SANDWICH SEMIGROUPS OF CLOSED FUNCTIONS

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ABSTRACT. The purpose of this paper is to find all K_1-minimal ideals of a sandwich semigroup of closed functions.

1. Introduction

It is assumed that all topological spaces discussed in this paper are T_1 spaces and have more than one point. Let X and U be topological spaces and let Y and V be nonempty subspaces of X and U, respectively. Fix a closed surjective function α from U onto X with $\alpha(V) = Y$. Denote by $\Gamma_\alpha(X, Y; U, V)$ the semigroup of all closed functions from X into U which also map Y into V where the multiplication fg of two such functions is $fg = f \circ \alpha \circ g$. It will be referred to as a *restrictive sandwich semigroup (of closed functions) with sandwich function* α.

K. D. Magill [4] investigated certain ideals of a restrictive semigroup of closed functions of a topological space and obtained all K_1-minimal ideals of the semigroup. R. Hofer [2,3] studied a restrictive semigroup of continuous functions of a certain topological space.

The purpose of this paper is to find all K_1-minimal ideals of a restrictive sandwich semigroup of closed functions of a topological space.

2. K_0-minimal ideals in $\Gamma_\alpha(X, Y; U, V)$

Those functions mapping X into U whose ranges consist of a finite number of points are closed functions and they play a very important role in the subsequent discussion. Suppose that f is such a function

and suppose, for example, that its range consists of two points u and v, where u and v are points of U. We find it useful to denote the function f by

$$\langle A, u; B, v \rangle,$$

where it is to be understood that A is the subset of X whose points are mapped by f into u, B is the subset whose points are mapped into v. In general, then, a function with finite range is given by "listing" the points of its range together with those subsets whose points are mapped into them. However, certain of these functions appear with enough frequency so that it is convenient to reserve special notation for them. For example, the functions of the form $\langle X_1, v; X_2, u \rangle$ play a particular important role, and consequently, appear open. Thus it is convenient to let

$$\langle X_1, v; X_2, w \rangle = X_{vw}.$$

We also let

$$\langle X_1, u_1; X_2, u_2; \cdots; X_n, u_n \rangle = X_u^n.$$

Note that X_u^n is an element of $\Gamma_\alpha(X, Y; U, V)$ if and only if for some subset $\{i_k\}_{k=1}^m$ of $\{i\}_{i=1}^n$, we have $\{u_{i_k}\}_{k=1}^m \subset V$ and $Y \subset \cup_{k=1}^m \{X_{i_k}\}$.

DEFINITION 2.1. Let X and U denote topological spaces and let Y and V be nonempty subsets of X and U, respectively. Let α denote a closed surjective function from U onto X with $\alpha(V) = Y$. By a *restrictive sandwich semigroup of closed functions with sandwich function* α we mean the semigroup $\Gamma_\alpha(X, Y; U, V)$ of all closed functions from X into U which also map Y into V where the multiplication fg of two closed functions is $fg = f \circ \alpha \circ g$.

LEMMA 2.2. *A restrictive sandwich semigroup* $\Gamma_\alpha(X, Y; U, V)$ *has a zero element if and only if* V *consists of one point.*

PROOF. It is easy.

DEFINITION 2.3. [1,4] Let I be an ideal of a semigroup S. An ideal J of S is said to be *I-minimal* if (i) J properly contains I, and (ii) if K is an ideal such that $I \subset K \subset J$, then either $K = I$ or $K = J$. If S contains a zero element 0 and $I = \{0\}$, we say that the ideal J is *0-minimal*.

Suppose that V consists of one point v. Since $\alpha(V) = Y$, we have $Y = \{y\}$ for some $y \in X$. It follows from Lemma 2.2 that $\Gamma_\alpha(X, Y; U, V)$ has a zero element 0 and hence $0 = \langle X, v \rangle$.

Now let us define

$$I^\alpha = \{X_{vu} = \langle X_1, v; X_2, u \rangle \mid y \in X_1, u \in U\}.$$

For each $u \in \alpha^{-1}(y) \setminus \{v\}$, let

$$I_u^\alpha = \{0\} \cup \{X_{vu} = \langle X_1, v; X_2, u \rangle \mid y \in X_1\}.$$

THEOREM 2.4. *Suppose that both X and U have more than one point. Then*

(1) *If $\alpha^{-1}(y) = \{v\}$, then I^α is the 0-minimal ideal which is contained in every non-zero ideal containing 0. In this case, I^α is not only 0-minimal, but is the only 0-minimal ideal of $\Gamma_\alpha(X, \{y\}; U, \{v\})$.*

(2) *If $\alpha^{-1}(y) \setminus \{v\} \neq \emptyset$, then for each $u \in \alpha^{-1}(y) \setminus \{v\}$, I_u^α is a 0-minimal ideal of $\Gamma_\alpha(X, \{y\}; U, \{v\})$.*

PROOF. (1) It is evident that if both X and U have more than one point, then I^α consists of more than one element. Suppose that $\alpha^{-1}(y) = \{v\}$. For every $f \in \Gamma_\alpha(X, \{y\}; U, \{v\})$ and any $X_{vu} \in I^\alpha \setminus \{0\}$,

$$fX_{vu} = \begin{cases} 0 & \text{if } f(\alpha(u)) = v, \\ \langle X_1, v; X_2, f(\alpha(u)) \rangle & \text{if } f(\alpha(u)) \neq v. \end{cases}$$

On the other hand, if $f^{-1}(\alpha^{-1}(X_2)) = \emptyset$, then $X_{vu}f = 0$ and if $f^{-1}(\alpha^{-1}(X_2)) \neq \emptyset$, then $X_{vu}f = \langle f^{-1}(\alpha^{-1}(X_1)), v; f^{-1}(\alpha^{-1}(X_2)), u \rangle$, which is an element of I^α. It follows that I^α is an ideal of $\Gamma_\alpha(X, \{y\}; U, \{v\})$. Now suppose that J is any non-zero ideal of $\Gamma_\alpha(X, \{y\}; U, \{v\})$. Then there exists a non-zero element f of J. It follows that $f(x) \neq v$ for some $x \in X \setminus \{y\}$. Since $\alpha^{-1}(y) = \{v\}$, we have $\alpha(f(x)) \neq y$. Take any element t of U such that $\alpha(t) = x$ because α is surjective. Let X_{vu} be an arbitrary element of I^α. Then

$$X_{vu} = \langle \{y\}, v; X \setminus \{y\}, u \rangle \circ \alpha \circ f \circ \alpha \circ X_{vt}$$
$$= \langle \{y\}, v; X \setminus \{y\}, u \rangle f X_{vt},$$

and hence $X_{vu} \in J$. Thus, $I^\alpha \subset J$ and hence the desired conclusion follows.

(2) Suppose now that $\alpha^{-1}(y) \setminus \{v\} \neq \emptyset$. For each $u \in \alpha^{-1}(y) \setminus \{v\}$, we can prove easily that I_u^α is an ideal of $\Gamma_\alpha(X, \{y\}; U, \{v\})$. Finally, we prove that the ideal I_u^α is 0-minimal. Now consider any ideal J containing zero element 0 properly and is contained the ideal I_u^α. Let f be any element of J. Then $f = X_{vu}^*$ for some disjoint nonempty subsets X_1^* and X_2^* of X with $y \in X_1^*$ and $X = X_1^* \cup X_2^*$. Let X_{vu} be any element of the ideal I_u^α. Choose an element x_1 in X_2^*. Then there exists an element p of $U \setminus \alpha^{-1}(y)$ such that $\alpha(p) = x_1$. Hence $X_{vu} = f \circ \alpha \circ \langle X_1, v; X_2, p \rangle$, which is an element of J. Thus, I_u^α is a subset of J and hence, I_u^α is a 0-minimal ideal. This completes the proof.

LEMMA 2.5. *Suppose that $\alpha^{-1}(y) = \{v\}$ and let $X_{vu} = \langle X_1, v; X_2, u \rangle$ $\in I^\alpha$. Then $X_1 = \{y\}$ if and only if for every non-zero element f of $\Gamma_\alpha(X, \{y\}; U, \{v\})$, we have $X_{vu}f \neq 0$.*

PROOF. Suppose that $X_1 = \{y\}$ and $f \neq 0$. Then $f(x) \neq v$ for some $x \in X$ and $f^{-1}(\alpha^{-1}(X_2)) = f^{-1}(U \setminus \{v\}) \neq \emptyset$. It follows that

$$X_{vu}f = X_{vu} \circ \alpha \circ f = \langle f^{-1}(\alpha^{-1}(X_1)), v; f^{-1}(\alpha^{-1}(X_2)), u \rangle \neq 0.$$

Now suppose that $X_1 \neq \{y\}$. Then there exists an element x' in X_1 with $x' \neq y$. Let v' be an element of U such that $\alpha(v') = x'$. Then $\langle \{y\}, v; X \setminus \{y\}, v' \rangle \neq \emptyset$ but $X_{vu}\langle \{y\}, v; X \setminus \{y\}, v' \rangle = 0$, which is a contradiction. Thus, we have $X_1 = \{y\}$.

It is well-known that if a semigroup has a minimal ideal, then that ideal is unique. We call it the *kernel* of the semigroup. The next result concerns the kernel of $\Gamma_\alpha(X, Y; U, V)$. Its proof is straightforward and will not be given.

LEMMA 2.6. $K_0 = \{\langle X, v \rangle \mid v \in V\}$ *is the kernel of* $\Gamma_\alpha(X, Y; U, V)$.

Suppose that $Y \neq X$ and $V \neq U$. Let

$$V_\alpha^2 = \{(v_1, v_2) \in V \times V \mid v_1 \neq v_2 \text{ and } \alpha(v_1) = \alpha(v_2)\}.$$

Assume that V_α^2 is a non-empty set. For each $(v_1, v_2) \in V_\alpha^2$, let

$$I_j(v_1, v_2) = \{\langle X_1, v_1; X_2, v_2\rangle \mid Y \subset X_j\},$$
$$K_0^j(v_1, v_2) = K_0 \cup I_j(v_1, v_2), \text{ for } j = 1, 2.$$

Note that if $V_\alpha^2 \neq \emptyset$, then we have $K_0^1(v_1, v_2) = K_0^2(v_2, v_1)$ for each $(v_1, v_2) \in V_\alpha^2$. Suppose that $V_\alpha^2 = \emptyset$ and $\alpha^{-1}(Y) \setminus V \neq \emptyset$, and let $u \in \alpha^{-1}(Y) \setminus V$. Then there exists a unique element v of V such that $\alpha(u) = \alpha(v)$. Define

$$K_1(v, u) = K_0 \cup \{\langle X_1, v; X_2, u\rangle \mid Y \subset X_1\}.$$

On the other hand, let

$$K_1 = K_0 \cup \{X_v^2 = \langle X_1, v_1; X_2, v_2\rangle \mid v_1, v_2 \in V \text{ and } Y \subset X_1\}.$$

We are now in a position to state and prove the main result of this section.

THEOREM 2.7. *Suppose that Y is a nonempty proper subset of X and V is a nonempty proper subset of U which has more than one point. Then we have the following :*

(1) *If $V_\alpha^2 \neq \emptyset$, then $K_0^1(v_1, v_2)$ and $K_0^2(v_1, v_2)$ are K_0-minimal ideals for each $(v_1, v_2) \in V_\alpha^2$.*

(2) *If $V_\alpha^2 = \emptyset$ and $\alpha^{-1}(Y) = V$, then K_1 is the only K_0-minimal ideal of $\Gamma_\alpha(X, Y; U, V)$.*

(3) *If $V_\alpha^2 = \emptyset$ and $\alpha^{-1}(Y) \setminus V \neq \emptyset$, then $K_1(u, v)$ is a K_0-minimal ideal of $\Gamma_\alpha(X, Y; U, V)$ for each $u \in \alpha^{-1}(Y) \setminus V$ and $v \in V$ with $\alpha(u) = \alpha(v)$.*

PROOF. (1) We will prove that $K_0^1(v_1, v_2)$ is a K_0-minimal ideal. The proof of the second case is similar to that of the first. Suppose that $V_\alpha^2 \neq \emptyset$ and let (v_1, v_2) be an arbitrary element of V_α^2. Let $f \in \Gamma_\alpha(X, Y; U, V)$ and let $X_v^2 \in K_0^1(v_1, v_2) \setminus K_0$. Then $X_v^2 = \langle X_1, v_1; X_2, v_2\rangle, Y \subset X_1$ and $\alpha(v_1) = \alpha(v_2)$. It follows that

$$fX_v^2 = f \circ \alpha \circ \langle X_1, v_1; X_2, v_2\rangle$$
$$= \langle X_1, f(\alpha(v_1)); X_2, f(\alpha(v_2))\rangle$$
$$= \langle X, f(\alpha(v_1))\rangle.$$

Moreover, we have

$$X_v^2 f = \begin{cases} A_v^2 & \text{if } A_2 \neq \emptyset, \\ \langle X, v_1 \rangle & \text{if } A_2 = \emptyset, \end{cases}$$

where $A_i = f^{-1}(\alpha - 1(X_i))$, $i = 1, 2$. Thus, $K_0^1(v_1, v_2)$ is an ideal of $\Gamma_\alpha(X, Y; U, V)$. One can easily show that $K_0^1(v_1, v_2)$ is K_0-minimal.

(2) Suppose that $V_\alpha^2 = \emptyset$ and $\alpha^{-1}(Y) = V$. It is clear that K_1 is an ideal of $\Gamma_\alpha(X, Y; U, V)$ which properly contains K_0. Now suppose that J is any ideal which properly contains K_0. Then there exists an element f in $J \setminus K_0$ such that $f(X)$ contains more than one point. Fix any $x_0 \in Y$. Then $f(x) \neq f(x_0)$ for some $x \in X$ and $\alpha(f(x)) \neq \alpha(f(x_0))$. Since $\alpha^{-1}(Y) = V$, there exists a unique element v_0 of V such that $\alpha(v_0) = x_0$. Take any $u \in U$ with $\alpha(u) = x$. Let X_v^2 be an arbitrary element of $K_1 \setminus K_0$ and let $T = X \setminus \{\alpha(f(x_0))\}$. Then

$$\begin{aligned} X_v^2 &= \langle \{\alpha(f(x_0))\}, v_1; T, v_2 \rangle \circ \alpha \circ f \circ \alpha \circ \langle X_1, v_0; X_2, v \rangle \\ &= \langle \{\alpha(f(x_0))\}, v_1; T, v_2 \rangle f \langle X_1, v_0; X_2, v \rangle. \end{aligned}$$

Since J is an ideal, we have $X_v^2 \in J$. Hence K_1 is contained in the ideal J and so, K_1 is K_0-minimal.

(3) The proof is similar to that of (1).

3. K_1-minimal ideals in $\Gamma_\alpha(X, Y; U, V)$

LEMMA 3.1. *Supose that $Y \neq X$ and $V \neq U$, Y and V have more than one point and that $V_\alpha^2 \neq \emptyset$. For each $(v_1, v_2) \in V_\alpha^2$, let $I^*(v_1, v_2)$ be the set of all functions whose range are v_1 and v_2. Let*

$$K_1^j(v_1, v_2) = K_0^j(v_1, v_2) \cup I^*(v_1, v_2), \text{ for } j = 1, 2$$

and

$$K_2^*(v_1, v_2) = K_1 \cup I^*(v_1, v_2).$$

Then $K_1^j(v_1, v_2)$ is a $K_0^j(v_1, v_2)$-minimal ideal of $\Gamma_\alpha(X, Y; U, V), j = 1, 2$, and $K_2^*(v_1, v_2)$ is K_1-minimal.

PROOF. It is straightforward to prove that $K_1^1(v_1, v_2)$ is an ideal. To show that $K_1^1(v_1, v_2)$ is a $K_0^1(v_1, v_2)$-minimal ideal, let J be any ideal which properly contains $K_0^1(v_1, v_2)$ and is contained in $K_1^1(v_1, v_2)$. Choose any element f of $J \setminus K_0^1(v_1, v_2)$. Then we have $f = X_v^2$ with both $X_1 \cap Y$ and $X_2 \cap Y$ are nonempty sets. Let g be any element of $K_1^1(v_1, v_2) \setminus K_0^1(v_1, v_2)$. Then $g = \langle G_1, v_1; G_2, v_2 \rangle$ with both $G_1 \cap Y$ and $G_2 \cap Y$ are nonempty sets. Suppose that $\alpha(v_1) = \alpha(v_2) \in X_1 \cap Y$. Since $f(X_2) = \{v_2\}$, we have $f(x_2) = v_2$ for some $x_2 \in X_2 \cap Y$. Next we choose an element v_2^* of V with $\alpha(v_2^*) = x_2$, since α is surjective. Thus, we have $g = f \circ \alpha \circ \langle G_1, v_1; G_2, v_2^* \rangle = f \langle G_1, v_1; G_2, v_2^* \rangle$, which is an element of J. It follows that $J = K_1^1(v_1, v_2)$. On the other hand, if $\alpha(v_1) = \alpha(v_2) \in X_2 \cap Y$, then we have the same results. Therefore, $K_1^1(v_1, v_2)$ is a $K_0^1(v_1, v_2)$-minimal ideal. Similarly, one can prove that $K_1^2(v_1, v_2)$ is a $K_0^2(v_1, v_2)$-minimal ideal and $K_2^*(v_1, v_2)$ is a K_1-minimal ideal.

LEMMA 3.2. *Let v and u be elements of V and $U \setminus V$, respectively. Let*

$$K_2(v, u) = K_1 \cup \{\langle X_1, v; X_2, u \rangle \mid Y \subset X_1\} \text{ if } \alpha(v) = \alpha(u)$$

and

$$K_2^{\sharp}(v, u) = K_1 \cup \{\langle X_1, v; X_2, u \rangle \mid Y \subset X_1\} \text{ if } \alpha(v) \neq \alpha(u) \text{ and } \alpha(u) \in Y.$$

Then they are K_1-minimal ideals.

LEMMA 3.3. *Suppose that Y is a subset of X and V is a subset of U which has more than one point. Then we have the following :*

(1) *If $\alpha^{-1}(Y) = V$, then $K_V = K_1 \cup \{X_{vu} \mid Y \subset X_1, v \in V \text{ and } u \in U\}$ is a K_1-minimal ideal.*

(2) *If $V_\alpha^2 = \emptyset$, then $K_2 = K_1 \cup \{X_v^2 \mid v_1, v_2 \in V\}$ is a K_1-minimal ideal.*

(3) *If $V_\alpha^2 = \emptyset$ and if V has more than two points, then $K_{2,3} = K_1 \cup \{X_v^3 \mid Y \subset X_1, v_1, v_2, v_3 \in V\}$ is a K_1-minimal ideal.*

LEMMA 3.4. *Suppose that $V_\alpha^2 \neq \emptyset$. Fix $(v_1, v_2) \in V_\alpha^2$. Suppose also that V has more than two points. Let $v_3 \in V$ be such that $v_3 \neq v_i$, for $i = 1, 2$. Now define $K_{2,3}^i(v_1, v_2, v_3)$ as follows : for each $i = 1, 2, 3$,*

$$K_{2,3}^i(v_1, v_2, v_3) = K_1 \cup \{ \langle X_1, v_1, X_2, v_2; X_3, v_3 \rangle \mid Y \subset X_i \}.$$

Then they are K_1-minimal ideals.

REMARK. In fact, we have

$$K_{2,3}^1(v_1, v_2, v_3) = K_{2,3}^2(v_2, v_1, v_3) \text{ and } K_{2,3}^3(v_1, v_2, v_3) = K_{2,3}^3(v_2, v_1, v_3).$$

THEOREM 3.5. *Suppose that Y is a nonempty proper subset of X and that V is a subset of U consisting of two points. Then we have the following :*

(1) *If $V_\alpha^2 \neq \emptyset$, then K_V is the only K_1-minimal ideal if $\alpha^{-1}(Y) = V$, and $K_2(v, u)$ is the type of K_1-minimal ideal if $\alpha^{-1}(Y) \setminus V \neq \emptyset$, where $(v, u) \in V \times (U \setminus V)$ with $\alpha(v) = \alpha(u)$.*

(2) *If $V_\alpha^2 = \emptyset$, then K_2 and K_V are the only K_1-minimal ideals if $\alpha^{-1}(Y) = V$, and K_2, $K_2(v, u)$ and $K_2^\sharp(v, u)$ are all types of K_1-minimal ideals if $\alpha^{-1}(Y) \setminus V \neq \emptyset$.*

PROOF. (1) Suppose that $V = \{v_1, v_2\}$ and $V_\alpha^2 \neq \emptyset$. Let J be any K_1-minimal ideal. Since $\alpha(V) = Y$, we have $Y = \{y\}$ for some $y \in X$. Since J properly contains K_1, there exists $f \in J$ such that $f \notin K_1$. If $f(X) \subset V$, then $f(X) = \{v_1, v_2\}$. Since $Y = \{y\}$, $f \in K_1$, which is a contradiction. Thus, $f(X) \not\subset V$. Then $f(x) \notin V$ for some $x \in X \setminus \{y\}$. Without loss of generality, we may assume that $f(y) = v_1$.

Suppose that $\alpha^{-1}(Y) = V$. Let $T = X \setminus \{\alpha(f(x))\}$. Then

$$\langle T, v_1; \{\alpha(f(x))\}, f(x) \rangle \circ \alpha \circ f = \langle X \setminus A, v_1; A, f(x) \rangle,$$

which is in $K_V \setminus K_1$, where $A = f^{-1}(\alpha^{-1}(\alpha(f(x))))$. Hence $K_V \cap J$ properly contains K_1. It follows that $J = K_V$.

Now suppose that $\alpha^{-1}(Y) \setminus V \neq \emptyset$. Then there exists $u \in \alpha^{-1}(Y) \setminus V$ such that $\alpha(u) = \alpha(v_1)$. If $f(x) \notin \alpha^{-1}(Y) \setminus V$, then $\alpha(f(x)) \neq \alpha(v_1) = y$. It follows that

$$\langle X \setminus \{\alpha(f(x))\}, v_1; \{\alpha(f(x))\}, u \rangle \circ \alpha \circ f = \langle X \setminus A, v_1; A, u \rangle,$$

which is an element of $(K_2(v_1, u) \cap J) \setminus K_1$, where $A = f^{-1}(\alpha^{-1}(\alpha(f(x))))$. Hence $K_2(v_1, u) \cap J$ properly contains K_1, which is a contradiction. Thus, $f(x) \in \alpha^{-1}(Y) \setminus V$. Consider an ideal $K_2(v_1, f(x))$. Choose any element u in U with $\alpha(u) = x$. Let X_1 be a proper subset of X containing y and let $X_2 = X \setminus X_1$. Let g be the function from X into U defined by $g = \langle X_1, v_1; X_2, u \rangle$. Then, $g \circ \alpha \circ f = \langle X_1, v_1; X_2, f(x) \rangle$, which is an element of $(K_2(v_1, f(x)) \cap J) \setminus K_1$. Thus, $J = K_2(v_1, f(x))$.

(2) Suppose that $V = \{v_1, v_2\}$ and $V_\alpha^2 = \emptyset$. Let J be any K_1-minimal ideal. Since $\alpha(V) = Y$, we have $Y = \{y_1, y_2\}$ for some $y_1, y_2 \in X$. Let $f \in J$ be such that $f \notin K_1$. Then we have four cases :

(a) $\alpha^{-1}(Y) = V$ and $f(X) \subset V$,
(b) $\alpha^{-1}(Y) = V$ and $f(X) \not\subset V$,
(c) $\alpha^{-1}(Y) \setminus V \neq \emptyset$ and $f(X) \subset V$
(d) $\alpha^{-1}(Y) \setminus V \neq \emptyset$ and $f(X) \not\subset V$.

Suppose that case (a) holds. Since $f \notin K_0$, $f(X) = \{v_1, v_2\}$. Then $f \in K_2 \setminus K_1$ and hence, $K_2 \cap J$ properly contains K_1. It follows that $J = K_2$.

Suppose that case (b) holds. Then $f(x) \notin V$ for some $x \in X \setminus Y$. Then,

$$\langle X \setminus \{\alpha(f(x))\}, v_1; \{\alpha(f(x))\}, f(x) \rangle \circ \alpha \circ f = \langle X \setminus A, v_1; A, f(x) \rangle,$$

which is an element of $(K_V \cap J) \setminus K_1$, where $A = f^{-1}(\alpha^{-1}(\alpha(f(x))))$. This implies that $J = K_V$.

Suppose that case (d) holds. Then $f(x) \notin V$ for some $x \in X \setminus Y$. Since $\alpha^{-1}(Y) \setminus V \neq \emptyset$, there exists $u \in \alpha^{-1}(Y) \setminus V$ such that $\alpha(u) = \alpha(v)$ for some $v \in V$, say $\alpha(u) = \alpha(v_1)$. We must have $\alpha(f(x)) \in Y$. We can show that $f(Y) = \{v\}$ for some $v \in V$. If $\alpha(v) = \alpha(f(x))$, then we have $J = K_2(v, f(x))$, and if $\alpha(v) \neq \alpha(f(x))$, then we must have $J = K_2^\sharp(v, f(x))$. This completes the proof.

THEOREM 3.6. *Suppose that Y is a nonempty proper subset of X and that V is a subset of U consisting of more than two points. Then we have the following :*

(1) *If $V_\alpha^2 = \emptyset$, then K_2, $K_{2,3}$ and K_V are all types of K_1-minimal ideals if $\alpha^{-1}(Y) = V$, and K_2, $K_2(v, u)$ and $K_2^\sharp(v, u)$ are all types of K_1-minimal ideals if $\alpha^{-1}(Y) \setminus V \neq \emptyset$.*

(2) *If $V_\alpha^2 \neq \emptyset$, then K_V, $K_2^*(v_1, v_2)$, and $K_{2,3}^i(v_1, v_2, v_3)$ are all types of K_1-minimal ideals if $\alpha^{-1}(Y) = V$, and $K_2(v, u)$, $K_2^*(v_1, v_2)$, $K_2^\sharp(v, u)$ and $K_{2,3}^i(v_1, v_2, v_3)$ are all types of K_1-minimal ideals if $\alpha^{-1}(Y) \setminus V \neq \emptyset$.*

PROOF. The proof is similar to that of Theorem 3.5.

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