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The Optimal Warranty Servicing for Repairable Products with Phase-Type Lifetime Distributions†

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Abstract

This paper considers warranty servicing for repairable products when product lifetimes are phase-type(PH) distributions. Two replace-repair strategies are analyzed based on renewal processes. The quantities of interest can be expressed in terms of the renewal function which, in general, is very difficult to evaluate. By exploiting properties of PH distributions we obtain simplifications to evaluate these performance measures. Numerical examples for four different PH distributions with typical hazard functions are presented and the results are discussed.

1. Introduction

Warranties are becoming an increasingly important part of the market place. A warranty is a contractual obligation incurred by a manufacturer or vendor in relation to product sales and services. Manufacturers use them as a powerful marketing tool. At other times, warranty is imposed on the manufacturer, by certain laws, to protect consumer interests. When a product is sold with a warranty, the manufacturer has to service it if the failure occurs during the warranty period. This consists of actions, termed warranty servicing, such as replacing a failed product with a new one for nonrepairable products and either repairing or replacing a failed product for repairable products. Product warranties can be modeled through the renewal process generated by product failures and the associated cost of actions at each failure. While the literature[1,4~7,10,13] dealing with

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warranties for nonrepairable products is large, the literature[1,9,11] for repairable products is somewhat limited. In most warranties, the performance measures are quite complicated and typically involve the renewal function and related functions. It is well known that the evaluation of the renewal function for an arbitrary probability distribution is very difficult. Outside the exponential family, computing these performance measures requires intricate numerical integration.

Phase-type(PH) distributions(Neuts, 1981) exhibit a wide range of qualitative features. By exploiting various properties of PH distributions, some studies[4~6,13] simplify the related renewal functions and render them amenable to numerical solutions. Kao and Smith(1993) simplified performance measures of Mamer's(1987) model and enhanced the applicability of useful but computationally unwieldy results. Rao(1995) developed evaluating algorithms for free replacement warranty and described the effects of various parameters on product warranty decisions. Kim(1996) analyzed free replacement and pro-rata warranty policies for products with renewable warranties and showed that irrespective of the pattern of hazard function, pro-rata warranty policies are preferable. Kao and Smith(1996) obtained simplifications for two hybrid warranty policies proposed by Nguyen and Murthy(1984a) and introduced computational approximations. But these studies are concerned with nonrepairable products.

This paper considers warranty servicing for repairable products with phase-type lifetime distributions and obtains simplifications to evaluate performance measures of interest in decision making. Since probability distributions of practical interest can be represented or approximated by phase type distributions (see Johnson and Taaffe, 1990 and O'Cinneide, 1990), requiring lifetime distributions to be of phase-type is not all restrictive.

The remainder of the paper is organized as follows. We briefly discuss phase-type distributions and phase-type renewal processes in Section 2. In Section 3, we analyze two replace-repair strategies and obtain simplifications to performance measures. In Section 4, we give numerical examples for four different PH distributions with typical hazard functions and provide qualitative informations. Finally, we conclude with a discussion on some further extensions.

2. Phase-Type Distributions and Renewal Processes

A probability distribution is said to be of phase-type if it can be described as the probability distribution of the time until absorption in a finite Markov chain(Neuts, 1981). Consider an m+1 state continuous-time Markov chain (CTMC) with initial probability row vector$(\boldsymbol{\beta}, \beta_{m+1})$, and infinitesimal generator Q

$$Q = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix}$$

where the m-vector $\boldsymbol{\beta}$ denotes ($\beta_1, \cdots, \beta_m$), S^0=-Se and e is a column vector of ones.

States 1, ⋯, m are transient, and state m+1 is absorbing. Distribution F corresponds to that of the time until absorption. The PH-distribution F with representation ($\boldsymbol{\beta}$, S) is denoted by PH($\boldsymbol{\beta}$, S) and is given by $F(x) = 1 - \boldsymbol{\beta} \exp(Sx)e$. Without loss of generality, we assume that $\beta_{m+1} = 0$, meaning that the time to absorption is always greater than zero. The i-th noncentral moment of PH($\boldsymbol{\beta}$, S) is given by

$$\mu_i = (-1)^i i! (\beta S^{-i} e), \text{ for } i \geq 0 \tag{1}$$

A renewal process in which the interarrival time distribution F is phase-type is known as a PH-renewal process(Neuts, 1981). Let ($\boldsymbol{\beta}$, S) be the representation of F. Associated with the PH-renewal process is an m-state irreducible, recurrent CTMC $\{X(t),\ t \geq 0\}$ with generator $Q^* = S + S^0\boldsymbol{\beta}$. Several important properties of PH-renewal process can be expressed in an efficiently computable form as follows :

· Let $N(t)$ denote the number of renewals in $(0, t]$. The renewal function $M(t) = E[N(t)]$ of the renewal process $\{N(t),\ t \geq 0\}$ is given (Neuts, 1981)by

$$M(t) = \zeta t - \zeta \ \boldsymbol{\beta} \exp(Q^* t) S^{-1} e - 1 \ , \ t \geq 0$$

where $\zeta = 1/\mu_1$ is the asymptotic renewal rate.

· Let $v(t) = \boldsymbol{\beta} \exp(Q^* t)$, for $t \geq 0$. We see that the jth element of the m-vector $v(t)$ gives $P\{X(t) = j\}$ given that the initial vector $v(0)$ is $\boldsymbol{\beta}$. Since $v(t) \rangle 0$ and $v(t)e = 1$, a distribution with representation $(v(t), S)$ is also of phase type. Let $\mu_i(t)$ denote i-th noncentral moment of PH$(v(t), S)$. $\mu_i(t)$ is given by (1) with $v(t)$ replacing $\boldsymbol{\beta}$ and note $\mu_1(0) = \mu_1$. The renewal function can be written as

$$M(t) = \zeta t + \zeta \mu_1(t) - 1, \ t \geq 0 \tag{2}$$

Define an m-vector $v(t,h) = v(t)\exp(Sh)$ for $t,\ h \geq 0$. Note $v(t) = v(t,0)$ and $v(t,h+z) = v(t,h)\exp(Sz)$ for all $t,\ h,\ z \geq 0$. Since $v(t,h) \geq 0$ and $v(t,h)e \leq 1$, a distribution with representation $(v(t,h),\ S)$ is also of phase type.

· Let $B(t)$ denote the excess life of the renewal process at time t and $F_\gamma(x)$ denote the corresponding distribution, i.e. $F_\gamma(x) = P\{B(t) \leq x\}$. The excess life distribution of a PH renewal process with interarrival time distribution PH$(\beta,\ S)$ is given (Kao and Smith, 1992) by

$$F_\gamma(x) = 1 - v(t)\exp(Sx)e, \ x \geq 0 \tag{3}$$

for any $t \geq 0$. In other words, F_γ is PH$(v(t),\ S)$.

3. Replace - Minimal Repair Strategies

For repairable products sold with free-replacement warranty, the manufacturer has the option of either repairing it or replacing it with a new one. The optimal strategy is one that minimizes the expected cost of servicing the warranty over the warranty period W. There are many different possible strategies with regard to replace-repair decisions (see Blischke and Murthy, 1994). This paper considers the following two relatively simple approaches proposed originally by Blischke and Murthy and corrects some representations of their model of the analysis in strategy 2.

Strategy 1 : A product is replaced by a new one if it fails in $(0,\ W - T_1]$ and subjected to minimal repair if it fails in $(W - T_1,\ W]$. The parameter $T_1 (0 \leq T_1 \leq W)$ is selected to minimize the expected cost of servicing the warranty.

Strategy 2 : A product is subjected to minimal repair if it fails in $(0,\ T_2]$ and is replaced by a new one if it fails in $(T_2,\ W]$. The parameter $T_2 (0 \leq T_2 \leq W)$ is selected to minimize the expected cost of servicing the warranty.

We carry out the analysis of these two strategies when the product lifetime is PH$(\beta,\ S)$.

3.1 Strategy 1

Replacements in $(0, W-T_1]$ occur according to a renewal process with the distribution of time between renewals given by F, if the time to replace is negligible. The expected number of replacements over this period is given by $M(W-T_1)$. To obtain the expected number of repairs in $(W-T_1, W]$, note that the time to first failure after $W-T_1$ is given by the excess life of the product in use at time $W-T_1$ with the distribution F_γ. Since failures over $(W-T_1, W)$ are repaired minimally, failures occur according to a nonhomogeneous Poisson process with the distribution F_γ, if the time to repair is negligible. The expected number of repairs over $(W-T_1, W)$ is given by $-\ln \overline{F}_\gamma(T_1)$, where $\overline{F}_\gamma(\cdot)=1-F_\gamma(\cdot)$ (Nguyen and Murthy, 1984b).

Let c_s and c_r be the manufacturing cost and the repair cost per unit. Then the expected cost of servicing the warranty under strategy 1, $C_1(T_1;W)$ is given by

$$C_1(T_1;W) = c_s M(W-T_1) - c_r \ln \overline{F}_\gamma(T_1).$$

When the time to failure distribution F is of phase type with representation $(\boldsymbol{\beta}, S)$ and order m, combining (2) and (3), we can evaluate the expected cost of servicing the warranty simplified considerably by following equation.

$$C_1(T_1;W) = c_s[\zeta(W-T_1)+\zeta\mu_1(W-T_1)-1] - c_r \ln[\nu(W-T_1, T_1)e] \tag{4}$$

The optimal T_1, T_1^* which minimizes $C_1(T_1;W)$ is obtained by examining the servicing costs at some critical points of T_1 and the end points. Some critical points of T_1 can be found in the next theorem 1.

Theorem 1 Provided they exist, some critical points of T_1 can be obtained by solving

$$-c_s\nu(W-T_1, T_1)e + c_r\nu(0, T_1)e = 0 \tag{5}$$

Proof. They can be obtained from $\dfrac{dC_1(T_1;W)}{dT_1} = 0.$

Since $\frac{d\mu_1(W-T_1)}{dT_1} = -\mu_1\nu(W-T_1)S^0+1$,

$\frac{d\ln\nu(W-T_1,T_1)e}{dT_1} = \frac{-\nu(W-T_1)S^0\beta\exp(ST_1)e}{\nu(W-T_1,T_1)e}$ and $\nu(W-T_1)S^0$ is a scalar,

we have the desired result.

3.2 Strategy 2

Since all failures in $[0, T_2)$ are repaired minimally, the expected number of minimal repairs over this period is given by $-\ln\overline{F}(T_2)$. The age of the product in use at time T_2 is simply T_2. Since products failing in the interval (T_2, W) are replaced by new ones, the number of replacements in (T_2, W) follows a delayed renewal process with the distribution for the first failure, $G(x)$, given by

$$G(x)=\frac{F(x+T_2)-F(T_2)}{\overline{F}(T_2)} \tag{6}$$

and the distribution for subsequent failures, $F(x)$. While Blischke and Murthy(1994) used $F_\gamma(x)$ instead of $G(x)$, $G(x)$ is needed because $G(x)$ is different from $F_\gamma(x)$. The expected number of replacements over the interval (T_2, W) is given by(Ross, 1970)

$$M_d(W-T_2) = G(W-T_2) + \int_0^{W-T_2} M(W-T_2-x)dG(x)$$

where $M(t)$ is the renewal function whose interarrival time distribution is F. By applying (6) and integrating by parts, $M_d(W-T_2)$ is written as follows.

$$M_d(W-T_2) = 1+ M(W-T_2) - \overline{F_\gamma}(T_2)/\overline{F}(T_2) \tag{7}$$

where $F_\gamma(\cdot)$ is the distribution of the excess life at time W-T_2. The total expected warranty service cost under strategy 2, $C_2(T_2, W)$ is given by

$$C_2(T_2; W) = c_s M_d(W - T_2) - c_r \ln \overline{F}(T_2)$$

When the time to failure distribution is of phase type, using (2), (3) and (7), the total expected warranty servicing cost can be obtained by the resulting equation.

$$C_2(T_2; W) = c_s[\zeta(W - T_2) + \zeta \mu_1(W - T_2) - \nu(W - T_2, T_2)e / \nu(0, T_2)e] - c_r \ln[\nu(0, T_2)e] \quad (8)$$

The optimal T_2, T_2^* which minimizes $C_2(T_2; W)$ can be obtained from the first order necessary condition.

Theorem 2 The first order necessary condition for strategy 2 is the same as that for strategy 1.

Proof. The necessary condition is obtained from $\frac{dC_2(T_2; W)}{dT_2} = 0$. Since $\nu(0, T_2)S^0$ is a scalar, we can obtain this equation.

$$-c_s \cdot \nu(W - T_2, T_2)e + c_r \nu(0, T_2)e = 0 \quad (9)$$

This equation is identical to (5) for strategy 1.

Note that the expected servicing costs are different as illustrated in the next section.

4. Numerical Examples

In the implementation of the results obtained in Section 3, we use a numerical method requiring the evaluation of matrix exponentials. In order to study warranty servicing, we consider different PH distributions based on the Erlang-2 distribution which Blischke and Murthy(1994) illustrated. Four different PH distributions with typical hazard functions as displayed in Fig.1, which are referred to as D1- D4 are chosen. We standardize these four distributions to have a mean lifetime of one. The parameters of these four distributions are described below.

D1 : Erlang-2(E_2) distribution with a rate of 2 in each phase

D2 : Exponential distribution with a rate 1

D3 : Hyperexponential with 2 phases(H_2) distribution with rates 2.82085228 and 0.50806659 and mixing probabilities 0.6 and 0.4

D4 : Mixture of an E_4 distribution with a rate of 8.385265 in each phase and E_2 distribution with a rate of 0.607932 in each phase with mixing probabilities of 0.814075 and 0.185925

D1, D2 and D3 are an Increasing-Failure-Rate (IFR) distribution, a Constant-Failure-Rate (CFR) distribution and a Decreasing-Failure-Rate (DFR) distribution, respectively. D4 is a mixture distribution that often arises in a number of reliability situations.

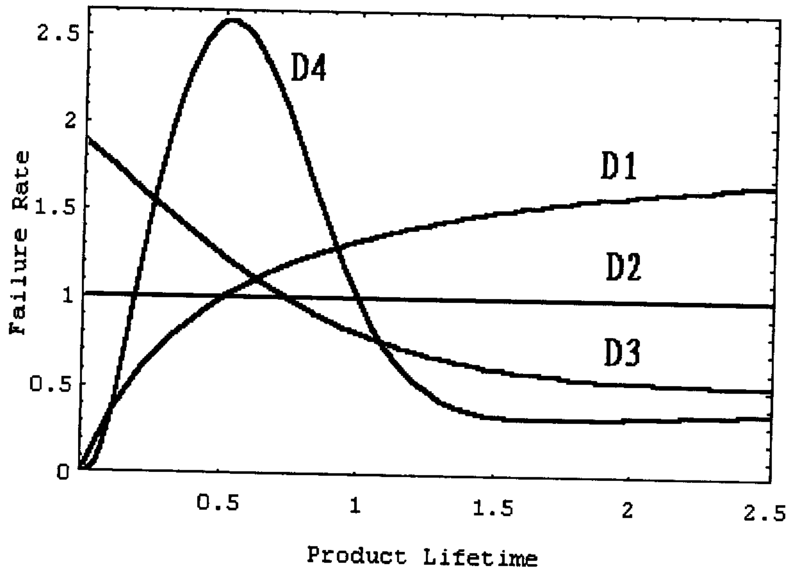

Fig. 1 Hazard Functions

The warranty period, W can be viewed as the multiples of the average product lifetime. Let W=0.5, 1, 2 years and c_s= \$100. The existence of solutions satisfying (5) or (9) depends on the distributional form and the ratio c_r/c_s. In general, they should be obtained by numerical methods. We use a search where values of c_r/c_s range from 0.00 to 1.00 in increments of 0.05 and values of

T range from 0.00 to W in increments of 0.01. Table 1 and 2 give results for strategy 1 and 2 when the distribution is D1 and $W=1$. For $c_r/c_s \leq 0.75$, there is no solution. This implies that the optimal decision is "always repair". For $0.80 \leq c_r/c_s < 1$, there are two solutions T^1 and T^2. T_1^* and T_2^* are the optimal decisions for strategy 1 and 2, respectively. $C_i(0;W)$ and $C_i(W;W)$ for i=1,2 are expected costs of servicing the warranty corresponding to "always replace" and "always repair". For $c_r/c_s = 0.8$, two solutions exist but those expected costs are larger than $C_i(W;W)$ for i=1,2. Hence, in this case $T_i^*=1.00$ for i=1,2. Note that an increase in c_r results in an decrease in T_1^* and T_2^* owing to the burden of repair cost. T_1^* and T_2^* are the same for both strategies.

For distribution D2 and D3, there is no solution i.e. the optimal decision is "always repair". And for distribution D4, there are two solutions for appropriate values of c_r/c_s and the results given in Table 3 and 4 are similar to those for distribution D1. Table 5 and 6 summarize the effect of warranty period on the optimal decisions for distribution D1 and D4. As the warranty period W increases, the optimal period over which minimal repair is to be performed decreases for distribution D1, but the opposite result occurs for distribution D4. These are caused by pattern of failure rates.

Table 1. Optimal Repair VS. Replace Decision for Strategy 1 : D1 and $W=1$

c_r/c_s	T^1	$C_1(T^1;W)$	T^2	$C_1(T^2;W)$	$C_1(0;W)$	$C_1(W;W)$	T_1^*	$C_1(T_1^*;W)$
0.75	-	-	-	-	75.458	67.604*	1.00	67.604
0.80	0.39	72.302	0.72	72.729	75.458	72.111*	1.00	72.111
0.85	0.23	73.925*	0.84	76.818	75.458	76.618	0.23	73.925
0.90	0.13	74.852*	0.91	81.174	75.458	81.125	0.13	74.852
0.95	0.06	75.321*	0.96	85.637	75.458	85.632	0.06	75.321

Table 2. Optimal Repair VS. Replace Decision for Strategy 2 : D1 and $W=1$

c_r/c_s	T^1	$C_2(T^1;W)$	T^2	$C_2(T^2;W)$	$C_2(0;W)$	$C_2(W;W)$	T_2^*	$C_2(T_2^*;W)$
0.75	-	-	-	-	75.458	67.604*	1.00	67.604
0.80	0.39	74.450	0.72	74.877	75.458	72.111*	1.00	72.111
0.85	0.23	75.111*	0.84	78.004	75.458	76.618	0.23	72.111
0.90	0.13	75.370*	0.91	81.691	75.458	81.125	0.13	75.370
0.95	0.06	75.448*	0.96	85.764	75.458	85.632	0.06	75.448

Table 3. Optimal Repair VS. Replace Decision for Strategy 1 : D4 and $W=1$

c_r/c_s	T^1	$C_1(T^1;W)$	T^2	$C_1(T^2;W)$	$C_1(0;W)$	$C_1(W;W)$	T_1^*	$C_1(T_1^*;W)$
0.70	-	-	-	-	130.806	116.508*	1.00	116.508
0.75	0.24	126.813	0.51	127.629	130.806	124.830*	1.00	124.830
0.80	0.16	128.485*	0.73	133.784	130.806	133.152	0.16	128.485
0.85	0.11	129.578*	0.84	141.604	130.806	141.474	0.11	129.578
0.90	0.07	130.282*	0.90	149.815	130.806	149.796	0.07	130.282
0.95	0.04	130.680*	0.95	158.119	130.806	158.118	0.04	130.680

Table 4. Optimal Repair VS. Replace Decision for Strategy 2 : D4 and $W=1$

c_r/c_s	T^1	$C_2(T^1;W)$	T^2	$C_2(T^2;W)$	$C_2(0;W)$	$C_2(W;W)$	T_2^*	$C_2(T_2^*;W)$
0.70	-	-	-	-	130.806	116.508*	1.00	116.508
0.75	0.24	130.237	0.51	131.053	130.806	124.830*	1.00	124.830
0.80	0.16	130.632*	0.73	135.933	130.806	133.152	0.16	130.632
0.85	0.11	130.763*	0.84	142.790	130.806	141.474	0.11	130.763
0.90	0.07	130.799*	0.90	150.332	130.806	149.796	0.07	130.799
0.95	0.04	130.805*	0.95	158.246	130.806	158.118	0.04	130.805

Table 5. Effect of Warranty Period on the Optimal Decisions : D1

c_r/c_s	$W=0.5$	$W=1$	$W=2$
0.65	always repair	always repair	always repair
0.70	always repair	always repair	T^* = 0.76
0.75	always repair	always repair	T^* = 0.50
0.80	always repair	always repair	T^* =0.33
0.85	always repair	T^* = 0.23	T^* = 0.21
0.90	always repair	T^* = 0.13	T^* = 0.13
0.95	always repair	T^* = 0.06	T^* = 0.06

Table 6. Effect of Warranty Period on the Optimal Decisions : D4

c_r/c_s	$W=0.5$	$W=1$	$W=2$
0.75	always repair	always repair	always repair
0.80	T^* = 0.13	T^* = 0.16	always repair
0.85	T^* = 0.09	T^* = 0.11	always repair
0.90	T^* = 0.06	T^* = 0.07	always repair
0.95	T^* = 0.03	T^* = 0.04	always repair

From a practical point of view, it is interesting to compare strategy 1 and 2. We represent the total expected servicing costs for distribution D1 ~ D4 in Fig.2 and Fig.3 corresponding to $W=0.5$ and $c_r=80$, and $W=1$ and $c_r=85$, respectively. In the case of D2 and D3, the advantage of minimal repair makes the expected servicing costs decrease. For D1 and D4, the functions globally increase or decrease depending on the warranty period or c_r/c_s. For the exponential distribution, D2, the total expected servicing costs of both strategies are same by the memoryless property. When products have an IFR distribution, D1 or the initial failure rate is high as in the case of D4, strategy 1 is more favorable. But strategy 2 is more favorable when products have an DFR distribution, D3. As might be expected, the total expected warranty servicing costs are arranged in order of magnitude in the cumulative hazard function.

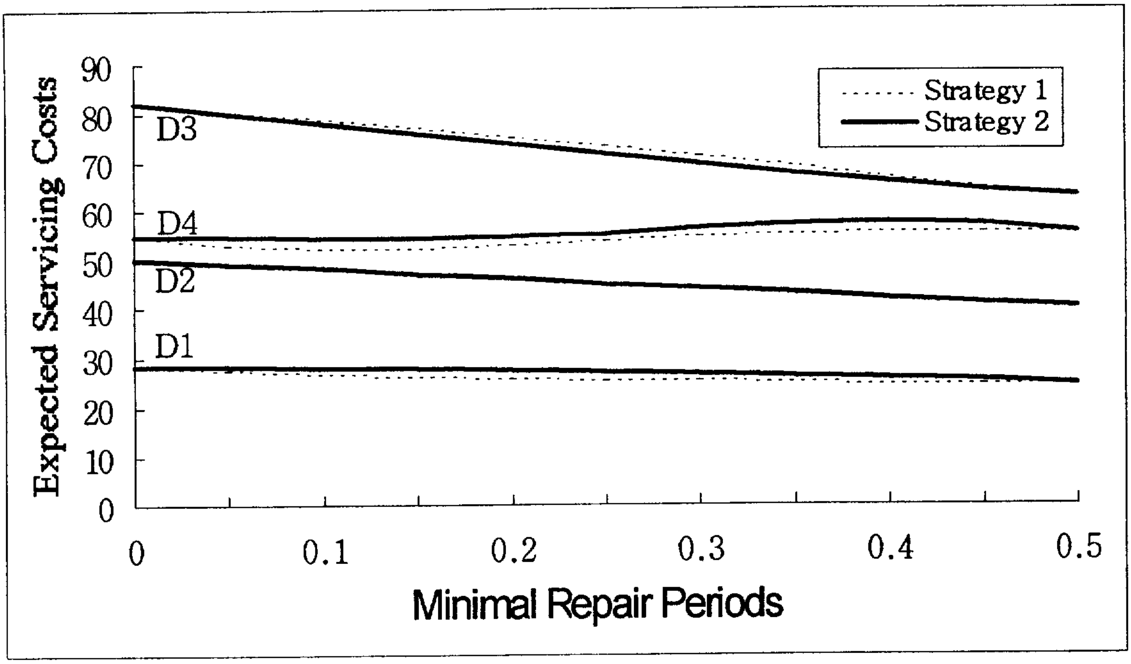

Fig. 2 Comparisons of Strategy 1 and 2 : $W=0.5$ and $c_r=80$

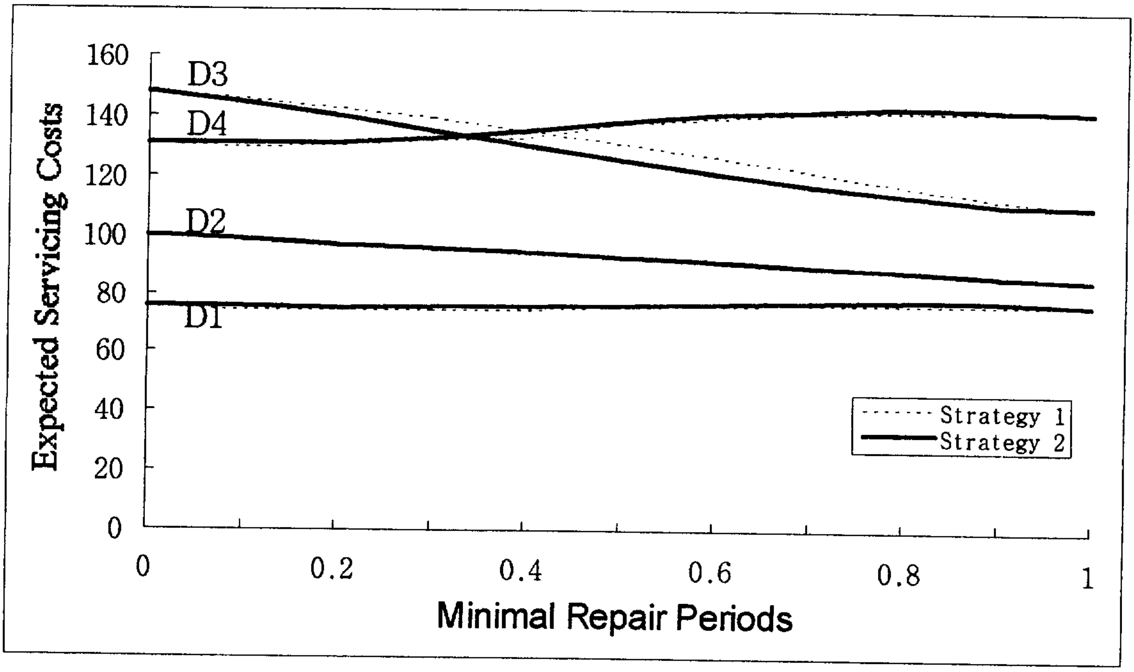

Fig. 3 Comparisons of Strategy 1 and 2 : $W=1$ and $c_r=85$

5. Conclusion

We have studied the optimal replace-repair strategy for warranty servicing when product lifetimes are phase-type. We examined two relatively simple strategies with a single parameter where the choice is between minimal repair and replacement. It is possible to formulate strategies that include different types of repair. The computational tractability of phase-type distributions reduces all the necessary evaluations to matrix inversion and the computation of matrix exponentials. Another advantage of PH distributions is that many probability distributions used in modeling product reliability can be approximated by PH representations. Fitting PH distributions is worthy of further study.

References

[1] Blischke, W. R. and D. N. P. Murthy, ***Warranty Cost Analysis,*** Marcel Dekker, New York, 1994.

[2] Johnson, M. A. and M. R. Taaffe, "Matching Moments to Phase Distribution : Density Function Shapes, " ***Stochastic Models,*** Vol. 6(1990), pp. 283-306.

[3] Kao, E. P. C. and M. S. Smith, "On Excess-, Current-, and Total-Life Distributions of Phase-Type Renewal Processes," ***Naval Research Logistics,*** Vol. 32(1992), pp. 789-799.

[4] Kao, E. P. C. and M. S. Smith, "Discounted and Per Unit Net Revenues and Costs of Product Warranty : The Case of Phase-Type Lifetime," ***Management Science,*** Vol. 39 (1993), pp. 1246-1255.

[5] Kao, E. P. C. and M. S. Smith, "Computational Approximations of Renewal process Relating to a Warranty Problem : The Case of Phase-Type Lifetimes," ***European Journal of Operational Research,*** Vol. 90(1996), pp. 156-170.

[6] Kim, H. G., "Warranty Policies for Non-Repairable Products with Phase-Type Lifetime Distributions," ***Journal of the Korean Society for Quality Management,*** Vol. 24(1996), pp. 1-9.

[7] Mamer, J. W., "Discounted and Per Unit Costs of Production Warranty," ***Management Science,*** Vol. 33(1987), pp. 916-930.

[8] Neuts, M. F., ***Matrix-Geometric Solutions in Stochastic Models : An Algorithmic Approach,*** The Johns Hopkins University Press, Baltimore, 1981.

[9] Nguyen, D. G. and D. N. P. Murthy, "Cost Analysis of warranty Policies," ***Naval Research Logistic Quarterly,*** Vol. 31(1984a), pp. 525-541.

[10] Nguyen, D. G. and D. N. P. Murthy, "A General Model for Estimating Warranty Costs for Repairable Products," ***IIE Transactions,*** Vol. 16(1984b), pp. 379-386.

[11] Nguyen, D. G. and D. N. P. Murthy, "Optimal Replace-Repair Strategy for Servicing Products Sold with Warranty," ***European Journal of Operational Research,*** Vol. 39(1989), pp. 206-212.

[12] O'Cinneide, C.A., "Characterization of Phase-Type Distributions," ***Stochastic Models,*** Vol. 6 (1990), pp. 1-57.

[13] Rao, B. M., "Algorithms for the Free Replacement Warranty with Phase-Type Lifetime Distributions," ***IIE Transactions,*** Vol. 27(1995), pp. 348-357.

[14] Ross, S. M., ***Applied Probability Models with Optimization Application,*** Holden-Day, San Francisco, 1970.