}$$

Now using the generating function of $G(N)$ let

$$g(t) = \frac{(t-1)}{\prod_{i=1}^{k}(1-\rho_i t)} = -1 + \sum_{n=1}^{\infty}[G(n-1) - G(n)]\, t^n. \tag{4.3}$$

From (4.3) observe that $\frac{g'(t)}{g(t)} = \frac{d}{dt}\ln(g(t)) = \frac{1}{t-1} + \sum_{i=1}^{k}\frac{\rho_i}{1-\rho_i t}$. Thus condition (4.1) is equivalent to the condition: $g'(1-1/N) = 0$. Therefore, if we assume (4.1) holds, we conclude that $v_0 = -1/N$. Note that (4.1) implies that $\rho_i < 1$ for all i, and in particular, all the poles of

g are to the right of the unique root of g. Thus, the hypotheses of the main conjecture are satisfied. The main conjecture now implies $v_j \leq -1/N$ for all j. In particular, since $r_0 = 1$, we conclude that $r_N \leq 0$. This means that $g^{(N)}(t) \geq 0$ for all t in the interval [0,1]. Since $g^{(N)}(0)/N! = G(N-1) - G(N)$, we conclude that $G(N-1) - G(N) \geq 0$, which concludes the proof of Theorem 4.1. ■

5. Proof of MCP for Polynomials (Theorem 4.2 part 1).

Let $g(x)$ be a polynomial of degree m with real coefficients. Assume $m \geq 3$ and that all the roots of g are real. Let $n = m-1$ and let C be the "center" of g, that is, C is the average of the roots of g. For $0 \leq j \leq n$, define

$$R_j = \max\{x \in \mathbf{R} : g^{(j)}(x) = 0\},$$
$$r_j = \min\{x \in \mathbf{R} : g^{(j)}(x) = 0\}.$$

It is clear that $r_0 \leq r_1 \leq \cdots \leq r_n = C = R_n \leq R_{n-1} \leq \cdots \leq R_0$. Theorem 4.2, part 1 can be restated as follows:

Theorem 5.1. The sequences $\{r_j\}$ and $\{R_j\}$ accelerate towards the center, C. That is, $r_j - r_{j-1} \leq r_{j+1} - r_j$ **and** $R_{j-1} - R_j \leq R_j - R_{j+1}$, for $1 \leq j \leq n-1$.

Lemma 5.1. Let $n \geq 2$ and let $\alpha_1, \cdots, \alpha_n$ be a decreasing sequence of real numbers. Let

$$P(x) = \prod_{i=1}^{n} (x - \alpha_i).$$

Let β be the largest root of $P'(x) = 0$. Then

$$\beta \geq \frac{(\alpha_1 + \alpha_2)}{2}.$$

Proof of Lemma 5.1: If $\alpha_1 = \alpha_2$, then $\beta = \alpha_1 = \alpha_2$, and the result is clear. Since $P'(\alpha_1) > 0$

and $P'(\alpha_2)<0$, we have $\alpha_1>\beta>\alpha_2$.

$$0=\frac{P'(\beta)}{P(\beta)}=\sum_{i=1}^{n}\frac{1}{\beta-\alpha_i}\geq\frac{1}{\beta-\alpha_1}+\frac{1}{\beta-\alpha_2},$$

which implies the result. ■

We now return to the proof of theorem 5.1. First we note that if $a,b,c\in\mathbf{R}$ with $ac\neq 0$ then it suffices to prove the theorem for $cg(ax+b)$. In fact, replacing $g(x)$ by $g(-x)$, it suffices to prove that $\{R_j\}$ accelerates. By induction on the degree of g, we need only show that $R_0-R_1\leq R_1-R_2$. Replacing $g(x)$ by $g(ax+b)$ for appropriate $a,b\in\mathbf{R}$, we may assume that $R_0=2$ and $R_2=0$. Thus, our task is to prove $R_1\geq 1$. If g has two or more non-negative roots, then Lemma 5.1 implies that $R_1\geq 1$. Thus we may assume that g can be written in the form:

$$g(x)=(x-2)\prod_{i=1}^{n}(1+\rho_i x),$$

with $\rho_i>0$. Let

$$G(\vec{\rho})=G(\rho_1,\cdots,\rho_n)=\sum_{i=1}^{n}\rho_i-2\sum_{i<j}\rho_i\rho_j. \tag{5.1}$$

Then $R_2=0$ implies $G(\vec{\rho})=0$. To show $R_1\geq 1$ it suffices to prove that $g'(1)<0$. Since $g(1)<0$, we are reduced to proving $g'(1)/g(1)\geq 0$, that is, if we define

$$F(\vec{\rho})=\sum_{i=1}^{n}\frac{\rho_i}{1+\rho_i},$$

then we must show that $G(\vec{\rho})=0$ implies $F(\vec{\rho})\geq 1$. Let

$$S=\{\vec{\rho}\in\mathbf{R}^n : G(\vec{\rho})=0,\ \rho_i\geq 0 \text{ for all } i \text{ and } \rho_i>0 \text{ for some } i\}$$

Lemma 5.2. Let $0<\varepsilon<1/2(n-1)$.

(i) If $\vec{\rho}\in S$ then $\rho_i>\varepsilon$ for some i.

(ii) If $\vec{\rho}\in S$ and $\rho_i>1$ for some i then $\rho_j>\varepsilon$ for some $j\neq i$.

Proof: For (i), observe that (5.1) is equivalent to $G(\vec{\rho})=\sum_{i=1}^{n}\rho_i(1-\sum_{j\neq i}\rho_j)$. Thus, if $\rho_i < 1/(n-1)$ for all i, $G(\vec{\rho})>0$. For (ii), assume that $\rho_1>1$, and assume that $\rho_i < 1/2(n-1)$ for all i. Then we have:

$$G(\vec{\rho})=\rho_1(1-2\sum_{i>1}\rho_i)+\sum_{i>1}\rho_i(1-\sum_{j\neq 1,i}\rho_j) > 0.$$ ■

Lemma 5.3. The function F achieves its minimum on the set S.

Proof: Note that if $\rho_i=1/(n-1)$ for all i, then $\vec{\rho}\in S$ and $F(\vec{\rho})=1$. Now choose K to be a large real number, and let

$$S(K)=\{\vec{\rho}\in S:\ \rho_i\leq K \text{ for all } i\}.$$

Then, by Lemma 5.2, the set S(K) is compact. Also, if $\vec{\rho}\notin S(K)$, then

$$F(\vec{\rho}) > \frac{K}{K+1}+\frac{\varepsilon}{\varepsilon+1} > 1$$

for K sufficiently large (again, by Lemma 5.2). This proves Lemma 5.3. ■

Let $S_{\min}$ be the set of points where F achieves its minimum value. Let $T\subseteq S_{\min}$ be the subset consisting of those points with the maximal number of components which are zero.

Lemma 5.4. Let $\vec{a}\in T$ be a point with a minimum number of distinct non-zero entries. Then all the non-zero entries of $\vec{a}$ are equal and $F(\vec{a})=1$.

Proof. Assume not: Then we may assume $a_1\neq a_2$ and $a_1a_2\neq 0$. Let $A=a_3+\cdots+a_n$ and $B=2\sum_{3\leq i<j}a_ia_j$ let $x=a_1+a_2$ and $y=a_1a_2$. Consider the function

$$Q(x,y)=G(\vec{a})=a_1(1-2A)+a_2(1-2A)+A-2a_1a_2-B=x(1-2A)-2y+A-B.$$

The domain of Q is

$$D=\{(x,y)\in \mathbf{R}^2:\ x^2\geq 4y\geq 0\}.$$

Let

$$f(x,y)=F(\vec{a})=\frac{a_1}{1+a_1}+\frac{a_2}{1+a_2}+C=\frac{x+2y}{1+x+y}+C,$$

where C is a constant depending only on $a_3,\cdots,a_n$. Now our assumptions imply that if we restrict $f(x,y)$ to the line segment $Q(x,y)=0$, $(x,y)\in D$, then the minimum of f occurs in the interior of the line segment. But since the first derivative of the function $f(x,[x(1-2A)+A-B]/2)$ is just $2(-5A+B+4)/(2Ax-3x-A+B-2)^2$, the function has no local minima (or maxima), a contradiction. Thus all the non-zero entries of $\vec{a}$ are equal, which yields $F(\vec{a})=1$. This proves the lemma, and hence the theorem. ■

For numerical evidence, let $g(x)=\prod_{i=0}^{9}(x-i)$. Then the center of the zeros of g is 4.5. We tabulate $v_j=r_{j+1}-r_j$ and $a_j=v_{j+1}-v_j$ in Table 5.1. As is seen in the table, $a_{j+1}>a_j$, which also implies $v_{j+1}>v_j$.

Table 5.1. Example of MCP for a Polynomial

j	r_j	v_j	a_j
0	0.0000	0.2925	0.0266
1	0.2925	0.3191	0.0307
2	0.6116	0.3498	0.0366
3	0.9614	0.3864	0.0452
4	1.3478	0.4316	0.0586
5	1.7794	0.4902	0.0819
6	2.2696	0.5721	0.1288
7	2.8417	0.7009	0.2565
8	3.5426	0.9574	
9	4.5000		

6. Proof of MCP for Certain Rational Functions (Theorem 4.2 part 2).

We first treat the case $k=2$ and $n=1$. Since the balanced case was already treated in Section 2, we may assume that the poles are simple. We may normalize g so that the poles are negative and such that the unique zero occurs at the origin. Then g may be written in terms of its partial fraction expansion:

$$g(t)=\left(\frac{1}{1+\mu_1 t}-\frac{1}{1+\mu_2 t}\right),$$

where μ_1 and μ_2 are positive real numbers. Then nth derivative of g is given by

$$g^{(n)}(t)=(-1)^n n!\left(\frac{\mu_1^n}{(1+\mu_1 t)^{n+1}}-\frac{\mu_2^n}{(1+\mu_2 t)^{n+1}}\right),$$

which implies

$$r_n=\frac{\mu_1^\alpha-\mu_2^\alpha}{(\mu_1\mu_2)^\alpha(\mu_1^{(1-\alpha)}-\mu_2^{(1-\alpha)})}=\frac{1}{\mu_2}\left[-1+\frac{a-1}{a-a^{x/(x+1)}}\right], \qquad (6.1)$$

where $\alpha=n/(n+1)$, $a=\mu_1/\mu_2$, and $x=n$. Without loss of generality, we may assume that $a>1$. Define $f(x)$ as follows:

$$f(x)=\frac{a-1}{a-a^{x/(x+1)}}. \qquad (6.2)$$

If we show that f is increasing convex in x, for $x\geq 0$, we will be done. We have

$$f'(x)=(a-1)\frac{a^{x/(x+1)}\ln(a)}{(a-a^{x/(x+1)})^2(x+1)^2}>0,$$

because $a>1$.

$$f''(x)=(a-A)(a-1)\ln(a)A\frac{(a-A)\ln(a)-2(x+1)[(a-A)-A\ln(a)(x+1)^{-1}]}{[(a-A)(x+1)]^4},$$

where $A=a^{x/(x+1)}$. Then we want to show

$$(a+a^{x/(x+1)})\ln(a)-2(x+1)(a-a^{x/(x+1)})\geq 0,$$

or,

$$\frac{\ln(a)(a+a^{x/(x+1)})}{a-a^{x/(x+1)}}\geq 2(x+1).$$

Let $h=1/(x+1)$. We must show

$$\frac{\ln(a)(a+a^{1-h})}{(a-a^{1-h})}\geq\frac{2}{h}.$$

That is,

$$\frac{\ln(a)(a^h+1)}{(a^h-1)}\geq\frac{2}{h}. \qquad (6.3)$$

Since $a^h=e^{h\ln(a)}$ and replacing h by $h/\ln(a)$, the inequality (6.3) becomes

$$\frac{e^h+1}{e^h-1} \geq \frac{2}{h},$$

or,

$$\frac{e^{h/2}+e^{-h/2}}{e^{h/2}-e^{-h/2}} \geq \frac{2}{h}.$$

Replacing h by $2h$ we are reduced to showing

$$\frac{e^h-e^{-h}}{e^h+e^{-h}} \leq h. \tag{6.4}$$

We must prove that the last inequality (6.4) holds for all $h \geq 0$. Since it holds for $h=0$, it suffices to prove the differentiated inequality:

$$1-\frac{(e^h-e^{-h})^2}{(e^h+e^{-h})^2} \leq 1,$$

which clearly holds. ■

Next we consider the case $k=2$ and $n=2$. Then we may normalize so that $g(t)$ is of the form

$$g(t)=\frac{(t-\alpha)(t-\beta)}{(1+\mu_1 t)(1+\mu_2 t)},$$

where μ_1, μ_2, α, and β are positive real numbers. Let

$$h(t)=g(t)-\frac{1}{\mu_1\mu_2}.$$

Then the root of h is smaller than the roots of g, but the roots of the derivatives of h coincide with the roots of the derivatives of g. Thus it is sufficient to prove the theorem for h. But h has a numerator of degree one, and thus we know the result holds for h. ■

As numerical examples we consider two functions and examine the behavior of $v_j=r_{j+1}-r_j$ and $a_j=v_{j+1}-v_j$ in Table 6.1 and 6.2 (we used MAPLE to obtain the successive derivatives and the zeros of the derivatives):

$g_1(x)=\dfrac{(x-1)}{(x+1)(x+2)}$ and $g_2(x)=\dfrac{(x-1)}{(x+1)(x+2)(x+3)(x+4)}$. The examples show the migrating property. However, compared with the polynomial case where $a_{j+1} > a_j \geq 0$, these rational functions show $0 \leq a_{j+1} < a_j$ and a_j approches to 0 for large j. Also note that

$g_1(x)$ is a function which is considered in Theorem 4.2, and $g_2(x)$ is a numerical example of the **main conjecture.**

Table 6.1: MCP for $g_1(x)$

j	r_j	v_j	a_j
0	1.00000	2.44949	0.01119
1	3.44949	2.46068	0.00281
2	5.91017	2.46349	0.00112
3	8.37366	2.46461	0.00057
4	10.83827	2.46518	0.00032
5	13.30345	2.46550	0.00020
6	15.76895	2.46570	0.00014
7	18.23465	2.46584	0.00008
8	20.70049	2.46592	0.00008
9	23.16641	2.46600	0.00005
10	25.63241	2.46605	0.00003
11	28.09846	2.46608	
12	30.56454		

Table 6.2: MCP for $g_2(x)$

j	r_j	v_j	a_j
0	1.00000	1.07182	0.00808
1	2.07182	1.07990	0.00392
2	3.15172	1.08382	0.00219
3	4.23554	1.08601	0.00136
4	5.32155	1.08737	0.00090
5	6.40892	1.08827	0.00063
6	7.49719	1.08890	0.00044
7	8.58609	1.08934	0.00035
8	9.67543	1.08969	0.00025
9	10.76512	1.08994	0.00021
10	11.85506	1.09015	0.00016
11	12.94521	1.09031	
12	14.03552		

7. Proof of THI in Special Cases (Theorem 4.3).

In this section, we prove Conjecture **THI** (stated in section 2) for $N=2$, 3, and 4. Theorem 4.3 can be restated as follows:

Theorem 7.1. Assume that

$$\sum_{i=1}^{k}\frac{\rho_i}{N-(N-1)\rho_i}=1\,. \tag{7.1}$$

Then $G(N-1)-G(N)\geq 0$ for $N=2$, 3, and 4.

Recall that $G(N,k,\vec{\rho})$ is the normalization constant of degree N in the k parameters $\rho_1,\cdots,\rho_k$. We shall often suppress $\vec{\rho}$ and k in the notation and write $G(N)=G(N,k,\vec{\rho})$. For $x_1,\cdots,x_l$ real numbers we define $G(N;x_1,\cdots,x_l)=G(N,k+l,\vec{\rho'})$, where $\vec{\rho'}=(\rho_1,\cdots,\rho_k,x_1,\cdots,x_l)$.

We adopt the convention $G(-1)=0$. For $n \geq 1$, let $A(n)=\sum_{i=1}^{k}\rho_i^n$. Let $\rho_{\max}=\max(\rho_1,\cdots,\rho_k)$ and $\rho_{\min}=\min(\rho_1,\cdots,\rho_k)$. Finally, we let $D(N)=G(N-1)-G(N)$. For the proof of the theorem we need several lemmas:

Lemma 7.1. Assume $N \geq 1$. Let

$$P=\sum_{i=1}^{k}\frac{\rho_i^2}{N-(N-1)\rho_i}.$$

Then condition (7.1) implies $(N-1)P=N-A(1)$.

Lemma 7.2. $\rho_{\min} \leq P \leq \rho_{\max}$.

Lemma 7.3. For every integer $r \geq 0$ we have

$$NG(N)=\sum_{l=1}^{r}A(l)G(N-l)+\sum_{j=1}^{k}G(N-r-1;\rho_j)\rho_j^{r+1}.$$

Lemma 7.4. For every integer $N \geq 1$ we have

$$N(N-1)D(N)=\sum_{i=1}^{k}\rho_i(N-A(1)-\rho_i(N-1))\,G(N-2;\rho_i).$$

Lemma 7.5. Condition (7.1) implies

$$D(N)=\sum_{i=1}^{k}\left[\frac{\rho_i}{N-(N-1)\rho_i}\right](P-\rho_i)^2\left[\frac{(N-1)}{N}\,G(N-2;\rho_i,P)-G(N-3;\rho_i,P)\right].$$

Remark: The case $r=N$ of Lemma 7.3 is due to Kobayashi [7]. We don't actually need Lemma 7.3 in the proof of Theorem 7.1; It is included since the proof is in the spirit of the generating function approach and the result may be of independent interest.

Proofs.

Proof of Lemma 7.1.

$$A(1)=\sum_{i=1}^{k}\rho_i=\sum_{i=1}^{k}\frac{\rho_i[N-(N-1)\rho_i]}{N-(N-1)\rho_i}$$

$$=N\sum_{i=1}^{k}\frac{\rho_i}{N-(N-1)\rho_i}-(N-1)\sum_{i=1}^{k}\frac{\rho_i^2}{N-(N-1)\rho_i}$$

$$=N-(N-1)P.$$

Proof of Lemma 7.2.

This follows from:

$$\sum_{i=1}^{k}\frac{\rho_{\min}\rho_i}{N-(N-1)\rho_i}\le\sum_{i=1}^{k}\frac{\rho_i^2}{N-(N-1)\rho_i}\le\sum_{i=1}^{k}\frac{\rho_{\max}\rho_i}{N-(N-1)\rho_i}.$$

Proof of Lemma 7.3.

Let

$$h(t)=\frac{1}{\prod_{i=1}^{k}(1-\rho_i t)}=\sum_{N=0}^{\infty}G(N)t^N.$$

and for $1\le i\le k$ let

$$h_i(t)=h(t)(1-\rho_i t)^{-1}.$$

Then we have

$$th'(t)=th(t)\sum_{i=1}^{k}\frac{\rho_i}{1-\rho_i t}=\sum_{i=1}^{k}h(t)\left[\sum_{l=1}^{r}\rho_i^l t^l+\frac{\rho_i^{r+1}t^{r+1}}{1-\rho_i t}\right]$$

$$=\left(\sum_{N=0}^{\infty}G(N)t^N\right)\left(\sum_{l=1}^{r}A(l)t^l\right)+\sum_{i=1}^{k}h_i(t)\rho_i^{r+1}t^{r+1}$$

Comparing coefficients of t^N we obtain the result.

Proof of Lemma 7.4. Note that

$$\sum_{N=0}^{\infty}D(N)t^N=(t-1)h(t).$$

Differentiating both sides with respect to t:

$$\sum_{N=1}^{\infty} ND(N)t^{N-1} = (t\,h(t))' - \sum_{i=1}^{k}\rho_i h_i(t) = h(t) + t\sum_{i=1}^{k}\rho_i h_i(t) - \sum_{i=1}^{k}\rho_i h(t) - \sum_{i=1}^{k}\rho_i^2 t h_i(t)$$

$$= (1-A(1))\,h(t) + \sum_{i=1}^{k}(\rho_i - \rho_i^2) t h_i(t)\,.$$

Differentiating once more with respect to t:

$$\sum_{N=2}^{\infty} N(N-1)D(N)t^{N-2} = (1-A(1))\sum_{i=1}^{k}\rho_i h_i(t) + \left[\sum_{i=1}^{k}\sum_{N=0}^{\infty}(\rho_i - \rho_i^2)G(N;\rho_i)t^{N+1}\right]'$$

Comparing coefficients of t^N we obtain the result.

Proof of Lemma 7.5. From Lemma 7.1 and 7.4, we see that condition (7.1) implies

$$ND(N) = \sum_{i=1}^{k}\left[\frac{\rho_i}{N-(N-1)\rho_i}\right](P-\rho_i)[G(N-2;\rho_i)(N-(N-1)\rho_i)]\,. \quad (7.2)$$

Since

$$\sum_{i=1}^{k}\left[\frac{\rho_i}{N-(N-1)\rho_i}\right](P-\rho_i) = 0\,,$$

we can subtract from the last bracketed term of (7.2) any expression which is independent of i, without changing the value of the sum.

Hence $ND(N) =$

$$\sum_{i=1}^{k}\left[\frac{\rho_i}{N-(N-1)\rho_i}\right](P-\rho_i)[G(N-2;\rho_i)(N-(N-1)\rho_i) - G(N-2;P)(N-(N-1)P)] \quad (7.3)$$

Comparing equation (7.3) with the expression in Lemma 7.5, we see that to prove the lemma it suffices to show

$$\frac{1}{N}[G(N-2;\rho_i)(N-(N-1)\rho_i) - G(N-2;P)(N-(N-1)P)]$$

$$= (P-\rho_i)\left[\frac{N-1}{N}G(N-2;\rho_i,P) - G(N-3;\rho_i,P)\right].$$

However, this follows upon substitution of the following identities into (7.3):

$$G(N-2;\rho_i) = G(N-2;\rho_i,x) - x\,G(N-3;\rho_i,x)\,,$$

$$G(N-2;x) = G(N-2;\rho_i,x) - \rho_i G(N-3;\rho_i,x)\,.$$

Proof of Theorem 7.1. By Lemma 7.5 it suffices to show that

$$\frac{N-1}{N} G(N-2;\rho_i, P) - G(N-3;\rho_i, P) \geq 0, \quad 0 \leq \rho_i \leq 1, \tag{7.4}$$

for each $i, i=1,\cdots,k$. Rewriting (7.4) as

$$G(N-2;x,P) \geq \frac{N}{N-1} G(N-3;x,P), \quad 0 \leq x \leq 1. \tag{7.5}$$

By equation (3) of Yao [18], $G(N-2;x,P)/G(N-3;x,P)$ is an increasing function of x. (Note that in equation (3) there is a small typographical error: The last term should be $\rho_1^{N-1}G(N-1)$ not $\rho_1^{N-1}G(1)$, but this does not affect the result of Yao [18]. Thus, it is sufficient to prove inequality (7.4) for $x=0$. That is, we have

$$G(N-2;P) \geq \frac{N}{N-1} G(N-3;P) \tag{7.6}$$

In the following we prove inequality (7.6) for $N=2$, 3, and 4.

$N=2$: $1 > 0$.

$N=3$: We have to show that $G(1;P) \geq 3/2$.

Proof. Since $G(1;P) = A(1) + P = A(1) + [3 - A(1)]/2 = 3/2 + A(1)/2 \geq 3/2$.

$N=4$: We need to show that $G(2;P) \geq (4/3)\, G(1;P)$.

Proof. Rewrite $G(2;P)$ as:

$$\begin{aligned} G(2;P) &= G(2) + G(1)P + P^2 = G(2) + P[G(1) + P] \\ &= G(2) + [(4 - A(1))/3]\,[G(1) + P] \\ &= 4/3[G(1) + P] + [3G(2) - A(1)G(1) - A(1)P]/3 \\ &= 4/3G(1;P) + [2G(2) - A(1)G(1)]/3 + [G(2) - A(1)P]/3 \geq 4/3\, G(1;P). \end{aligned}$$

The last inequality follows from the fact that $2G(2) = A(1)G(1) + A(2) \geq A(1)G(1)$, and $G(2) \geq G(1)\rho_{max} \geq A(1)P$ by Lemma 7.2. ■

8. Numerical Testing.

In this section we provide some numerical evidence which supports our main conjecture. First it is easy to calculate TH_n by the convexity property of our computational procedure (see Section 2). That is, a simple binary type of searching is sufficient to get TH_n. For the exact TH, we may use the two tailed recursion proposed by [3]. In fact, we tested numerous cases with the parameters drawn from the random number generator (page 1195 in [10]) to detect any counter examples, without any success. To demonstrate numerical examples here we consider k = 5, 10, 15, and 20. For each k we increase the population from 10 to 50 and with fixed values of $\rho_i = 1/i$, $i = 1, \cdots, k$. In Table 8.1, TH, TH_n, and TH_w are tabulated.

Table 8.1: Numerical Comparison of Throughputs

N \ k	5			10			15			20		
	TH	TH_n	TH_w	TH	TH_n	TH_w	TH	TH_n	TH_w	TH	TH_n	TH_w
10	.9981	.9773	.8921	.9959	.9668	.8843	.9940	.9601	.8794	.9921	.9551	.8758
20	.9999	.9945	.9476	.9999	.9922	.9457	.9999	.9908	.9445	.9999	.9898	.9436
30	1.000	.9976	.9656	1.000	.9966	.9647	1.000	.9961	.9642	1.000	.9957	.9638
40	1.000	.9987	.9744	1.000	.9981	.9739	1.000	.9978	.9736	1.000	.9976	.9734
50	1.000	.9991	.9796	1.000	.9988	.9793	1.000	.9986	.9791	1.000	.9985	.9790

9. Concluding Remarks.

In this paper we proposed a new procedure which is conjectured to yield a lower throughput bound of queueing network. We showed that the procedure is numerically stable and produces a sharper bound than that of FPM by Whitt. We also reformulated the queueing network problem into a rather general mathematical problem called "migrating critical points" and proved it in some special cases including polynomials. Our proposed procedure may be applied to the evaluation of telecommunication network protocols such as the sliding window [14] and the analysis of the flexible manufacturing systems [2]. We hope that the conjectures stated in this paper would become the theorems in the near future.

References.

[1] Boas, R. P. and A. R. Reddy, "Zeros of Succesive Derivatives of Entire Functions", ***Journal of Math. Ann. and Appl.,*** Vol. 42 (1973), pp. 466-473.

[2] Buzacott, J. and J. Shanthikumar, ***Stochastic Models of Manufacturing System,*** Prentice Hall, 1993.

[3] Buzen, J. P., "Computational algorithms for closed queueing networks with exponential servers", ***Comm. ACM,*** Vol. 16 (1973), pp. 527-531.

[4] Craven, T., G. Csordas, and W. Smith, "The Zeros of Derivatives of Entire Functions and the Polya-Wiman Conjecture", ***Ann. of Math.,*** Vol. 125 (1987), pp. 405-431.

[5] Gordon, J. J., "The evaluation of Normalizing Constants in Closed Queueing Networks", ***Oper. Res.,*** Vol. 38 (1990), pp. 863-869.

[6] Harrison, P. G., "On Normalizing Constants in Queueing Networks", ***Oper. Res.,*** Vol. 33 (1985), pp. 464-468.

[7] Kobayashi, H., Modeling and Analysis: ***An Introduction to System Performance Evaluation Methodology,*** Addison Wesley, 1978.

[8] Knessl, C. and C. Tier, "Asymptotic Expansions for Closed Queueing Networks with Multiple Job Classes", ***IEEE Tran.,*** on Computers, Vol. 41 (1992), pp. 480-488.

[9] Macdonald, I. G., ***Symmetric Functions and Hall Polynomials,*** Oxford Univ. Press, 1979.

[10] Park, S. and K. Miller, "Random Number Generators: Good Ones Are Hard To Find", ***Comm. of the ACM,*** Vol. 31 (1988), pp. 1192-1201.

[11] Polya, G., "On the Zeros of the Derivatives of a Function and Its Analytic Character", ***Bull. Amer. Math. Soc.,*** Vol. 49 (1943), pp. 178-191.

[12] Reddy, A.R., "On the Derivatives of Zeros of Entire Functions", ***Proc. of Amer. Math. Soc,*** Vol. 45 (1974), pp. 105-112.

[13] Reiser, M. and S. S. Lavenberg, "Mean-Value Analysis of Closed Multi-chain Queueing Networks", ***Journal of the ACM,*** Vol. 27 (1980), pp. 313-322.

[14] Sauer, C. H. and K. M. Chandy, ***Computer Systems Performance Modeling,*** Prentice Hall, 1981.

[15] Schwartz, M., ***Telecommunication Networks: Protocols, Modeling and Analysis,*** Addison Wesley, 1987.

[16] Sheil-Small, T., "On the Zeros of the Derivatives of Real Entire Functions and Wiman's Conjecture", ***Ann. of Math.,*** Vol. 129 (1989), pp. 179-193.

[17] Whitt, W., "Open and Closed Models of Queues", ***AT&T Bell Lab. Tech. J.",*** Vol. 63 (1984),

pp. 1911-1979.

[18] Yao, D. D., "Some Properties of the Throughput Function of Closed Networks of ueues", ***Oper. Res. Letters,*** Vol. 3 (1985), pp. 313-317.

[19] Zahorjan, J., K. C. Sevcik, D. L. Eager, and B. Galler, "Balanced Job Bound Analysis ofQueueing Networks", ***Comm. of the ACM,*** Vol. 25 (1982), pp. 134-141.