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CONJUGACY SEPARABILITY OF FREE PRODUCTS WITH AMALGAMATION

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ABSTRACT. We first prove a criterion for the conjugacy separability of free products with amalgamation where the amalgamated subgroup is not necessarily cyclic. Applying this result, we show that free products of finite number of polycyclic-by-finite groups with central amalgamation are conjugacy separable. We also show that polygonal products of polycyclic-by-finite groups, amalgamating central cyclic subgroups with trivial intersections, are conjugacy separable.

1. Introduction

A group G is said to be *conjugacy separable* if, for each pair of nonconjugate elements $x, y \in G$, there exists a finite homomorphic image $\overline{G}$ of G such that the images of x, y in $\overline{G}$ are not conjugate in $\overline{G}$. Conjugacy separability of groups is related to the conjugacy problem in the study of groups. In fact, Mostowski [11] proved that finitely presented conjugacy separable groups have solvable conjugacy problem. Since Stebe [15] and Dyer [3] studied the conjugacy separability of generalized free products of free or nilpotent groups, the conjugacy separability of generalized free products of various groups, amalgamating a cyclic subgroup, has been studied in [7, 8, 9, 12, 13, 16]. But, when the amalgamated subgroup is not cyclic, the conjugacy separability of generalized free products is not much known. In this paper, we give a criterion (Theorem 2.3) for generalized free products amalgamating a subgroup, not necessarily cyclic, to be conjugacy separable. Applying this result, we show that generalized free

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products of finite number of polycyclic-by-finite groups, amalgamating a subgroup in the center, are conjugacy separable. Moreover, from our criterion, we can easily prove the main result in [6] that polygonal products of polycyclic-by-finite groups, amalgamating central cyclic subgroups with trivial intersections, are conjugacy separable.

Throughout this paper we use standard notations and terminology.

The letter G always denotes a group and 1 denotes the identity of a group.

$\{x\}^G$ denotes the set of all conjugates of x in G.

H^G denotes the normal closure of H in G.

$x \sim_G y$ means x, y are conjugate in G.

$N \lhd_f G$ means N is a normal subgroup of finite index in G.

We denote by $E *_H F$ the generalized free products of E and F amalgamating the subgroup H. If $x \in G = A *_H B$ then $\|x\|$ denotes the free product length of x in G.

Let S be a subset of G. Then G is said to be *S-separable* if, for each $x \in G \backslash S$, there exists $N \lhd_f G$ such that $x \notin NS$.

If G is $\{1\}$-separable, then G is said to be *residually finite.*

If G is H-separable for each finitely generated subgroup H of G, then G is called *subgroup separable.*

We shall make extensive use of the following results.

THEOREM 1.1. [10, Theorem 4.6] *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*
(1) If $\|x\| = 0$, then $\|y\| \leq 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_A h_r \sim_B x$.
(2) If $\|x\| = 1$, then $\|y\| = 1$ and, either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.
(3) If $\|x\| \geq 2$, then $\|x\| = \|y\|$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.

THEOREM 1.2. [3, Theorem 4] *If A and B are conjugacy separable and H is finite, then $A *_H B$ is conjugacy separable.*

REMARK 1.3. *Let $u_1u_2\cdots u_r$ and $v_1v_2\cdots v_s$ be reduced words in $E *_H F$. Then $u_1u_2\cdots u_r = v_1v_2\cdots v_s$ if and only if $r = s$ and there exist $h_i \in H$ such that $u_1 = v_1h_1$, $u_2 = h_1^{-1}v_2h_2, \ldots, u_{r-1} = h_{r-2}^{-1}v_{r-1}h_{r-1}$, and $u_r = h_{r-1}^{-1}v_r$.*

2. A criterion

In this section we prove a criterion for the conjugacy separability of generalized free products of groups amalgamating a subgroup. Then, in next sections, we apply this criterion to show that certain groups are conjugacy separable.

Let $G = E *_H F$. For normal subgroups $M \lhd E$ and $L \lhd F$ such that $M \cap H = L \cap H$, there is a natural homomorphism

$$\phi_{M,L} : E *_H F \longrightarrow E/M *_{\overline{H}} F/L,$$

where $\overline{H} = MH/M = LH/L$.

DEFINITION 2.1. Let $G = E *_H F$ and let $\Lambda = \{(M_i, L_i) : i \in I\}$ be a set of normal subgroups of E and F such that $M_i \cap H = L_i \cap H$ satisfying:

D1. for $(M_{\alpha_i}, L_{\alpha_i}) \in \Lambda$, we have $(\cap_{i=1}^n M_{\alpha_i}, \cap_{i=1}^n L_{\alpha_i}) \in \Lambda$;

D2. for each $(M, L) \in \Lambda$, $\phi_{M,L}(G) = E/M *_{\overline{H}} F/L$ is conjugacy separable;

D3. for $x, y \in E$ such that $x \not\sim_E y$ (or $x, y \in F$ and $x \not\sim_F y$), there exists $(M, L) \in \Lambda$ such that $Mx \not\sim_{E/M} My$ (or $Lx \not\sim_{F/L} Ly$);

D4. for $x \in E$ such that $\{x\}^E \cap H = \emptyset$ (or $x \in F$ and $\{x\}^F \cap H = \emptyset$), there exists $(M, L) \in \Lambda$ such that $\{Mx\}^{E/M} \cap MH/M = \emptyset$ (or $\{Lx\}^{F/L} \cap LH/L = \emptyset$);

D5. $\cap_{(M,L)\in\Lambda} MHeH = HeH$ and $\cap_{(M,L)\in\Lambda} LHfH = HfH$ for $e \in E$ and $f \in F$.

Then we call Λ a *compatible filter* of the generalized free product $G = E *_H F$.

Note that much simple filtrations were used by G. Baumslag [2] in the study of the residual finiteness of generalized free products.

LEMMA 2.2. *If $G = E *_H F$ has a compatible filter Λ, then G is residually finite.*

PROOF. Let $1 \neq x \in G$. Since $E/M *_{\overline{H}} F/L$ is conjugacy separable by D2, it is residually finite. Hence it suffices to find a suitable $(M, L) \in \Lambda$ such that $\overline{x} \neq 1$ in $\overline{G} = E/M *_{\overline{H}} F/L$.

First, suppose $x \in E \cup F$, say $x \in E$. Since $x \neq 1$, $x \not\sim_E 1$. Thus, by D3, there exists $(M, L) \in \Lambda$ such that $Mx \not\sim_{E/M} M1$. This clearly implies that $\overline{x} \neq 1$ in $\overline{G} = E/M *_{\overline{H}} F/L$.

Second, suppose $x \notin E \cup F$. Without loss of generality, we assume $x = e_1 f_1 \cdots e_n f_n$, where $e_i \in E \backslash H$ and $f_i \in F \backslash H$. Now it follows from D5 that $\cap_{(M,L)\in\Lambda} MH = H$ and $\cap_{(M,L)\in\Lambda} LH = H$. Hence there exist $(M_i, L_i) \in \Lambda$ and $(M'_i, L'_i) \in \Lambda$ such that $e_i \notin M_i H$ and $f_i \notin L'_i H$ for each i. Let $M = \cap_{i=1}^n (M_i \cap M'_i)$ and $L = \cap_{i=1}^n (L_i \cap L'_i)$. Then, by D1, we have $(M, L) \in \Lambda$, $e_i \notin MH$ and $f_i \notin LH$ for all i. Thus $\|\overline{x}\| = \|x\| > 1$ in $\overline{G} = E/M *_{\overline{H}} F/L$, hence $\overline{x} \neq 1$. □

THEOREM 2.3. *Let $G = E *_H F$ have a compatible filter Λ satisfying:*

D6. *if there exist a reduced word $u_1 \cdots u_{r-1} u_r \in E *_H F$ $(r \geq 2)$ and $h \in H$ such that $u_1 \cdots u_{r-1} h u_r \notin H u_1 \cdots u_{r-1} u_r H$, then there exists $(M, L) \in \Lambda$ such that $\phi_{M,L}(u_1 \cdots u_{r-1} h u_r) \notin \phi_{M,L}(H u_1 \cdots u_{r-1} u_r H)$, where $\phi_{M,L}(G) = E/M *_{\overline{H}} F/L$;*

D7. *for each $(M, L) \in \Lambda$, if $x, y \in H$ and $\overline{x} \sim_{\overline{E}} \overline{y}$ (or $\overline{x} \sim_{\overline{F}} \overline{y}$), where $\overline{E} = E/M$ (or $\overline{F} = F/L$), then $\overline{x} = \overline{y}$.*

Then $G = E *_H F$ is conjugacy separable.

PROOF. Let $x, y \in G$ such that $x \not\sim_G y$. Without loss of generality we can assume that x and y are of minimal lengths in their conjugacy classes in G. Since G is residually finite by Lemma 2.2, we may assume $x \neq 1 \neq y$. To prove our result, we shall find $(M, L) \in \Lambda$ such that $\overline{x} \not\sim_{\overline{G}} \overline{y}$, where $\overline{G} = E/M *_{\overline{H}} F/L$. Then, by D2, $\overline{G}$ is conjugacy separable, whence there exists $\overline{K} \triangleleft_f \overline{G}$ such that $\overline{K}\overline{x} \not\sim_{\overline{G}/\overline{K}} \overline{K}\overline{y}$. Let $K = \phi^{-1}_{M,L}(\overline{K})$ be the preimage of $\overline{K}$ in G. Then we have $K \triangleleft_f G$ and $Kx \not\sim_{G/K} Ky$, as required.

Case 1. $\|x\| = 0 = \|y\|$. By D3, there exists $(M, L) \in \Lambda$ such that $\overline{x} \not\sim_{\overline{E}} \overline{y}$, where $\overline{E} = E/M$. Now, if $\overline{x} \sim_{\overline{G}} \overline{y}$ in $\overline{G} = E/M *_{\overline{H}} F/L$ then, by Theorem 1.1, there is a sequence $\overline{h}_1, \overline{h}_2, \ldots, \overline{h}_r$ of elements in $\overline{H}$ such that $\overline{y} \sim_{\overline{E}} \overline{h}_1 \sim_{\overline{F}} \overline{h}_2 \sim_{\overline{E}} \cdots \sim_{\overline{E}} \overline{h}_r \sim_{\overline{F}} \overline{x}$. Hence, by D7, we have $\overline{y} = \overline{h}_1 = \cdots = \overline{h}_r = \overline{x}$. This clearly contradicts to the choice of M. Hence $\overline{x} \not\sim_{\overline{G}} \overline{y}$ in $\overline{G} = E/M *_{\overline{H}} F/L$.

Case 2. $\|x\| = 0$ *and* $\|y\| = 1$ *(or* $\|y\| = 0$ *and* $\|x\| = 1$*), say,* $y \in E\backslash H$ *and* $x \in H$. Since y is of minimal length 1 in its conjugacy class, we have $\{y\}^E \cap H = \emptyset$. By D4, there exists $(M, L) \in \Lambda$ such that $\{Mx\}^{E/M} \cap MH/M = \emptyset$. Let $\overline{G} = E/M *_{\overline{H}} F/L$. Then, by Theorem 1.1, $\overline{y}$ is of minimal length 1 in its conjugacy class in $\overline{G}$. Hence $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 3. $\|x\| \neq \|y\|$ *and* $\|x\| \geq 2$ *(or* $\|y\| \geq 2$*).* Since x is of minimal length in its conjugacy class, it is cyclically reduced. Let $x = e_1 f_1 \cdots e_n f_n$ (say, other cases being similar), where $e_i \in E \backslash H$ and $f_i \in F \backslash H$. Then, as in the proof of Lemma 2.2, we can find $(M_1, L_1) \in \Lambda$ such that $\|\phi_{M_1,L_1}(x)\| = \|x\| = 2n$ in $\phi_{M_1,L_1}(G) = E/M_1 *_{\hat{H}} F/L_1$. Similarly, we can find $(M_2, L_2) \in \Lambda$ such that $\|\phi_{M_2,L_2}(y)\| = \|y\|$ in $\phi_{M_2,L_2}(G) = E/M_2 *_{\hat{H}} F/L_2$. Let $(M, L) = (M_1 \cap M_2, L_1 \cap L_2)$. Then $(M, L) \in \Lambda$ and, in $\overline{G} = E/M *_{\overline{H}} F/L$, we have $\|\overline{x}\| = \|x\|$ and $\|\overline{y}\| = \|y\|$. Moreover $\overline{x}$ is cyclically reduced and is of minimal length $2n$ in its conjugacy class and $\|\overline{x}\| \neq \|\overline{y}\|$. Hence, by Theorem 1.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 4. $\|x\| = 1 = \|y\|$.

(i) $x, y \in E \backslash H$ (or $x, y \in F \backslash H$). Since x is of minimal length 1 in its conjugacy class, $\{x\}^E \cap H = \emptyset$ and $x \not\sim_E y$. By D3, there exists $(M_1, L_1) \in \Lambda$ such that $M_1 x \not\sim_{E/M_1} M_1 y$ and, by D4, there exists $(M_2, L_2) \in \Lambda$ such that $\{M_2 x\}^{E/M_2} \cap M_2 H/M_2 = \emptyset$. As before, let $(M, L) = (M_1 \cap M_2, L_1 \cap L_2)$. Then $(M, L) \in \Lambda$ and, in $\overline{G} = E/M *_{\overline{H}} F/L$, we have $\overline{x} \not\sim_{\overline{E}} \overline{y}$ and $\{\overline{x}\}^{\overline{E}} \cap \overline{H} = \emptyset$. Hence $\overline{x}$ is of minimal length 1 in its conjugate class and $\overline{x} \not\sim_{\overline{E}} \overline{y}$. Thus, by Theorem 1.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

(ii) Suppose $x \in E \backslash H$ and $y \in F \backslash H$ (or $x \in F \backslash H$ and $y \in E \backslash H$). As in (i) above, there exists $(M, L) \in \Lambda$ such that $\{\overline{x}\}^{\overline{E}} \cap \overline{H} = \emptyset$ and $\{\overline{y}\}^{\overline{F}} \cap \overline{H} = \emptyset$, where $\overline{E} = E/M$ and $\overline{F} = F/L$. Let $\overline{G} = E/M *_{\overline{H}} F/L$, then $\overline{x}, \overline{y}$ are of minimal length 1 in their conjugacy classes with $\overline{x} \in \overline{E}$ and $\overline{y} \in \overline{F}$. Hence, by Theorem 1.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 5. *Suppose* $\|x\| = \|y\| = 2n$. Let $x = e_1 f_1 \cdots e_n f_n$ and $y = e'_1 f'_1 \cdots e'_n f'_n$, where $e_j, e'_j \in E \backslash H$ and $f_j, f'_j \in F \backslash H$ for all j. As in the proof of Lemma 2.2, we can find $(M_0, L_0) \in \Lambda$ such that $e_j, e'_j \notin M_0 H$ and $f_j, f'_j \notin L_0 H$ for all j. Since $x \not\sim_G y$, we have $x \not\sim_H y^*$ for all cyclic permutation y^* of y. Thus each of the equations

$$(j): \; e'_j f'_j \cdots e'_n f'_n e'_1 f'_1 \cdots e'_{j-1} f'_{j-1} = h^{-1} e_1 f_1 \cdots e_n f_n h$$

has no solution $h \in H$. We shall find $(M_i, L_i) \in \Lambda$ such that $M_i \subset M_0$, $L_i \subset L_0$ and the equation

$$\phi_{M_j,L_j}(j): \; \phi_{M_j,L_j}(e'_j f'_j \cdots f'_n e'_1 f'_1 \cdots f'_{j-1}) = \phi_{M_j,L_j}(h^{-1} e_1 f_1 \cdots e_n f_n h)$$

has no solution $\phi_{M_j,L_j}(h) \in \phi_{M_j,L_j}(H)$ for each j. Then, for $M = \cap_{i=1}^n M_i$ and $L = \cap_{i=1}^n L_i$, we have $\|\tilde{x}\| = \|x\| = \|y\| = \|\tilde{y}\|$ and $\tilde{x} \not\sim_{\tilde{H}} \tilde{y}^*$ for any cyclic permutation $\tilde{y}^*$ of $\tilde{y}$, where $\tilde{G} = E/M *_{\tilde{H}} F/L$. Hence we have $\tilde{x} \not\sim_{\tilde{G}} \tilde{y}$ as required.

Here we only consider the case $j = 1$, since the others are similar.

(1) There exists i such that $e'_i \notin He_iH$ (or $f'_i \notin Hf_iH$). Then, by D5, we can find $(P, Q) \in \Lambda$ such that $e'_i \notin PHe_iH$. Let $(M_1, L_1) = (M_0 \cap P, L_0 \cap Q)$. In $\overline{G} = E/M_1 *_{\overline{H}} F/L_1$, we have $\overline{e}'_i \notin \overline{H}\overline{e}_i\overline{H}$ and hence $\overline{y} \notin \overline{H}\overline{x}\overline{H}$. Thus clearly $\overline{x} \not\sim_{\overline{H}} \overline{y}$.

Hence, from now, we suppose $e'_i \in He_iH$ and $f'_i = Hf_iH$ for all i.

(2) There exists i such that $e'_1f'_1 \cdots f'_{i-1}e'_i \in He_1f_1 \cdots f_{i-1}e_iH$, but $e'_1f'_1 \cdots e'_if'_i \notin He_1f_1 \cdots e_if_iH$. Then $e'_1f'_1 \cdots f'_{i-1}e'_i = h_1e_1f_1 \cdots f_{i-1}e_ih_2$ and $f'_i = h_3f_ih_4$ for some $h_k \in H$. Since $e'_1f'_1 \cdots e'_if'_i \notin He_1f_1 \cdots e_if_iH$, we have $e_1f_1 \cdots e_ih_2h_3f_i \notin He_1f_1 \cdots e_if_iH$. Hence, by D6, there exists $(P, Q) \in \Lambda$ such that $\phi_{P,Q}(e_1f_1 \cdots e_ih_2h_3f_i) \notin \phi_{P,Q}(He_1f_1 \cdots e_if_iH)$, where $\phi_{P,Q}(G) = E/P *_{\tilde{H}} F/Q$. As before, let $(M_1, L_1) = (M_0 \cap P, L_0 \cap Q)$. In $\overline{G} = E/M_1 *_{\overline{H}} F/L_1$, we have $\|\overline{x}\| = \|x\| = \|y\| = \|\overline{y}\| = 2n$ and $\overline{e_1f_1 \cdots e_ih_2h_3f_i} \notin \overline{He_1f_1 \cdots e_if_iH}$. This follows that $\overline{e'_1f'_1 \cdots e'_if'_i} \notin \overline{He_1f_1 \cdots e_if_iH}$, hence $\overline{y} \notin \overline{H}\overline{x}\overline{H}$. Thus clearly we have $\overline{x} \not\sim_{\overline{H}} \overline{y}$.

(3) $y \in HxH$ but $x \not\sim_H y$. Then $e'_1f'_1 \cdots e'_nf'_n = h_1e_1f_1 \cdots e_nf_nh_2$ for some $h_1, h_2 \in H$ and $h_1h_2 \neq 1$. By D3, we can find $(P, Q) \in \Lambda$ such that $Ph_1h_2 \not\sim_{E/P} P1$. As before, let $(M_1, L_1) = (M_0 \cap P, L_0 \cap Q)$. Then, in $\overline{G} = E/M_1 *_{\overline{H}} F/L_1$, we have $\overline{y} \not\sim_{\overline{H}} \overline{x}$. Otherwise, $\overline{e'_1f'_1 \cdots e'_nf'_n} = \overline{h}^{-1}\overline{e_1f_1 \cdots e_nf_n}\overline{h}$ for $\overline{h} \in \overline{H}$. Thus $\overline{e_1f_1 \cdots e_nf_n} = \overline{h}\overline{h}_1\overline{e_1f_1 \cdots e_nf_n}\overline{h}_2\overline{h}^{-1}$. This follows from D7 that $\overline{e}_1 = \overline{hh_1}\overline{e}_1(\overline{hh_1})^{-1}$, $\overline{f}_1 = \overline{hh_1}\overline{f}_1(\overline{hh_1})^{-1}$, ..., $\overline{e}_n = \overline{hh_1}\overline{e}_n(\overline{hh_1})^{-1}$, and $\overline{f}_n = \overline{hh_1}\overline{f}_n\overline{h}_2\overline{h}^{-1}$. Hence, by D7 again, $(\overline{hh_1})^{-1} = \overline{h}_2\overline{h}^{-1}$. Thus $\overline{h_1h_2} = 1$, whence $h_1h_2 \in M_1 \subset P$, a contradiction. Therefore $\overline{x} \not\sim_{\overline{H}} \overline{y}$, where $\overline{G} = E/M_1 *_{\overline{H}} F/L_1$.

This completes our proof that there exists $(M_1, L_1) \in \Lambda$ such that $M_1 \subset M_0$, $L_1 \subset L_0$ and the equation $\phi_{M_1,L_1}(1)$ has no solution. Thus $\overline{x} \not\sim_{\overline{H}} \overline{y}$ in $\overline{G} = E/M_1 *_{\overline{H}} F/L_1$.

This completes the proof. □

3. Amalgamating central subgroups

In this section we show that generalized free products of finite many polycyclic-by-finite groups, amalgamating a central subgroup, are conjugacy separable.

LEMMA 3.1. *Let E, F be conjugacy separable and $H \subset Z(E) \cap Z(F)$. If E/M and F/M are also conjugacy separable for each $M \triangleleft_f H$, then the set $\Lambda = \{(M, M) : M \triangleleft_f H\}$ forms a compatible filter of $E *_H F$.*

PROOF. Clearly D1 holds. Since E/M and F/M are conjugacy separable and since $\overline{H} = MH/M = H/M$ is finite, $E/M *_{\overline{H}} F/M$ is conjugacy separable by Theorem 1.2.

For D3, let $x, y \in E$ such that $x \not\sim_E y$. Since E is conjugacy separable, there exists $N \triangleleft_f E$ such that $Nx \not\sim_{E/N} Ny$. Let $M = H \cap N$. Then $M \triangleleft_f H$ and $M \subset N$, hence $Mx \not\sim_{E/M} My$. Similarly, if $x, y \in F$ such that $x \not\sim_F y$, then there exists $M \triangleleft_f F$ such that $Mx \not\sim_{F/M} My$.

For D4, let $x \in E$ and $\{x\}^E \cap H = \emptyset$. Since $H \subset Z(E)$, this implies $x \notin H$. Then $x \notin MH = H$ for all $M \triangleleft_f H$. Thus, we have $\{Mx\}^{E/M} \cap H/M = \emptyset$. Similarly, if $x \in F$ and $\{x\}^F \cap H = \emptyset$, then $\{Mx\}^{F/M} \cap H/M = \emptyset$ for all $M \triangleleft_f H$.

For D5, clearly $\cap_{(M,M)\in\Lambda} MHeH = HeH$ and $\cap_{(M,M)\in\Lambda} MHfH = HfH$ for $e \in E$ and $f \in F$, since $M \subset H$. Therefore $\Lambda = \{(M, M) : M \triangleleft_f H\}$ is a compatible filter of $E *_H F$. □

THEOREM 3.2. *Let E, F be conjugacy separable and $H \subset Z(E) \cap Z(F)$. If E/M and F/M are also conjugacy separable for each $M \triangleleft_f H$, then the generalized free product $G = E *_H F$ is conjugacy separable.*

PROOF. Since $\Lambda = \{(M, M) : M \triangleleft_f H\}$ is a compatible filter of $E *_H F$ by Lemma 3.1, we only need to show that D6 and D7 in Theorem 2.3 hold. But, since $H \subset Z(E) \cap Z(F)$, we have $H \subset Z(G)$. Hence clearly D6 and D7 hold. Therefore G is conjugacy separable by Theorem 2.3. □

Any homomorphic image of polycyclic-by-finite groups are polycyclic-by-finite. And polycyclic-by-finite groups are conjugacy separable [4] and subgroup separable [14, p.148]. Thus, applying Theorem 3.2 inductively, we have the next result.

COROLLARY 3.3. *Let E_i be polycyclic-by-finite groups, where $i = 1, \dots, n$, and $H \subset Z(E_i)$ for each i. Then the generalized free product of the E_i, amalgamating H, is conjugacy separable.*

4. On polygonal products

In this section we apply our criterion, Theorem 2.3, to polygonal products of polycyclic-by-finite groups, amalgamating central cyclic subgroups with trivial intersections. The conjugacy separability of those polygonal products was originally known by [6] but the proof there was quite complicate. Here, using our criterion, we shortly show that those polygonal products are conjugacy separable. Of course we use some lemmas in [6] unavoidably. Terminology of polygonal products can be found in [1, 6].

THEOREM 4.1. *Let P be the polygonal product of the polycyclic-by-finite groups A_0, A_1, $\ldots$, A_m ($m \geq 3$), amalgamating the central subgroups $\langle a_1\rangle, \ldots, \langle a_m\rangle, \langle a_0\rangle$ with trivial intersections. Then P is conjugacy separable*

PROOF. Let P_0 be the subgroup $\langle a_0, a_1, \ldots, a_m\rangle$ of P. Then P_0 is the polygonal product of abelian groups $\langle a_0, a_1\rangle, \langle a_1, a_2\rangle, \ldots, \langle a_m, a_0\rangle$ amalgamating cyclic subgroups $\langle a_1\rangle, \ldots, \langle a_m\rangle, \langle a_0\rangle$, with trivial intersections. Moreover, P_0 is a graph product of cyclic groups $\langle a_1\rangle, \ldots, \langle a_m\rangle, \langle a_0\rangle$. Hence P_0 is conjugacy separable by [5, p.104]. For $1 \leq i \leq m+1$, let $P_i = (\cdots((P_0 *_{B_m} A_m) *_{B_{m-1}} A_{m-1})\cdots) *_{B_{m-i+1}} A_{m-i+1}$, where $B_j = \langle a_j, a_{j+1}\rangle$ and $a_{m+1} = a_0$. Then P_i is the polygonal product of $\langle a_0, a_1\rangle$, $\cdots$, $\langle a_{m-i}, a_{m-i+1}\rangle$, A_{m-i+1}, $\cdots$, A_m amalgamating the central subgroups $\langle a_1\rangle, \ldots, \langle a_{m-i+1}\rangle, \ldots, \langle a_0\rangle$, with trivial intersections, and $P_{i+1} = P_i *_{B_{m-i}} A_{m-i}$. Thus $P = P_{m+1} = P_m *_{B_0} A_0$. Hence we obtain P as taking suitable generalized free products starting from P_0. Thus, since P_0 is conjugacy separable, for an induction assuming that every polygonal product of polycyclic-by-finite groups $\langle c_0, c_1\rangle, C_1, \ldots, C_m$, amalgamating the central subgroups $\langle c_1\rangle, \ldots, \langle c_m\rangle, \langle c_0\rangle$ with trivial intersections, is conjugacy separable, we shall show that $P = P_m *_{B_0} A_0$ is conjugacy separable. By assumption, P_m is conjugacy separable.

To apply our criterion for $P = P_m *_{B_0} A_0$, we shall show that

$$\Lambda = \{(\langle a_0^s, a_1^t\rangle^{P_m}, \langle a_0^s, a_1^t\rangle) : s, t > 1\}$$

forms a compatible filter for $P = P_m *_{B_0} A_0$.

Note that $\langle a_0^{s_1}, a_1^{t_1}\rangle^{P_m} \cap \langle a_0^{s_2}, a_1^{t_2}\rangle^{P_m} = \langle a_0^s, a_1^t\rangle^{P_m}$ and $\langle a_0^{s_1}, a_1^{t_1}\rangle \cap \langle a_0^{s_2}, a_1^{t_2}\rangle = \langle a_0^s, a_1^t\rangle$ where s, t are the least common multiples of s_1, s_2 and t_1, t_2. Hence D1 holds.

By induction hypothesis, $\overline{P_m} = P_m/\langle a_0^s, a_1^t\rangle^{P_m}$ is conjugacy separable for any $s,t > 1$, since $\overline{P_m}$ is the polygonal product of $\langle \overline{a_0}, \overline{a_1}\rangle$, $A_1/\langle a_1^t\rangle$, A_2, $\ldots$, A_{m-1}, $A_m/\langle a_0^s\rangle$, amalgamating the subgroups $\langle \overline{a}_1\rangle$, $\langle a_2\rangle$, $\ldots$, $\langle a_m\rangle$, $\langle \overline{a}_0\rangle$. Now $\overline{B}_0 = \langle a_0, a_1\rangle/\langle a_0^s, a_1^t\rangle$ is finite. Thus, for any $s,t > 1$, $\overline{P} = P/\langle a_0^s, a_1^t\rangle^P = \overline{P_m} *_{\overline{B}_0} \overline{A}_0$ is conjugacy separable by Theorem 1.2, where $\overline{A}_0 = A_0/\langle a_0^s, a_1^t\rangle$. Thus D2 holds.

As in the proof of Lemma 3.1, D3 holds, since P_m, A_0 are conjugacy separable. Lemma 3.5 in [6] shows that D4 holds for P_m. Since $B_0 = \langle a_0.a_1\rangle \subset Z(A_0)$, D4 holds for A_0 as in Lemma 3.1. Lemma 3.9 in [6] shows that $\cap_{s,t}\langle a_0^s, a_1^t\rangle^{P_m} B_0 x B_0 = B_0 x B_0$ for $x \in P_m$. Thus D5 holds for P_m. Since $A = \langle a_0.a_1\rangle \subset Z(A_0)$ and A_0 is subgroup separable, D5 holds for A_0 as in Lemma 3.1. Therefore Λ is a compatible filter for $P = P_m *_{B_0} A_0$.

The condition D6 for $P = P_m *_{B_0} A_0$ was considered in [6, p.305]. D7 for P_m directly follows from Lemma 3.2 in [6] and D7 clearly holds for A_0, since $B_0 \subset Z(A_0)$. Hence $P = P_m *_{B_0} A_0$. is conjugacy separable by Theorem 2.3. This completes our proof. □

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