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A RIGIDITY THEOREM FOR REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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Abstract. The purpose of this paper is to prove a rigidity theorem for real hypersurfaces in a complex projective space.

1. Introduction

Let $P_n(C)$ be a n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature $4c$. It is an open question whether a real hypersurface in $P_n(C)$ has rigidity or not. More precisely, if M is a $(2n-1)$-dimensional Riemannian manifold and ι,$\hat{\iota}$ are two isometric immersions of M into $P_n(C)$, then are ι and $\hat{\iota}$ congruent ? To this problem, Y.-W. Choe, B. H. Kim, H. S. Kim, H. Song, Y. J. Suh, R. Takagi and the second author gave some partial solutions (see [1], [2] and [4]). On the other hand, R. Takagi ([5] and [6]) classified all homogeneous real hypersurfaces in $P_n(C)$ which are orbits under analytic subgroups of the projective unitary group $PU(n+1)$ in $P_n(C)$. These homogeneous real hypersurfaces in $P_n(C)$ are locally congruent to one of the six model spaces of type A_1,A_2, B,C,D and E (for details, see Theorem A in [5]). An almost contact structure is naturally introduced on a real hypersurface in $P_n(C)$ from the complex structure of $P_n(C)$, and all the structure vector fields of the model spaces are principal directions of the spaces. The purpose of this paper is to give a partial solution of the rigidity problem mentioned above . Namely we shall prove

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THEOREM 1. *Let M be a $(2n-1)$-dimensional Riemannian manifold, and let $\hat{\iota}$ and ι be two isometric immersions of M into $P_n(C)(n \geq 3)$. If the structure vector fields of $\hat{\iota}$ and ι are principal directions, then the two structure vector fields coincide up to sign on M,*

THEOREM 2. *Let M be a $(2n-1)$-dimensional Riemannian manifold, and let $\hat{\iota}$ and ι be two isometric immersions of M into $P_n(C)(n \geq 3)$. If the structure vector fields of $\hat{\iota}$ and ι are principal directions and the type number of $(M, \hat{\iota})$ or (M, ι) is not equal to 2 at every point of M, then $\hat{\iota}$ and ι are rigid, that is, there exists an isometry ϕ of $P_n(C)$ such that $\phi \circ \iota = \hat{\iota}$.*

2. Preliminaries on real hypersurfaces

Let ι be an isometric immersion of a $(2n-1)$-dimensional Riemannian manifold M into the complex projective space $P_n(C)$ with the metric of constant holomorphic sectional curvature $4c$. For a local orthonormal frame field $\{e_1, e_2, ..., e_{2n-1}\}$ of M, we denote its dual 1−forms by θ_i. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} of M are defined by

$$d\theta_i + \sum \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0, \tag{2.1}$$

$$\Theta_{ij} = d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj} \tag{2.2}$$

respectively, where and in the sequel the indices $i, j, k, l, ...$ run over the range $\{1, 2, ..., 2n-1\}$, unless otherwise stated. We denote the components of the shape operator or the second fundamental tensor A of (M, ι) by A_{ij}, and put $\psi_i = \sum A_{ij}\theta_j$. The rank of A is called the *type number* of (M, ι). For the complex structure J of $P_n(C)$, we put $J_{ij} \circ \iota = \phi_{ij}$ and $J_{2ni} \circ \iota = \xi_i$. Then we have the equations of Gauss and Weingarten

$$\Theta_{ij} = \psi_i \wedge \psi_j + c\theta_i \wedge \theta_j + c\sum(\phi_{ik}\phi_{jl} + \phi_{ij}\phi_{kl})\theta_k \wedge \theta_l, \tag{2.3}$$

$$d\psi_i + \sum \psi_j \wedge \theta_{ji} = c\sum(\xi_j\phi_{ik} + \xi_i\phi_{jk})\theta_j \wedge \theta_k \tag{2.4}$$

respectively, where (ϕ_{ij}, ξ_k) is the almost contact structure on M. $\xi = (\xi_i)$ is called a *structure vector field* of ι. The tensor fields $A = (A_{ij}), \phi = (\phi_{ij})$ and $\xi = (\xi_i)$ on M satisfy

$$A_{ij} = A_{ji}, \quad \phi_{ij} = -\phi_{ji}, \tag{2.5}$$

$$\sum \phi_{ik}\phi_{kj} = -\delta_{ij} + \xi_i\xi_j, \quad \sum \xi_j\phi_{ji} = 0, \quad \sum \xi_i^2 = 1, \tag{2.6}$$

$$d\phi_{ij} = \sum(\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki}) - \xi_i\psi_j + \xi_j\psi_i, \tag{2.7}$$

$$d\xi_i = \sum(\xi_j\theta_{ji} - \phi_{ji}\psi_j). \tag{2.8}$$

For another isometric immersion $\hat\iota$ of M into $P_n(C)$, we shall denote the differential forms and tensor fields of $(M, \hat\iota)$ by the same symbol as ones in (M, ι) but with a hat. Since the canonical 1−forms, connection forms and curvature forms are independent of the choice of immersions, it follows from (2.3) that

$$\begin{aligned} &A_{ik}A_{jl} - A_{il}A_{jk} + c(\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) \\ &= \hat{A}_{ik}\hat{A}_{il} - \hat{A}_{il}\hat{A}_{jk} + c(\hat\phi_{ik}\hat\phi_{jl} - \hat\phi_{il}\hat\phi_{jk} + 2\hat\phi_{ij}\hat\phi_{kl}). \end{aligned} \tag{2.9}$$

As for the rigidity of (M, ι) and $(M, \hat\iota)$, the following is known and will be used later.

THEOREM A ([1]). *Let M be a $(2n-1)-$ dimensional Riemannian manifold, and $\hat\iota$ and ι be two isometric immersions of M into $P_n(C) (n \geq 3)$. If the two structure vector fields coincide up to sign on M and the type number of $(M, \hat\iota)$ or (M, ι) is not equal to 2 at every point of M, then $\hat\iota$ and ι are rigid, that is, there exists an isometry ϕ of $P_n(C)$ such that $\phi \circ \iota = \hat\iota$.*

3. Proof of Theorems

Since the structure vector fields ξ of ι and $\hat{\xi}$ of $\hat{\iota}$ are principal directions, we have

$$\sum A_{ij}\xi_j = \alpha\xi_i \quad and \quad \sum \hat{A}_{ij}\hat{\xi}_j = \hat{\alpha}\hat{\xi}_i. \tag{3.1}$$

It is known ([3]) that the principal curvatures α and $\hat{\alpha}$ are constant on M. Since $\hat{A} = (\hat{A}_{ij})$ is a symmetric matrix and $\hat{\xi}$ is a principal direction, we can choose a local orthonormal frame field $\{e_1 = \hat{\xi}, e_2, ..., e_{2n-1}\}$ of M such that $\hat{A}$ is diagonalized with respect to the frame field, that is,

$$\hat{A}_{ij} = \nu_i\delta_{ij}, \tag{3.2}$$

where ν_i are principal curvature of $\hat{\iota}(M)$ and $\nu_1 = \hat{\alpha}$. It follows from this frame field and (2.6) that

$$\hat{\xi}_1 = 1, \quad \hat{\xi}_a = 0 \quad and \quad \hat{\phi}_{i1} = 0, \tag{3.3}$$

where and in the sequel the indices $a, b, c, ..$ run over the range $\{2, 3, ..., 2n -1\}$. From (2.9) and (3.2) we have

$$\begin{aligned} &A_{ik}A_{jl} - A_{il}A_{jk} + c(\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) \\ =&\nu_i\nu_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + c(\hat{\phi}_{ik}\hat{\phi}_{jl} - \hat{\phi}_{il}\hat{\phi}_{jk} + 2\hat{\phi}_{ij}\hat{\phi}_{kl}). \end{aligned} \tag{3.4}$$

Putting $i = a, j = b, k = c$ and $l = 1$ into (3.4) and using (3.3), we obtain

$$A_{ac}A_{b1} - A_{a1}A_{bc} + c(\phi_{ac}\phi_{b1} - \phi_{a1}\phi_{bc} + 2\phi_{ab}\phi_{c1}) = 0. \tag{3.5}$$

If we multiply (3.4) by ξ_l and use (2.6) and (3.1), we get

$$\begin{aligned} &c(\hat{\phi}_{ik}\sum \hat{\phi}_{jl}\xi_l - \hat{\phi}_{jk}\sum \hat{\phi}_{il}\xi_l + 2\hat{\phi}_{ij}\sum \hat{\phi}_{kl}\xi_l) \\ =&\alpha(\xi_j A_{ik} - \xi_i A_{jk}) + \nu_i\xi_i\nu_j\delta_{jk} - \nu_i\xi_j\nu_j\delta_{ik}. \end{aligned} \tag{3.6}$$

Now we shall prove

LEMMA 3.1. *If the principal curvature α vanishes on M, then the two structure vector fields ξ and $\hat{\xi}$ coincide up to sign on M.*

PROOF. It follows from $\alpha = 0$ and (3.6) that

$$\hat{\phi}_{ik} \sum \hat{\phi}_{jl}\xi_l - \hat{\phi}_{jk} \sum \hat{\phi}_{il}\xi_l + 2\hat{\phi}_{ij} \sum \hat{\phi}_{kl}\xi_l = 0 (k \neq i, j). \tag{3.7}$$

Multiplying (3.7) by ξ_i and summing up it for $i (\neq k)$, we have

$$\Big(\sum \hat{\phi}_{jk}\xi_k - \hat{\phi}_{ji}\xi_i\Big) \sum \hat{\phi}_{il}\xi_l = 0 (i \neq j) \tag{3.8}$$

by use of (2.5) and (2.6). If $\sum \hat{\phi}_{jk}\xi_k - \hat{\phi}_{ji}\xi_i = 0$ for $i \neq j$, then, by summing up for $i (\neq j)$, we obtain $(2n-3) \sum \hat{\phi}_{jk}\xi_k = 0$. Therefore (3.8) implies that

$$\sum \hat{\phi}_{ik}\xi_k = 0 \quad \text{or} \quad \sum \hat{\phi}_{jk}\xi_k = 0 \quad \text{for} \quad i \neq j.$$

If there are non-zero components of the vector field $\hat{\phi}\xi$, then (3.8) shows that there is only one non-zero component of $\hat{\phi}\xi$, say $\sum \hat{\phi}_{2k}\xi_k \neq 0$. By putting $i, j \neq 2$ and $k = 2$ into (3.7), it is easily seen that

$$\hat{\phi}_{ij} = 0 \quad \text{for} \quad i, j \neq 2.$$

This means that the rank of $(\hat{\phi}_{ij})$ is not greater than 2. Since the rank of $(\hat{\phi}_{ij})$ is equal to $2n - 2 \geq 4$, the above argument is contrary. Thus all the components of $\hat{\phi}\xi$ vanish, that is, $\hat{\phi}\xi = 0$ on M. This shows that $\xi = \pm\hat{\xi}$, and completes the proof. □

From now on we consider the case where the principal curvature α does not vanish on M. Putting $i = a, k = b$ and $j = 1$ into (3.6) and using (3.3), we have

$$\xi_1 A_{ab} - \xi_a A_{b1} = \frac{\hat{\alpha}}{\alpha}\xi_1 \nu_a \delta_{ab}. \tag{3.9}$$

If we multiply (3.5) by ξ_c and make use of (2.5), (2.6) and (3.1), then we obtain

$$\xi_a A_{b1} = \xi_b A_{a1}. \tag{3.10}$$

LEMMA 3.2. *If the principal curvature α does not vanish on M, then the two structure vector fields ξ and $\hat{\xi}$ coincide up to sign on M.*

PROOF. Assume that the two structure vector fields ξ and $\hat{\xi}$ do not coincide on M, that is, $\xi_1^2 \neq 1$ on M. Then, multiplying (3.10) by ξ_b and using (3.1), we obtain

$$A_{a1} = \frac{\alpha - A_{11}}{1 - \xi_1^2}\xi_a\xi_1. \tag{3.11}$$

We see from (3.11) that $\xi_1 \neq 0$ on M. In fact, if $\xi_1 = 0$ on M, we have $A_{a1} = 0$ for any index a. It follows from (3.5) that

$$\phi_{ac}\phi_{b1} - \phi_{bc}\phi_{a1} + 2\phi_{ab}\phi_{c1} = 0.$$

Putting $b = c$ into the above equation, we get $\phi_{ab}\phi_{b1} = 0$. If $\phi_{ab} = 0$, then it is easily seen that the rank of (ϕ_{ij}) is not greater than 2 and it is contrary. If $\phi_{b1} = 0$, then it follows from (2.6) that $\xi_1^2 = 1$ and it also contradicts. Substituting (3.11) into (3.9), we have

$$A_{ab} = \frac{\alpha - A_{11}}{1 - \xi_1^2}\xi_a\xi_b + \frac{\hat{\alpha}}{\alpha}\nu_a\delta_{ab}. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.5), we also obtain

$$c(\phi_{ac}\phi_{b1} - \phi_{a1}\phi_{bc} + 2\phi_{ab}\phi_{c1}) = \frac{\hat{\alpha}}{\alpha}\frac{\alpha - A_{11}}{1 - \xi_1^2}(\xi_a\nu_b\delta_{bc} - \xi_a\nu_a\delta_{ac})\xi_1,$$

from which

$$\phi_{ac}\phi_{b1} - \phi_{a1}\phi_{bc} + 2\phi_{ab}\phi_{c1} = 0 \quad \text{for} \quad c \neq a, b. \tag{3.13}$$

If we multiply (3.13) by $\xi_b (b \neq c)$ and make use of (2.5) and (2.6), then we have

$$(\phi_{ab}\xi_b + \phi_{a1}\xi_1)\phi_{b1} = 0 \quad \text{for} \quad a \neq b. \tag{3.14}$$

If $\phi_{ab}\xi_b + \phi_{a1}\xi_1 = 0$ for $a \neq b$, then it is easily seen from the multiplication of this equation by ξ_b that $2(n-2)\phi_{a1}\xi_1 = 0$. Therefore (3.14) implies that $\phi_{a1} = 0$ or $\phi_{b1} = 0$ for $a \neq b$. If $\phi_{a1} = 0$ for any index a, then we get $\xi_1^2 = 1$ and it is contrary. Therefore there are some entries such that $\phi_{a1} \neq 0$. But (3.14) shows that there is only one non-zero entry, say $\phi_{21} \neq 0$. Putting $a, b \neq 2$ and $c = 2$ into (3.13), we have $\phi_{ab} = 0$ for $a, b \neq 2$. This implies that the rank of (ϕ_{ij}) is not greater than 2, and it also contradicts. Thus this completes the proof. □

PROOF OF THEOREMS. *Theorem 1 follows from Lemmas 3.1 and 3.2, and Theorem 2 is immediate from Theorems 1 and A.*

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