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CONDITION OF PSEUDOHYPERBOLIC STRUCTURE

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ABSTRACT. The paper presents results on the perturbation problem of invariant manifolds of differential equations. It is well-known that if there is a pseudohyperbolic structure on an invariant manifold then one is strongly indestructible. The set of strongly indestructible invariant manifolds is wider than the set of persistent (normally hyperbolic) manifolds. The following theorem is main result of the paper: if the condition of transversality holds on an invariant manifold, except, possibly, for the non-degenerate strong sources and non-degenerate strong sinks, then there is the pseudohyperbolic structure on the invariant manifold. From this it follows the conditions for the indestructibility of locally non-unique invariant manifolds. An example is considered.

1. Introduction

Let us consider a system of autonomous differential equations

$$\dot{u} = F_0(u), \quad u \in R^n \tag{1}$$

where F_0 is a smooth vector field. Hereafter "smooth" means C^1 smooth. We suppose M_0 to be a compact connected C^1 submanifold invariant for the flow system (1), $k = codim M_0$. In the space of smooth bounded vector fields $\{F\}$ we introduce a topology generated by the C^1 norm $\|F\|_1$:

$$\|F\| = \sup_{u\in R^n} |F(u)|, \quad |\partial F(u)| = \max_{|v|=1} |\partial F(u)v|,$$
$$\|\partial F\| = \sup_{u\in R^n} |\partial F(u)|, \quad \|F\|_1 = \|F\| + \|\partial F\|.$$

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In order to introduce a topology in the space of smooth compact manifolds $\{M\}$ C^1 close to M_0 we consider a construction of a tubular neighborhood of M_0. Let $E = \{E(x) : x \in M_0\}$ be a smooth field of planes such that $dimE(x) = codimM_0$ and $E(x)$ is transversal to M_0 at x. Thus, we have a vector bundle E with M_0 as a base.

LEMMA 1 [2]. *For any vector bundle E there exists a vector bundle E_1 such that the direct sum $E \oplus E_1 = E^*$ is diffeomorphic to $M_0 \times R^{n+1}$, i.e., E^* is trivial.*

THEOREM 1. *A normally hyperbolic compact invariant manifold is indestructible.*

THEOREM 2. *A persistent invariant manifold is normally hyperbolic.*

In the condition of Theorem 1, a perturbed invariant manifold M possesses the property of local uniqueness. Thus Theorems 1,2 give necessary and sufficient conditions for a manifold to be persistent. However there are equations for which indestructible manifolds are locally non-unique, and hence, they are not persistent. For example, the equilibrium point $x = 0$ of the equation $\dot{x} = x^3$, $x \in R$, is indestructible and is not locally unique by Mane [6]. In the paper we will consider a set of indestructible manifolds without the supposition of their local uniqueness.

2. Pseudohyperbolic structure

In order to state sufficient conditions of the strong indestructibility we have to introduce the following notions. Stable and unstable subspaces at a point $x \in M_0$ are defined as

$$\begin{aligned} E^s(x) = \{y \in T_xN : &|\partial U(t,x)y| \to 0, \\ &|\partial U(t,x)y||(\partial U_0)^{-1}(t,x)| \to 0, \text{ as } t \to +\infty\}, \\ E^u(x) = \{y \in T_xN : &|\partial U(t,x)y| \to 0, \\ &|\partial U(t,x)y||(\partial U_0)^{-1}(t,x)| \to 0, \text{ as } t \to -\infty\}, \end{aligned}$$

We say that the condition of transversality is valid at $x \in M_0$ if

$$T_xN = T_xM_0 + E^s(x) + E^u(x).$$

The dimensions of stable and unstable subspaces may depend on x and $dim E^s(x) + dim E^u(x) \geq codim M_0$. It should be noted that if the transversality condition holds on the whole manifold M_0 and the sum is direct, *i.e.*,

$$TM_0 + E^s + E^u = TM_0 \oplus E^s \oplus E^u,$$

then M_0 is normally hyperbolic. It is clear that the normal hyperbolicity condition implies the transversality condition. As th example from [8] show, the converse is not true, in general.

Let V be a trajectorically convex set of (1) contained in M_0, *i.e.*, V is such that the intersection of each orbit J of (1) and V is an interval $\Delta(J) = \{u = x(t) : t \in (\alpha, \beta)\}$, where $x(t)$ is a solution corresponding to J (the cases $\alpha = -\infty$, $\beta = \infty$ are not excluded). Points $E_n = x(\alpha)$, $-\infty < \alpha$ and $E_x = x(\beta)$, $\beta < \infty$ are named an entrance and an exit, respectively, $\beta - \alpha = |\Delta(J)|$ is called a length of the arc $\Delta(J)$. Henceforth, we will only consider closed trajectorically convex sets V, for which the disposition of J and the boundary ∂V of V satisfies one of the following conditions:

(i) $J \cap \partial V = \phi$,

(ii) $J \cap \partial V$ is formed by one or two points - the entrance and the exit,

(iii) $J \cap \partial V = J \cap V = \Delta$.

The set $P(V) = \{\cup \Delta(J) : J \cap \partial V = J \cap V = \Delta(J)\}$ is called a lateral manifold of V. Then the equation (1) may be considered as a perturbation of the linearized system defined on the vector bundle E:

$$\begin{aligned} \dot{x} &= f(x) + D_y f_0(x, 0) y, \\ \dot{y} &= \phi(x) y, \end{aligned} \tag{2}$$

where $D_y f_0$ stands for the partial derivative of f_0 with respective to y. If system (2) has an invariant bundle $E|_V$, $V \subset M_0$, then the invariance of $E|_V$ yields $D_y f_0(x, 0) = 0$, *i.e.*, system (2) on $E|_V$ takes the form

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{y} &= \phi(x) y. \end{aligned} \tag{3}$$

Let $(X(t, x) + C(t, x)y,\ A(t, x)y)$ be a solution of system (2), where the sum is realized by the exponential map on M_0. Obviously, a solution of system (3) is of the form $(X(t,x),\ A(t,x)y)$.

DEFINITION 1. System (2) is called *pseudohyperbolic* on the manifold M_0 if there exists an open covering $\Xi = \{V_i : i = -1, 0, 1, \cdots, k, k+1\}$ of M_0 such that

(i) each V_i is a regular trajectorically convex set with respect to system (2), V_{-1} is a sufficiently small neighborhood of strong non-degenerate sources and V_{k+1} is a sufficiently small neighborhood of strong non-degenerate sinks at which the transversality condition is violated,

(ii) for each i, there exists an invariant continuous decomposition of the bundle $E|_{V_i} = E_i$ into a direct sum $E_i = E_i^s \oplus E_i^u$ and $i = \dim E_i^s$ if $-1 < i < k = 1$,

(iii) if $p \in V_i$, $U(t,p) \in V_j \backslash V_i$, then

$$DU(t,p)E_i^s(p) \subset E_j^s(U(t,p)) \quad \text{as} \quad t > 0,$$
$$DU(t,p)E_i^u(p) \subset E_j^u(U(t,p)) \quad \text{as} \quad t < 0;$$

(iv) there exist a, $\lambda > 0$ such that

(a) if $p \in V_i$, $t > 0$, $U(t,p) \in V_i$, then for each i

$$|DU_i^s(t,p)||(DU^0(t,p))^{-1}| \leq a\exp(-\lambda t)$$

and

$$|DU_i^s(t,p)| \leq a\exp(-\lambda t) \quad \text{as} \quad i \neq -1,$$

(b) if $p \in V_i$, $t < 0$, $U(t,p) \in V_i$, then for each i

$$|DU_i^u(t,p)||(DU^0(t,p))^{-1}| \leq a\exp(\lambda t)$$

and

$$|DU_i^u(t,p)| \leq a\exp(\lambda t) \quad \text{as} \quad i \neq k+1,$$

where $DU_i^s = DU|_{E_i^s}$, $DU_i^u = DU|_{E_i^u}$, $DU^0 = DU|_{TM_0}$.

The sets $V_i, 0 \leq i \leq k$, are called sets of hyperbolicity. The covering $\Xi = \{V_i\}$ and decompositions $\{E_i = E_i^s \oplus E_i^u\}$, for which conditions $(i) - (iv)$ are satisfied, are called a pseudohyperbolic structure.

The following theorem was proved in the work [15].

THEOREM 3. *If there is a pseudohyperbolic structure on M_0 then M_0 is strongly indestructible.*

Our goal is to prove :

THEOREM 4. *If the transversality condition holds on the invariant manifold M_0 of (1) except, possibly, for strong sinks and strong sources with nonzero indices, then the equation (1) on M_0 possesses the pseudohyperbolic structure. In addition, if strong sinks and strong sources are non-degenerate, then the pseudohyperbolic structure is non-degenerate.*

From Theorems 3 and 4 it follows

THEOREM 5. *If the transversality condition holds on the compact manifold M_0 of (1) except, possibly, for non-degenerate strong sinks and non-degenerate strong sources M_0 is strong indestructible.*

To make the proof transparent, we proceed with a summary of its main fragments using the following example.

EXAMPLE. Normally nonhyperbolic invariant manifold on which the transversality condition holds.

In R^3, consider a differential equation, leaving, in the plane $\{x_1 = 0\}$, the unit circle $S = M_0$ invariant. On M_0, there are three equilibrium points: $A(0,0,1), B(0,1,0)$ and $C(0,0,-1)$. The linearized equation at A is of the form

$$\dot{x}_1 = 2x_1, \quad \dot{x}_2 = x_2, \quad \dot{x}_3 = 2x_3.$$

Thus, the Lyapunov exponent along M_0 is 1 and those along the subspace normal to M_0 are 2. The linearized equation at B is of the form

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = x_2, \quad \dot{x}_3 = 0$$

and is of the form $\dot{x}_3 = -x_3^2$ in a neighborhood of B on M_0. Thus, the system is hyperbolic on the normal subspace at B. The linearized equation at C is of the form

$$\dot{x}_1 = -2x_1, \quad \dot{x}_2 = -x_2, \quad \dot{x}_3 = -2x_3.$$

We have that the stable E^s and unstable E^u subspaces at A, B and C are the following:

(i) $E^u(A)$ is the plane $\{x_2 = 0\}$ normal to M_0, $E^s(A) = \{0\}$,
(ii) $E^u(B)$ is the straight line $\{x_3 = 0, x_1 = 0\}$, $E^s(B) = \{x_3 = 0, x_2 = 0\}$,
(iii) $E^u(C) = \{0\}$, $E^s(C) = \{x_2 = 0\}$.

Thus, the transversality condition holds at A, B and C. The stable subspace is defined by the behavior of the differential $DU(t,p)$ as $t \to +\infty$ and the unstable subspace is defined by the behavior of $DU(t,p)$ as $t \to -\infty$. Hence, for a point $p \in M_0$ between A and B, $E^u(p)$ is defined by B. Consequently, $E^u(p)$ is 2-dimensional disk transversal to M_0 and $E^s(p)$ is a straight line. Thus, $E^s(p) + E^u(p) + TM_o(p) = R^3$ and the transversality condition holds at each point of M_0. The covering Xi consists of V_0, V_1, V_2, which are neighborhoods of A, B and C, respectively. The set V_0 has two points of exit and has no point of entrance, V_1 has a point of entrance and a point of exit, V_2 has two points of entrance. The stable and unstable subbundles E_i^s, E_i^u, $(i = 1, 2)$ are the extensions of the stable and the unstable subspaces of A, B, C, respectively. We may choose E_1^s, E_i^u so that the inclusion $E_1^u \subset E_0^u$ holds on $V_0 \cap V_1$ and the inclusion $E_1^s \subset E_2^s$ holds on $V_0 \cap V_1$.

3. The proof of theorem 4

The proof of theorem 4 is based on a sequence of lemmas stated below. Previously, we need to give a notation of the chain-recurrent set of equation (1) on M_0 [1]. The set of all chain-recurrent points is called the chain-recurrent set of equation (1) on M_0 and is denoted by CR.

LEMMA 2[3]. *Let the transversality condition hold on an invariant closed set $Q \subset M_0$. Then*

(i) *the set $\Lambda = \{p \in Q : E_p^s + E_p^u = E_p^s \oplus E_p^u\}$ is invariant and closed,*

(ii) *the decomposition* $TR^n|_\Lambda = TM|_\Lambda + E^s|_\Lambda + E^u|_\Lambda$ *is continuous on* Λ,

(iii) *each set* $\Lambda_i = \{p \in \Lambda : dim E^s_p = i\}, (i = 0, 1, \cdots, k = codim M_0)$ *is invariant and closed,*

(iv) *every chain-recurrent orbit contained in* Q *is contained in* Λ,*i.e.,* $Q \cap CR \subset \Lambda$,

(v) *if there exists an orbit* ϕ *such that* $\phi \in \Lambda$, $\phi(t) \to \Lambda_{i_1}$ *for* $t \to -\infty$ *and* $\phi(t) \to \Lambda_{i_2}$ *for* $t \to +\infty$ *then* $i_2 > i_1$.

LEMMA 3[14]. *Let the transversality condition hold on an invariant closed set* $Q \subset M_0$. *For each* Λ_i *there exist* a *and* $\lambda > 0$ *such that for* $p \in \Lambda_i$,

$$|DU^s(t,p)| < a\exp(-\lambda t),$$
$$|DU^s(t,p)||(DU^0(t,p))^{-1}| < a\exp(-\lambda t),$$

for $t > 0$, *and*

$$|DU^u(t,p)| < a\exp(\lambda t),$$
$$|DU^u(t,p)||(DU^0(t,p))^{-1}| < a\exp(\lambda t),$$

for $t < 0$.

LEMMA 4[4]. *Let the transversality condition hold on an invariant closed set* $Q \subset M_0$. *If an orbit* γ *is contained in a sufficiently small neighborhood of* Λ_i *then the transversality condition holds on* γ, $E^s(p) + E^u(p) = E^s(p) \oplus E^u(p)$ *for* $p \in \gamma$, *and* $\dim E^s(p) = i$, *i.e.,* $\gamma \subset \Lambda_i$.

We obtained the following lemma from lemma 2[14].

LEMMA 5[14]. *If the neighborhood* V_i *of* Λ_i *is sufficiently small then there exists*

(i) *an invariant decomposition*

$$TR^n|_{V_i} = TM_0|_{V_i} \oplus E^s_i \oplus E^u_i,$$

where $E^s_i|_{\Lambda_i} = E^s|_{\Lambda_i}$ *and* $E^u_i|_{\Lambda_i} = E^u|_{\Lambda_i}$.

(ii) *constants a and $\lambda > 0$ such that*

$$|DU^s(t,p)| < a\exp(-\lambda t),$$
$$|DU^s(t,p)||(DU^0(t,p))^{-1}| < a\exp(-\lambda t),$$

for $t > 0$, $p \in V_i$, $U(t,p) \in V_i$,

$$|DU^u(t,p)| < a\exp(\lambda t),$$
$$|DU^u(t,p)||(DU^0(t,p))^{-1}| < a\exp(\lambda t),$$

for $t < 0$, $p \in V_i$, $U(t,p) \in V_i$.

COMPLETION OF THE PROOF OF THEOREM 4. The constructed family $\{V_i\}$ form a covering of M_0. Conditions (i)–(iv) of the pseudohyperbolic structure hold by construction. As for condition (iv), notice that for each initial neighborhood V_i this condition holds by Lemma 5. Since a new neighborhood is obtained as an extension of V_i at a finite time τ, the inequalities involved in condition (iv) also hold on this new neighborhood with the same constant λ, while a constant a may increase.

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