Comm. Korean Math. Soc. **12** (1997), No. 1, pp. 157–163

ON THE SYMMETRIC SIERPINSKI GASKETS

HYUN-JONG SONG AND BYUNG-SIK KANG

ABSTRACT. Based on a n-regular polygon P_n, we show that $r_n = 1/(2\sum_{j=0}^{[(n-4)/4]+1}\cos 2j\pi/n)$ is the ratio of contractions $f_i (1 \le i \le n)$ at each vertex of P_n yielding a symmetric gasket G_n associated with the just-touching I.F.S. $\mathcal{G}_n = \{f_i | 1 \le i \le n\}$. Moreover we see that for any odd n, the ratio r_n is still valid for just-touching I.F.S. $\mathcal{H}_n = \{f_i \circ R | 1 \le i \le n\}$ yielding another symmetric gasket H_n where R is the π/n-rotation with respect to the center of P_n.

1. Introduction

Figures of Sierpinki gaskets, e.g., G_3, G_5 and G_6 are available in the literature[1, 2, 3], and one can easily see that the contraction ratio for those gaskets has a uniform description $1/2(1+\cos 2\pi/n)$. But one would immediately realize that this formula works only up to $n = 8$. Proper contraction ratio r_n for arbitrary G_n may be known to the experts in fractal geometry, but to authors' knowledge none has ever pointed it out in the literature. Furthermore, one of the authors accidently discovered that the ratio r_5 is still valid for the I.F.S. $\mathcal{H}_5 = \{f_i \circ R | 1 \le i \le 5\}$ to be just-touching. Hence in this paper we report the just-touching scale r_n for G_n and show that this works on the just-touching scale for H_n if n is odd.

Trying with the ratio r_n, one can see that H_n is totally disconnected if $n \equiv 2 (\mathrm{mod}\ 4)$ and overlapping if $n \equiv (\mathrm{mod}\ 4)$. In a sbusequent paper, we shall show how to determine just-touching scales for these exceptional cases.

Received June 2, 1996. Revised November 2, 1996.
1991 AMS Subject Classification: Primary 28A80, Secondary 58F08.
Key words and phrases: Fractal gasket, Scaling factor, Iterated function system.

2. Scaling factors for gaskets

Let V_i, $1 \leq i \leq n$ be counterclockwise indexed vertices of P_n and take a sub-regular n-gan Q_n of P_n sharing the common center O and having a vertex W_i, $1 \leq i \leq n$ corresponding to V_i respectively. Here we consider under what scale law suitable translated versions Q_n^i of Q_n in the direction of V_i are fitted into P_n, i.e., Q_n^i and P_n have only edges or verties in common. Indeed we see that the desire incident relationship among Q_n^i depends on the number $[\frac{n-4}{4}]$, where $[x]$ is the gaussian integer of x.

Due to the n-fold cyclic symmetry, it is sufficient to consider the incidence of Q_n^i and Q_n^2. It is clear that if Q_n^1 and Q_n^2 have either only a vertex in common or only on one edge in common then Q_n^i are fitted into P_n. Let W_i^j be the vertex of Q_n^i corresponding to W_j by the translation.

For simplicity of our argument, we relabel the vertices W_1^j of Q_n^1 by $X_1 = W_1^3$, $W_2 = W_1^4, \cdots$ the vertices W_2^j of Q_n^2 by $Y_1 = W_2^n, Y_2 = W_2^{n-1}, \cdots$ as the following figures.

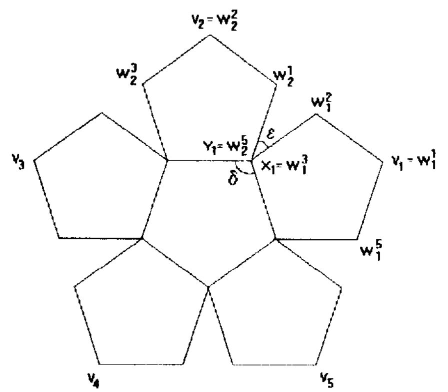

FIGURE 1; The sub pentagons $Q_5^i (1 \leq i \leq 5)$ fitted into a pentagon P_5

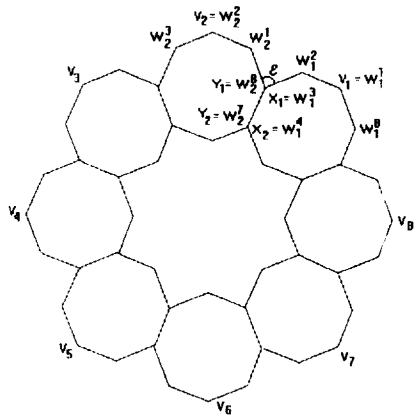

FIGURE 2; The sub octagons $Q_8^i (1 \leq i \leq 8)$ fitted into a octagon P_8

Then we have the following lemma.

LEMMA 1. *A necessary condition for Q_n^i to be fitted into P_n is that*

(1) *Q_n^1 and Q_n^2 have only a vertex $X_k (= Y_k)$ in common when $k = [(n-4)/4] + 1$ if $n \not\equiv 0 (mod\ 4)$*
(2) *Q_n^1 and Q_n^2 have only on edge $X_k X_{k+1} (= Y_k Y_{k+1})$ in common when $k = [(n-4)/4]$ if $n \equiv 0 (mod\ 4)$.*

PROOF. Suppose Q_n^1 and Q_n^2 share a vertex X_k in common as shown in figure 3.

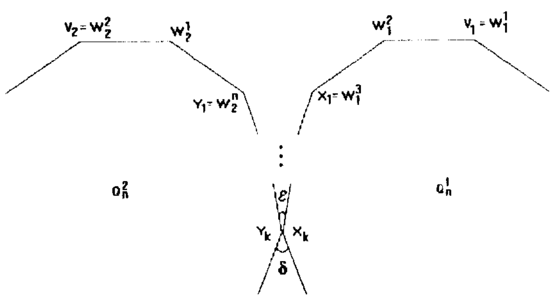

FIGURE 3; A portion of the sub-n-gons Q_n^1 and Q_n^2 fitted into a n-gon P_n

Then an angle

$$\varepsilon = \angle(Y_{k-1}X_kX_{k-1}) = \pi - \frac{4(k-1)\pi}{n} - \frac{4\pi}{n} = \frac{(n-4k)\pi}{n}$$

must be larger than 0 and an anlge

$$\delta = \angle(Y_{k+1}X_kX_{k+1}) = 2\pi - \frac{2(n-2)\pi}{n} - \epsilon = \frac{(4k-n+4)\pi}{n}$$

must be greater than or equal to 0. Hence we have

$$\frac{n-4}{4} \le k < \frac{n}{4}.$$

Therefore $k = [(n-4)/4] + 1$ if $n \not\equiv 0 (\mathrm{mod}\ 4)$ and $k = [(n-4)/4]$ if $n \equiv 0(\mathrm{mod}\ 4)$.

For $n \equiv 0(\mathrm{mod}\ 4)$, $\delta = 0$ and hence Q_n^1 and Q_n^2 have a common edge X_kX_{k+1}. □

From Lemma 1, we can determine the exact ratio r_n between an edge of Q_n and that of P_n under which Q_n^i are fitted into P_n. We omit details of its computation.

LEMMA 2. *Given ratio* $r_n = 1/2 \sum_{j=0}^{[(n-4)/4]+1} \cos 2j\pi/n$ *of contraction* f_i *at each vertex* V_i *of* $P_n, Q_n^i = f_i(P_n)$ *are fitted into* P_n *and so* $\{f_i | 1 \le i \le n\}$ *is the I.F.S. of the just touching gasket.*

REMARK. Note that the ratio r_n is still valid for $n = 3$ and $n = 4$. For instance the case of $n = 3$, the ratio is $\frac{1}{2}$ and so constructs the Sierpinski triangle. The case of $n = 4$, the last term in the summation of r_4 equals to 0 and the ratio is $\frac{1}{2}$ and so constructs the Peano curve.

Now we discuss construction of gaskets with the iterated function system of which π/n-rotation R gets involved. It seems rather difficult to investigate approximating set of the target image directly through I.F.S. $\mathcal{F}_1 = \{f_i \circ R | 1 \le i \le n\}$.

We replace $\mathcal{F}_1$ by an equivalent I.F.S. $\mathcal{F}_2 = \{R_i \circ f_i | 1 \le i \le n\}$, where R_i is π/n-rotation with respect to the centeı $O_i = f_i(O)$ of Q_n^1 with which we can easily figure out the behavior of appoximating sets.

LEMMA 3. *Under the same contraction ratio, $\mathcal{F}_1$ and $\mathcal{F}_2$ yield the same target image.*

Indeed approximating set of $\mathcal{F}_2$ at each level can be obtained by π/n-rotation of each sub-regular n-gon with respect to its center belong to the same leveled approximating set of $\mathcal{F}_0 = \{f_i | 1 \leq i \leq n\}$. In particular, if we take the ratio r_n of f_i in Lemma 2 and if n is odd, then it is easy to see that each approximating set of $\mathcal{F}_2$ consists of subregular n-gons sharing vertices one another.

Hence we have

THEOREM. *Under the contraction ratio r_n of f_i in Lemma 2, for any $n \geq 3$ (resp. for any odd $n \geq 3$) the iterated function system $\{f_i | 1 \leq i \leq n\}$ (resp. $\{f_i \circ R_i | 1 \leq i \leq n\}$) yields a just-touching target image G_n (resp. H_n) of which fractal dimension is $\frac{ln(1/n)}{ln r_n}$.*

One would be interested in the ratio of $\mathcal{F}_2$ (or $\mathcal{F}_1$) yielding the just-touching gasket H_n for even n. Trying with the ratio r_n, we can see that H_{4k+2} is totally disconnected whereas H_{4k} is overlapping. Note that the first level approximating set of H_{4k+2} is disconnected and that of H_{4k} is overlapping under the given contraction ratio of $\mathcal{F}_2$. Thus one requires another family of approximating sets heading towards the just touching target image. This will be treated in a subsequent paper. Here we only introduce the ratio $s_4 = (-1+\sqrt{5})/2\sqrt{2} = .437016\cdots$ and $s_6 = (-1+\sqrt{5})/2\sqrt{3} = .3568221\cdots$ of H_4 and H_6 together with target images for the interests of readers.

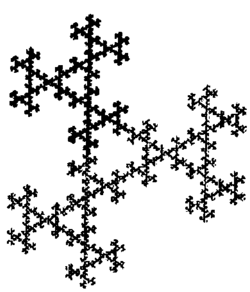

FIGURE 4; H_3 (ratio = 1/3)

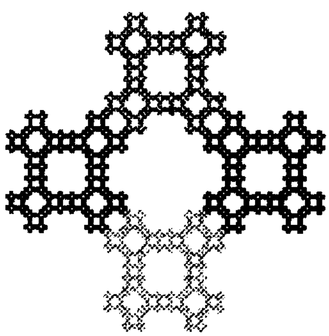

FIGURE 5; H_4 (ratio = 0.437016...)

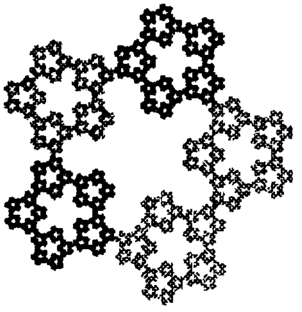

FIGURE 6; H_5 (ratio = 0.3819661...)

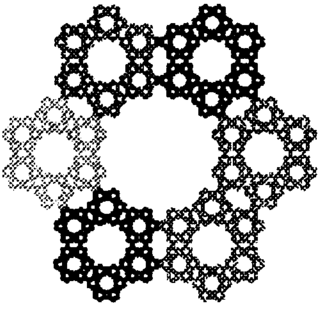

FIGURE 7; H_6 (ratio = 0.3568221...)

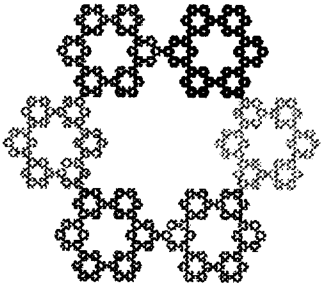

FIGURE 8; G_6 (ratio =1/3)

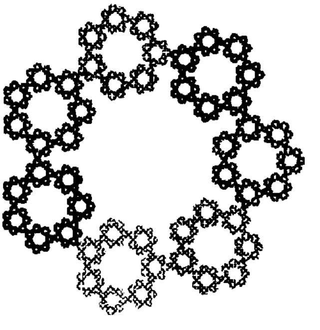

FIGURE 9; G_7 (ratio = 0.3079785...)

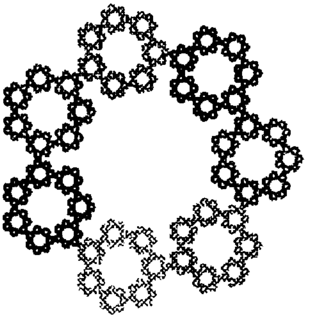

FIGURE 10; G_8 (ratio = 0.2928933...)

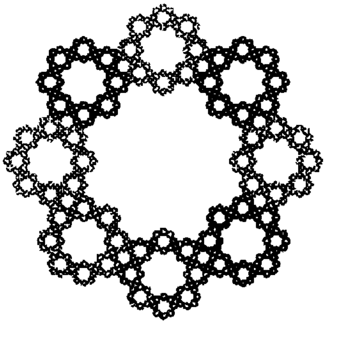

FIGURE 11; G_9 (ratio = 0.2577728...)

References

1. M. Barnsley, *Fractals Everywhere 2nd edition*, Academic Press, New York, 1993.
2. G. Edgar, *Measure, Topology, and Fractal Geometry*, Springer Verlag, New York, 1990.
3. B. B. Mandelbrot, *The fractal geometry of nature*, W. H. Freeman and Company, New York, 1983.

Hyun-Jong Song
Department of Applied Mathematics
Pukyung National University
Pusan 608-737, Korea
e-mail: hjsong@dolphin.pknu.ac.rk

Byung-Sik Kang
Department of Computational Mathematics
Kosin Univesity
Pusan 606-701, Korea
e-mail: bskang@sdg.kosin.ac.kr