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KAEHLER SUBMANIFOLDS WITH $RS = 0$ IN A COMPLEX PROJECTIVE SPACE

JONG-IK HYUN

ABSTRACT. Our study focuses on the condition under which a subspace of complex projective space can become an Einstein space. We prove that a subspace becomes an Einstein space if it's codimension is less than $n-1$ and its curvature tensor and Ricci tensor satisfies Ryan's condition.

0. Indroduction

Ryan, P. J. [3] and Takahashi,T. [4] has studied complex hypersurfaces in a complex space form satisfying the condition

$$R(X,Y)S = 0 \tag{0.1}$$

for any vectors X and Y of the hypersurfaces, where R denotes the curvature tensor, S is the Ricci tensor and $R(X,Y)$ operates on the tensor algebra as a derivation. Ryan proved that these hypersurfaces are Einstein if the ambient space is not complex euclidean, which was generalized by Kon, M. [1] in the case of Kaehler submanifolds in a complex space form of constant negative holomorphic sectional curvature. On the other hand, Takahashi also verified that such hypersurfaces are cylindrical if the ambient space is complex euclidean. The purpose of this note is to prove the following.

THEOREM. *Let M be an n-dimensional Kaehler submanifold immersed in an $(n+q)$- dimensional complex projective space PC_{n+q}. If M satisfies the condition* (0.1) *and the codimension q is less than $n-1$, then M is Einstein.*

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1. Kaehler submanifolds in PC_{n+q}

Let M be an n-dimensional Kaehler manifold and I an isometric and holomorpic immersion of M into an $(n+q)$- dimensional complex projective space $PC_{n+q}(c)$ of constant holomorphic section curvature c.

We call such I simply a *Kaehler immersion.* Throughout this note, M may be identified with $I(M)$, because the argument is local. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}, \cdots, \mathbf{e}_{n+q}$ be a unitary frame field in $PC_{n+q}(c)$ in such a way that, restricted to M, $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are tangent to M. It's dual coframe field $\mathbf{w}^1, \cdots, \mathbf{w}^n, \mathbf{w}^{n+1}, \cdots, \mathbf{w}^{n+q}$ consists of complex valued linear differential forms of type $(1,0)$ on M such that

$$\mathbf{w}^\alpha = 0 \qquad (\alpha = n+1, n+2, \cdots, n+q) \tag{1.1}$$

and $\mathbf{w}^1, \cdots, \mathbf{w}^n, \mathbf{w}^{-1}, \cdots, \mathbf{w}^{-n}$ are linearly independent. Greek indices run over the range $n+1, \cdots, n+q$. The induced Kaehler metric g on M is given by $g = 2\sum_i \mathbf{w}^i \bigotimes \mathbf{w}^{-i}$ and $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a unitary frame of M and $\mathbf{w}^1, \cdots, \mathbf{w}^n$ is a coframe field of $\mathbf{e}_1, \cdots, \mathbf{e}_n$. Associated to the frame $\mathbf{e}_1, \cdots, \mathbf{e}_n, e_{n+1}, \cdots, \mathbf{e}_{n+q}$ there exist complex valued differential forms $\mathbf{w}^A_B$, which are usually called *connection forms* on $PC_{n-q}(c)$, such that

$$d\mathbf{w}^A + \sum_B \mathbf{w}^A_B \wedge \mathbf{w}^B = 0, \qquad \mathbf{w}^A_B + \bar{\mathbf{w}}^B_A = 0, \tag{1.2}$$

$$d\mathbf{w}^A_B + \sum_C \mathbf{w}^A_C \wedge \mathbf{w}^C_B = \Omega^A_B, \tag{1.3}$$

$$\Omega^A_B = \sum_{C,D} R^A_{BC\bar{D}} \mathbf{w}^C \wedge \bar{\mathbf{w}}^D,$$

where Ω^A_B denotes the curvature form and $R^A_{BC\bar{D}}$ denotes the curvature tensor on $PC_{n+q}(c)$, which are given by

$$R^A_{BC\bar{D}} = \frac{c}{2}(\delta^A_B \delta_{CD} + \delta^A_C \delta_{BD}), \tag{1.4}$$

because $PC_{n+q}(c)$ is of constant holomorphic sectional curvature c. Here the capital letters run over the range $1, \cdots, n, n+1, \cdots, n+q$.

It follows from (1.2) and the Cartan's lemma that the exterior derivative of (1.1) gives

$$\mathbf{w}_i^\alpha = \sum_j h_{ij}^\alpha \mathbf{w}^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{1.5}$$

where the small letters run over the range $1, \cdots, n$. Then the quadratic form $\sum_{i,j} h_{ij}^\alpha \mathbf{w}^i \otimes \mathbf{w}^j$ is called the *second fundamental form* of M in the direction of $\mathbf{e}^\alpha$. Since $\mathbf{w}^\alpha = 0$ again,(1.2) and (1.3) become

$$d\mathbf{w}^i + \sum_j \mathbf{w}_j^i \wedge \mathbf{w}^j = 0, \tag{1.6}$$

$$d\mathbf{w}_j^i + \sum_k \mathbf{w}_k^i \wedge \mathbf{w}_j^k = \Omega_j^i, \tag{1.7}$$

$$\Omega_j^i = \sum_{k,j} R^i_{jk\bar{l}} \mathbf{w}^k \wedge \bar{\mathbf{w}}^l,$$

where $\mathbf{w}_j^i$ (resp.Ω_j^i) denotes the connection (resp.curvature) form on M and $R^i_{jk\bar{l}}$ denotes the curvature tensor on M. It follows form (1.4), (1.5) and (1.7) that we have the equation of Gauss

$$R^{\bar{i}}_{jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_\alpha h_{jk}^\alpha \bar{h}_{il}^\alpha, \tag{1.8}$$

Now, with respect to these frames, the Ricci form S can be expressed

$$S = \sum_{k,l} (R_{k\bar{l}} \mathbf{w}^k \otimes \bar{\mathbf{w}}^l + R_{\bar{k}l} \bar{\mathbf{w}}^k \otimes \mathbf{w}^l), \tag{1.9}$$

where the Ricci tensor $R_{k\bar{l}}$ is defined by $R_{k\bar{l}} = \sum_i R^{\bar{i}}_{jk\bar{l}}$ and is satisfies $R_{k\bar{l}} = R_{\bar{l}k} = \bar{R}_{l\bar{k}}$. Because of (1.8), $R_{k\bar{l}}$ is given by

$$R_{k\bar{l}} = \frac{n+1}{2} c\delta_{kl} - \sum_{\alpha,i} h_{ki}^\alpha \bar{h}_{il}^\alpha. \tag{1.10}$$

2. Proof of Theorem

In this secion, let M be an n-dimensional Kaehler submanifold immersed holomorphically into $PC_{n+q}(c)$. We assume that M satisfies the condition (0.1).

In our notations, this condition is equivalent to

$$\sum_k R_{k\bar{j}}\Omega_i^k + \sum_k R_{i\bar{k}}\bar{\Omega}_j^k = 0. \tag{2.1}$$

substituting (1.7) and (1.10) into (2.1), we have the equation

$$\begin{aligned} &c\sum_{\alpha,r}(h_{ir}^\alpha \bar{h}_{rl}^\alpha \delta_{jk} - h_{kr}^\alpha \bar{h}_{rj}^\alpha \delta_{il}) \\ &+2\sum_{\alpha,\beta,r,s}(h_{ik}^\beta \bar{h}_{lr}^\beta h_{rs}^\alpha \bar{h}_{sj}^\alpha - h_{kr}^\beta \bar{h}_{rs}^\alpha h_{si}^\alpha \bar{h}_{jl}^\beta) = 0. \end{aligned} \tag{2.2}$$

Let H^α be an $n \times n$ matrix with its components (h_{ij}^α). Then, for a suitable choice of the frame $\mathbf{e}_1, \cdots, \mathbf{e}_n$ a matrix $\sum_\alpha H^\alpha \bar{H}^\alpha$ can be orthogonalized as follows:

$$\sum_\alpha H^\alpha \bar{H}^\alpha = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \tag{2.3}$$

Since the matrix is a positive semi-definite Hermitian one, the eigenvalues $\lambda_1, \cdots, \lambda_n$ are non-negative real valued functions on M. Moreover, we have

$$\sum_{\alpha,i} h_{ki}^\alpha \bar{h}_{il}^\alpha = \lambda_k \delta_{kl}. \tag{2.4}$$

From (2.4), (2.1) becomes

$$c(\lambda_i - \lambda_k)\delta_{il}\delta_{jk} + 2(\lambda_j - \lambda_i)\sum_\alpha h_{ik}^\alpha \bar{h}_{lj}^\alpha = 0. \tag{2.5}$$

It follows from this equation that the equations

$$(2.6)\qquad \begin{cases} (\lambda_i-\lambda_j)\left(\sum_\alpha h_{ij}^\alpha \bar{h}_{ij}^\alpha - \dfrac{c}{2}\right)=0, \\ (\lambda_i-\lambda_j)\sum_\alpha h_{ik}^\alpha \bar{h}_{lj}^\alpha = 0 \qquad \text{unless } i=l \text{ and } j=k. \end{cases}$$

are obtained.

We may suppose that $\lambda_1,\cdots,\lambda_p$ are all distinct eigenvalues of $\sum_\alpha H^\alpha \bar{H}^\alpha$. Let $n_1,\cdots,n_p$ be the multiplicities of $\lambda_1,\cdots,\lambda_p$ respectively, where p is a function on M. If $p=1$ everywhere on M, then M is exactly Einstein. Suppose that $p \geqq 2$ at a point x of M. Then it follows from (2.6) that

$$(2.7)\qquad \begin{cases} \sum_\alpha h_{ij}^\alpha \bar{h}_{ij}^\alpha = \dfrac{c}{2} & \text{if } \lambda_i \neq \lambda_j, \\ \sum_\alpha h_{ik}^\alpha \bar{h}_{lj}^\alpha = 0 & \text{if } \lambda_i \neq \lambda_j \text{ and } (k,l)\neq(i,j) \text{ or } (j,i). \end{cases}$$

Let h_{ij} be a vector in C^q defined by $h_{ij}=(h_{ij}^{n+1},\cdots,h_{ij}^{n+q})$. Consider the set $\{h_{ij}\,;\,\lambda_i\neq\lambda_j\}$ consisting of $\sum_{r<s}^p n_r n_s$ vectors in C^q. The equations (2.7) mean that they are linearly independent. Accordingly, because of $\sum_{r=1}^p n_r = n$, we have

$$(2.8)\qquad q \geqq \sum_{r<s}^{p} n_r n_s \geqq n-1,$$

where the second equality holds if $p=2$ and n_1 is equal to 1 or $n-1$.

This completes the proof.

REMARK. As is well showed at Remark 4.2 in [2], the product manifold of $PC_1(c)$ and $PC_{n-1}(c)$ is an n-dimensional Kaehler manifold and it is imbeddend into a $(2n-1)$ -dimensional complex projective space $PC_{2n-1}(c)$. Then $PC_1(c)\times PC_{n-1}(c)$ satisfies the condition (0.1), but if $n\geqq 3$, then it is not Einstein. These implies that the estimate of the codimension is best possible.

REMARK. The Proof in this section can be discussed in the similar manner, through the ambient space is a complex space form of constant negative holomorphic sectional curvature.

In this case, the first equation of (2.7) implies that M is Einstein. This is a brief proof of Kon's result.

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Department of Mathematics Education
Cheju National University of Education
Cheju 690-060, Korea