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ON SOME BASIC PROPERTIES OF THE INHOMOGENEOUS QUASI-BIRTH-AND-DEATH PROCESS

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ABSTRACT. The basic theory of the quasi-birth-and-death process is extended to a process which is inhomogeneous in levels. Several key results in the standard homogeneous theory hold in a more general context than that usually stated, in particular not requiring positive recurrence. These results are subsumed under our development. The treatment is entirely probabilistic.

1. Introduction

We consider a countable, irreducible, recurrent Markov chain whose one-step transition matrix P is of block-Jacobi form, that is, P may be partitioned as

$$P = \begin{bmatrix} B_0 & C_0 & 0 & 0 & \cdots \\ A_1 & B_1 & C_1 & 0 & \cdots \\ 0 & A_2 & B_2 & C_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}, \tag{1.1}$$

where the diagonal matrices B_i are finite and square but not necessarily of the same size.

A Markov chain prescribed by a transition matrix of form (1.1) for which

$$B_i \equiv B \quad \text{and} \quad C_i \equiv C \quad \text{for} \quad i \geq 1 \quad \text{and} \quad A_i \equiv A \quad \text{for} \quad i \geq 2 \tag{1.2}$$

is usually termed a quasi-birth-and-death process (QBD). The same term is used also for the corresponding continuous-time process for which

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the infinitesimal generator Q^* admits such a decomposition (1.1) with appropriate components satisfying (1.2). We take up the continuous-time version in Section 4.

Such processes have been extensively studied since the seminal work of Evans [6] and Wallace [19] and its significant development by Neuts (see [15] for a comprehensive account). The algorithmic solution, which possesses what is usually termed a matrix-geometric form, has made use of three matrices G, R and U, relations between which were derived by Hajek [7] and Latouche [12]. The most efficient solution to date has been provided recently by Latouche and Ramaswami [13], who also list a rich variety of interesting applications in the literature. Under a condition of level-crossing information completeness an alternative geometric prescription is given by Beuerman and Coyle [2] (see also [1], [4], [5]).

The solution of the case in which (1.2) fails to hold but the matrices in (1.1) are all $m \times m$ has been considered by the author in an earlier article [16] by rather different methods. The solution procedure of [16] arises out of an exploration of some basic relations between multi-term linear recurrence relations and extended continued fractions. See also Hanschke [8,9] for the use of the continued fraction method on the repeated call-attempt and related problems. A general class of these problems is also treated by Hanschke using generalized continued fractions in [10].

In this paper we extend the basic solution parameters G, R and U to cover the inhomogeneous quasi-birth-and-death process (IQBD). In the next section we show that in their place there are sequences $(R_i)_{i \geq 0}$, $(G_i)_{i \geq 1}$ and $(U_i)_{i \geq 1}$. These carry over the probabilistic interpretations of the QBD case and the arguments required often parallel closely those of that case. Where appropriate we give outlines only of the proofs involved.

In the sequel the partitioning

$$\mathbf{v} \;=\; (v_0, v_1, v_2, \cdots)$$

of an infinite row vector will be assumed conformable with (1.1) without further comment. Similarly we write

$$\mathbf{e} \;=\; (e_0, e_1, e_2, \cdots)^T,$$

where $\mathbf{e}$ is the infinite column vector each of whose entries is unity. As the order is always clear from the context we represent identity matrices generically by I and zero vectors by 0.

2. Basic Parameters

In accordance with customary usage, if the submatrix B_i occurs in rows and columns $r, r+1, \cdots, r+s$ of P, we refer to states $r, r+1, \cdots, r+s$ as being the *phases* $0, 1, \cdots, s$ in *level* i. The state $r+j$ is also specified in terms of its level and phase as (i, j). By (1.1), the process is then skip-free in levels.

2.1 THE RATE MATRICES R_i. In the customary notation (see Chung [3]), let ${}_iP^{(n)}_{i,j;i+k,\nu}$ be the taboo probability that, starting in (i, j), the chain visits state $(i+k, \nu)$ at time n without returning to level i in between. For $k \geq 1$, the sum $\sum_{n=1}^{\infty} {}_iP^{(n)}_{i,j;i+k,\nu}$ then represents the mean number of visits to $(i+k, \nu)$ before returning to level i, given the chain starts in (i, j). We define $R(i, i+k)$ to be the matrix (in general rectangular) with this quantity as its (j, ν) entry and set $R_i = R(i, i+1)$. We remark that by a well-known result of Markov chain theory (see Karlin [11, Chapter 5, Theorem 3.3]) the matrices $R(i, i+k)$ must be finite whenever the Markov chain is recurrent, although in the QBD context this property is usually stated only in the positive recurrent case.

LEMMA 1. *If the chain is recurrent, then for* $k \geq 1$

$$R(i, i+k) = R_i R_{i+1} \cdots R_{i+k-1}.$$

PROOF. Arguing as in Neuts [15, Lemma 1.2.2] with careful attention to levels, we derive

$$R(i, i+k) = R(i, i+k-1) R(i+k-1, i+k),$$

whence the result follows by induction. □

LEMMA 2. *If the chain is recurrent, then the matices R_i satisfy the recurrence relations*

$$R_i = C_i + R_i B_{i+1} + R_i R_{i+1} A_{i+2} \quad (i \geq 0).$$

PROOF. Application of the theorem of total probability for passage from level i to level $i+1$ in n steps with i a taboo level, followed by summation over n yields

$$R_i = C_i + R(i, i+1) B_{i+1} + R(i, i+2) A_{i+2} \quad (i \geq 0).$$

The claim now follows from Lemma 1. □

THEOREM 1. *If the chain is recurrent and* $\mathbf{x} = (x_0, x_1, \cdots)$ *a positive, left-invariant vector, then*

$$x_{i+1} = x_i R_i \quad (i \geq 0).$$

PROOF. By the theorem of total probability the n-step transition probability from state $(i+1, j)$ to itself can be decomposed for $n \geq 1$ as

$$P^{(n)}_{i+1,j;i+1,j} = {}_i P^{(n)}_{i+1,j;i+1,j} + \sum_{\nu} \sum_{r=0}^{n} P^{(r)}_{i+1,j;i,\nu} \cdot {}_i P^{(n-r)}_{i,\nu;i+1,j}. \tag{2.1}$$

The sum of the leading term on the right-hand side over n from 1 to m tends to a finite limit as $m \to \infty$ while the corresponding sum for the left-hand side diverges. Hence

$$\frac{\sum_{n=1}^{m} {}_i P^{(n)}_{i+1,j;i+1,j}}{\sum_{n=1}^{m} P^{(n)}_{i+1,j;i+1,j}} \to 0 \quad \text{as} m \to \infty.$$

Further, with the customary taboo notation

$$\frac{\sum_{n=1}^{m} P^{(n)}_{i+1,j;i,\nu}}{\sum_{n=1}^{m} P^{(n)}_{i+1,j;i+1,j}} \to \sum_{n=1}^{\infty} {}_{i+1,j} P^{(n)}_{i+1,j;i,\nu} \equiv {}_{i-1,j} P^{*}_{i+1,j;i,\nu}$$

(see Karlin [11, Chapter 5, Theorem 2.2]), while

$$\sum_{n=1}^{\infty} {}_iP^{(n)}_{i,\nu;i+1,j} = (R_i)_{\nu,j}.$$

Hence by a basic convolution result (see Karlin [11, Chapter 5, Lemma 2.2]) we deduce from (2.1) that

$$1 = \sum_{\nu} {}_{i+1,j}P^*_{i+1,j;i,\nu}(R_i)_{\nu,j}. \tag{2.2}$$

By Karlin [11, Chapter 5, Theorems 3.3 and 3.4] an irreducible, recurrent Markov chain has (up to a scale factor) a unique positive left-invariant vector

$$\mathbf{v} = (v_0, v_1, \cdots).$$

Normalized to $(v_{i+1})_j = 1$, its version in our context can be written as

$$(v_k)_\rho = {}_{i+1,j}P^*_{i+1,j;k,\rho} \quad ((k,\rho) \neq (i+1,j)).$$

Hence if $\mathbf{x}$ is an arbitrarily-scaled version of $\mathbf{v}$, then (2.2) reads

$$(x_{i+1})_j = \sum_{\nu}(x_i)_\nu(R_i)_{\nu,j},$$

and the theorem follows. □

2.2 THE FUNDAMENTAL MATRICES G_i. For $i \geq 1$, let G_i be the matrix whose (j,k) entry is the probability that, starting from state (i,j), the process eventually reaches level $i-1$ and enters it at phase k. The results of the next lemma are just the theorem of total probability applied to first passage probabilities from states in level i to level $i-1$ with and without taboo levels and do not require recurrence or even irreducibility of the chain.

LEMMA 3. *The fundamental matrices G_i satisfy the recursive relations*

$$G_i = A_i + B_i G_i + C_i G_{i+1} G_i \quad (i > 0).$$

If ${}_N G_i$ is the probability matrix corresponding to G_i when $N(> i)$ is also a taboo level, then

$$\begin{aligned} {}_N G_{N-1} &= A_{N-1} + B_{N-1} \cdot {}_N G_{N-1}, \\ {}_N G_i &= A_i + B_i \cdot {}_N G_i + C_i \cdot {}_N G_{i+1} \cdot {}_N G_i \quad (0 < i < N-1). \end{aligned}$$

RRMARK. By a simple probabilistic argument the sequence $({}_N G_i)_{N>i}$ is monotone non-decreasing entrywise with increasing N and ${}_N G_i \uparrow G_i$ as $N \to \infty$.

Lemma 3 provides a scheme for the numerical calculation of G_i. Choose N large, where $i \leq \ell < N-1$. Since ${}_N G_\ell$ is substochastic,

$$B_\ell \cdot {}_N G_\ell e_{\ell-1} \leq B_\ell e_\ell,$$

where inequality is interpreted entrywise. By the irreducibility of P the matrices B_{N-1} and $B_\ell + C_\ell \cdot {}_N G_\ell$ are strictly substochastic. Hence $I - B_{N-1}$ and $I - B_\ell - C_\ell \cdot {}_N G_{\ell+1}$ are invertible and we have the recursion

$$\begin{aligned} {}_N G_{N-1} &= (I - B_{N-1})^{-1} A_{N-1}, \\ {}_N G_\ell &= (I - B_\ell - C_\ell \cdot {}_N G_{\ell+1})^{-1} A_\ell \quad (i \leq \ell < N-1) \end{aligned}$$

for the successive determination of ${}_N G_{N-1}, {}_N G_{N-2}, \ldots, {}_N G_i$. The inverses will have nonnegative entries, so the recursion does not suffer numerical errors induced by subtractions.

For $\epsilon > 0$ suitably small, the fundamental matrices may be calculated with

$$\| e_{i-1} - {}_N G_i e_{i-1} \| < \epsilon$$

as a stopping criterion.

2.3 THE MATRICES U_i. For $i \geq 1$, let U_i be the matrix whose (j, ν) entry is the probability, with $i-1$ as a taboo level, that starting from (i, j) the chain eventually revisits level i and enters it through (i, ν). Clearly U_i is always strictly substochastic. We have the following result.

THEOREM 2. *If the chain is recurrent, then*

$$
\begin{aligned}
R_i &= C_i(I - U_{i+1})^{-1} \quad (i \geq 0), \\
G_i &= (I - U_i)^{-1} A_i \quad (i \geq 1), \\
U_i &= B_i + C_i G_{i+1} \ = \ B_i + R_i A_{i+1} \quad (i \geq 1), \\
R_i &= C_i(I - B_{i+1} - C_{i+1} G_{i+2})^{-1} \quad (i \geq 0), \\
G_i &= (I - B_i - R_i A_{i+1})^{-1} A_i \quad (i \geq 1), \\
U_i &= B_i + C_i(I - U_{i+1})^{-1} A_{i+1} \quad (i \geq 1).
\end{aligned}
$$

PROOF. The first three relations follow by the argument of Section 1 of Latouche [12], giving due care to the level labellings. The remaining three relations may then be deduced from them.

The last relation may also be deduced directly from a probabilistic argument similar to the others given by Latouche. □

Much as for the quantities G_i there is an iterative procedure for the determination of the U_i. Set ${}_N U_i$ for the probability matrix corresponding to U_i when level N is also taboo. Then we have

$$
\begin{aligned}
{}_N U_{N-1} &= B_{N-1}, \\
{}_N U_\ell &= B_\ell + C_\ell (I - {}_N U_{\ell+1})^{-1} A_{\ell+1} \quad (i \leq \ell < N-1).
\end{aligned}
$$

3. Equilibrium distribution of probability

The chain is recurrent if G_j is stochastic for some value of j, in which case G_i is stochastic for all i. In the case of positive recurrence we may proceed to calculate the ergodic probability vector

$$\pi = (\pi_0, \pi_1, \cdots).$$

The fundamental matrices may be calculated as noted in Section 2.2 and the rate matrices then determined in accordance with Theorem 2. A more numerically effective procedure is available using a technique of Latouche and Ramaswami [13]. This is outlined in Section 5.

THEOREM 3. *For a recurrent chain*

(a) *the matrix* $B_0 + R_0A_1$ *is stochastic;*

(b) *in a positive recurrent chain the vector* π_0 *is a positive, left-invariant eigenvector of* $B_0 + R_0A_1$ *normalized by*

$$\pi_0 \left[e_0 + \sum_{i=0}^{\infty} R_0R_1 \cdots R_i e_{i+1} \right] = 1. \tag{3.1}$$

PROOF. Since return to level 0 is certain for any initial state $(0, j)$, the theorem of total probability provides

$$\sum_\nu (B_0)_{j,\nu} + \sum_\nu \sum_\rho {}_0P^*_{0,j;1,\rho}(A_1)_{\rho,\nu} = 1 \quad \forall j,$$

on conditioning on the last state entered before level 0 is reached. That is,

$$(B_0 + R_0A_1)e_0 = e_0$$

and (a) is proved.

With positive recurrence, Theorem 1 gives

$$\pi_i = \pi_0 R_0 \cdots R_{i-1} \quad (i \geq 1).$$

Relation (3.1) is thus merely the standard normalization condition required for positive recurrence.

Further, the global balance conditions provide

$$\pi_0 = \pi_0 B_0 + \pi_1 A_1.$$

Substitution of $\pi_0 R_0$ for π_1 completes the demonstration of part (b). □

Since P is stochastic, we have

$$B_0e_0 + C_0e_1 = e_1, \tag{3.2}$$

$$A_ie_{i-1} + B_ie_i + C_ie_{i+1} = e_i \quad (i \geq 1). \tag{3.3}$$

Hence

$$(B_0 + R_0A_1)e_0 = (e_0 - C_0e_1) + R_0(e_1 - B_1e_1 - C_1e_2),$$

so that by Theorem 3

$$R_0 e_1 = (C_0 + R_0 B_1) e_1 + R_0 C_1 e_2.$$

Also from Lemma 2

$$R_0 e_1 = C_0 e_1 + R_0 B_1 e_1 + R_0 R_1 A_2 e_1.$$

A comparison of the last two equations provides

$$R_0 C_1 e_2 = R_0 R_1 A_2 e_1. \tag{3.4}$$

Similarly we have from (3.3) that for each $m \geq 1$

$$\begin{aligned}
(B_0 + R_0 A_1) e_0 &= (e_0 - C_0 e_1) + R_0(e_1 - B_1 e_1 - C_1 e_2) \\
&\quad + \sum_{i=1}^{m} R_0 R_1 \cdots R_i (e_{i+1} - A_{i+1} e_i - B_{i+1} e_{i+1} - C_{i+1} e_{i+2}) \\
&= \left(e_0 + \sum_{i=0}^{m} R_0 \cdots R_i e_{i+1} \right) - (C_0 + R_0 B_1 + R_0 R_1 A_2) e_1 \\
&\quad - \sum_{i=0}^{m-2} R_0 \cdots R_i (C_{i+1} + R_{i+1} B_{i+2} + R_{i+1} R_{i+2} A_{i+3}) e_{i+2} \\
&\quad - R_0 \cdots R_{m-1} C_m e_{m+1} - R_0 \cdots R_m (B_{m+1} e_{m+1} + C_{m+1} e_{m+2}) \\
&= \left(e_0 + \sum_{i=0}^{m} R_0 \cdots R_i e_{i+1} \right) - \sum_{i=0}^{m-1} R_0 \cdots R_i e_{i+1} \\
&\quad - R_0 \cdots R_{m-1} C_m e_{m+1} - R_0 \cdots R_m (B_{m+1} e_{m+1} + C_{m+1} e_{m+2}).
\end{aligned}$$

By Theorem 3 this simplifies to

$$R_0 \cdots R_m e_{m+1} = R_0 \cdots R_{m-1} (C_m e_{m+1} + R_m B_{m+1} e_{m+1} + R_m C_{m+1} e_{m+2}),$$

and Lemma 2 provides much as before that

$$R_0 \cdots R_m C_{m+1} e_{m+2} = R_0 \cdots R_{m+1} A_{m+2} e_{m+1}. \tag{3.5}$$

Taken together, (3.4) and (3.5) may be interpreted to give the following result.

COROLLARY. *Suppose the chain is recurrent and starts at any state in level 0. Then for each $i \geq 0$, the mean number of crossings from level $i+1$ to level i before the first return to level 0 equals the mean number of crossings in the reverse direction.*

This result has been noted for a QBD case by Neuts in connection with (1.2.20) in [15].

4. Inhomogeneous QBDs in continuous time

Suppose an irreducible, stable, conservative Markov process in continuous time has infinitesimal generator Q^* which may be ascribed a block partitioning

$$Q^* = \begin{bmatrix} B_0^* & C_0^* & 0 & 0 & \cdots \\ A_1^* & B_1^* & C_1^* & 0 & \cdots \\ 0 & A_2^* & B_2^* & C_2^* & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

The off-diagonal elements of Q^* are nonnegative and the diagonal elements strictly negative. We have also

$$B_0^* e_0 + C_0^* e_1 = 0, \tag{4.1}$$

$$A_i^* e_{i-1} + B_i^* e_i + C_i^* e_{i+1} = 0 \quad (i \geq 1). \tag{4.2}$$

The analogue to the process P' considered by Neuts [15, Section 1.7] need not be defined as $\sup_i \max_j (-B_i^*)_{j,j}$ may happen not to be finite. However we may work in terms of an analogue to the other auxiliary process used by Neuts.

Put $\Delta_i = -\text{diag } B_i^* \quad (i \geq 0)$ and

$$A_i = \Delta_i^{-1} A_i^*, \quad B_i = \Delta_i^{-1} B_i^* + I, \quad C_i = \Delta_i^{-1} C_i^*.$$

With these block entries, the matrix P given by (1.1) then represents the one-step transition matrix of the process considered immediately after its successive transitions. We verify readily that (4.1), (4.2) translate into (3.2) and (3.3).

Suppose that P is recurrent. If its rate matrices are R_i $(i \geq 0)$ and its positive, left-invariant vector $\mathbf{x}$, define

$$R_i^* = \Delta_i R_i \Delta_{i+1}^{-1} \quad \text{and} \quad x_i^* = x_i \Delta_i^{-1}.$$

Since P is the jump chain of Q^*, by a result of Pollett and Taylor [17] the recurrence of P implies the regularity of Q^*. As $\mathbf{x}$ is a positive, left-invariant vector of P, it satisfies the global balance equations

$$\begin{aligned} x_0 &= x_0 B_0 + x_1 A_1, \\ x_i &= x_{i-1} C_{i-1} + x_i B_i + x_{i+1} A_{i+1} \quad (i > 0), \end{aligned}$$

which translate into

$$\begin{aligned} x_0^* B_0^* + x_1^* A_1^* &= 0, \\ x_{i-1}^* C_{i-1}^* + x_i^* B_i^* + x_{i+1}^* A_{i+1}^* &= 0 \quad (i > 0). \end{aligned}$$

Hence $\mathbf{x}^*$ is a positive, left-invariant vector of Q^*. If

$$h \equiv \sum_{i=0}^{\infty} x_i^* e_i < \infty,$$

then by a result of Miller [14] the regularity of Q^* implies that the minimal process corresponding to Q^* is positive recurrent and has stationary distribution of probability $h^{-1}\mathbf{x}^*$.

We remark that the relations

$$\begin{aligned} (B_0 + r_0 A_1) e_0 &= e_0, \\ x_0 (B_0 + r_0 A_1) &= x_0, \\ x_{i+1} &= x_i R_i \end{aligned}$$

translate repectively into

$$\begin{aligned} (B_0^* + R_0^* A_1^*) e_0 &= 0, \\ x_0^* (B_0^* + R_0^* A_1^*) &= 0, \\ x_{i+1}^* &= x_i^* R_i^*. \end{aligned}$$

5. The efficient calculation of G_i and R_i

We have noted already how the quantities G_i may be calculated. Theorem 3 provides a means of deriving the other key quantities used to complete the solution of the IQBD *via* Theorem 2. Rather more efficient means exist. A substantial paper of Latouche and Ramaswami [13] investigates how this may be done for a QBD and cites earlier literature on this question. (See also Ramaswami [18] for an elegant extension to block-$M/G/1$ systems of an idea of Burke.) It is not our present intention to treat this question in depth for an IQBD. We shall, however, indicate in outline how the use of one of the seminal ideas of [13] leads to explicit formulae for the G_i and R_i, respectively.

Let P be an irreducible IQBD partitioned according to (1.1). For $i > 0$ and $n \geq 1$, let $A_i(2n)$ denote the matrix whose (j, ℓ) entry is the probability that, starting in state $(i2^{n-1}, j)$, the IQBD reaches level $(i-1)2^{n-1}$ before level $(i+1)2^{n-1}$ and enters it through phase ℓ. For $i \geq 0$ and $n \geq 1$, $C_i(2n)$ denotes the corresponding matrix with the roles of levels $(i-1)2^{n-1}$, $(i+1)2^{n-1}$ reversed. Then the matrix

$$P(2n) = \begin{bmatrix} 0 & C_0(2n) & 0 & 0 & \cdots \\ A_1(2n) & 0 & C_1(2n) & 0 & \cdots \\ 0 & A_2(2n) & 0 & C_2(2n) & \cdots \\ . & . & . & . & \cdots \end{bmatrix}$$

is evidently the transition matrix of an IQBD.

If we observe $P(2n)$ at its evenly-labelled levels, we derive a new IQBD

$$P(2n+1) = \begin{bmatrix} B_0(2n+1) & C_0(2n+1) & 0 & 0 & \cdots \\ A_1(2n+1) & B_1(2n+1) & C_1(2n+1) & 0 & \cdots \\ 0 & A_2(2n+1) & B_2(2n+1) & C_2(2n+1) & \cdots \\ . & . & . & . & \cdots \end{bmatrix}$$

with

$$\begin{aligned} A_i(2n+1) =& A_{2i}(2n)A_{2i-1}(2n) \quad (i \geq 1), \\ B_0(2n+1) =& C_0(2n)A_1(2n), \\ B_i(2n+1) =& C_{2i}(2n)A_{2i+1}(2n) + A_{2i}(2n)C_{2i-1}(2n) \quad (i \geq 1), \\ C_i(2n+1) =& C_{2i}(2n)C_{2i+1}(2n) \quad (i \geq 0). \end{aligned}$$

Further observing of $P(2n+1)$ at instants of change of level provides an IQBD which is seen easily to be $P(2n+2)$ and is given by

$$A_i(2n+2) = [I - B_i(2n+1)]^{-1} A_i(2n+1) \quad (i \geq 1),$$
$$C_i(2n+2) = [I - B_i(2n+1)]^{-1} C_i(2n+1) \quad (i \geq 0).$$

From the definition of $P(2)$, the original IQBD P can be represented in this sequence of processes as $P(1)$.

The fundamental matrices of these processes are simply related. For $i > 0$, $n \geq 1$ write $G_i(n)$ for the fundamental matrix of $P(n)$ corresponding to level i. Then by obvious probabilistic arguments

$$G_i(2n+1) = G_i(2n+2) \quad (i \geq 1), \tag{5.1}$$

$$G_{2i+1}(2n) = A_{2i+1}(2n) + C_{2i+1}(2n) G_{i+1}(2n+1) \quad (i \geq 0), \tag{5.2}$$

$$G_{2i}(2n) = A_{2i}(2n) + C_{2i}(2n) G_{2i+1}(2n) G_{2i}(2n) \quad (i > 0). \tag{5.3}$$

On setting $i = 0$ in (5.2), we have *via* (5.1) that

$$G_1(2n) = A_1(2n) + C_1(2n) G_1(2n+2).$$

A simple induction gives for each $m \geq 1$ that

$$G_1 \equiv G_1(1) \geq A_1(2) + \sum_{i=1}^{m} C_1(2) C_1(4) \cdots C_1(2i) A_1(2i+2).$$

The right-hand side represents ${}_kG_1$ for $k = 2^{m+1}$ and so converges to the left-hand side as $m \to \infty$. Hence we have

$$G_1 = A_1(2) + \sum_{i=1}^{\infty} C_1(2) C_1(4) \cdots C_1(2i) A_1(2i+1).$$

A similar argument provides

$$G_1(2n) = A_1(2n) + \sum_{i=n}^{\infty} C_1(2n) C_1(2n+2) \cdots C_1(2i) A_1(2i+2) \quad (n \geq 1). \tag{5.4}$$

We have further from (5.1)-(5.3) that

$$\begin{aligned} G_{2i+1}(2n) &= A_{2i+1}(2n) + C_{2i+1}(2n)G_{i+1}(2n+2) \quad (i \geq 0), \\ G_{2i}(2n) &= [I - C_{2i}(2n)G_{2i+1}(2n)]^{-1}A_{2i}(2n). \end{aligned}$$

These two relations may be used in a finite recursion to derive $G_i(= G_i(2))$ in terms of some $G_1(2n)$ $(n \geq 1)$, which may then by evaluated from (5.4). That is, for odd j, the first one of the above equations gives $G_{2i+1}(2n)$ in terms of $G_{i+1}(2n+2)$, and for even j, the second one gives $G_{2i}(2n)$ in terms of $G_{2i+1}(2n)$ and so by the first one in terms of $G_{i+1}(2n+2)$, which has a lower subscript. So in either case, for $j > 2$, we get $G_j(2n)$ in terms of $G_k(2n+2)$ with $k < j$.

We have also similar arguments to $R_i(n)$ for the recursive expression which is probabilistic, and they are given by the following equations.

$$R_i(2n+1) = G_i(2n+2) \quad (i \geq 0), \tag{5.5}$$

$$R_{2i+1}(2n) = C_{2i+1}(2n) + A_{2i+1}(2n)R_i(2n+1) \quad (i \geq 0), \tag{5.6}$$

$$R_{2i}(2n) = C_{2i}(2n) + A_{2i}(2n)R_{2i-1}(2n)R_{2i}(2n) \quad (i \geq 1). \tag{5.7}$$

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