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GENERALIZATIONS OF LIMIT THEOREMS BY A.V. SKOROKHOD

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ABSTRACT. In order to study the limits of sequences appearing in, for example, stochastic process, A.V. Skorokhod has defined new function space topologies. We compare these topologies with the topology of compact convergence, the topology of pointwise convergence and others.

1. Introduction

Continuous convergence in function spaces has been much studied in the literature and the structure of function spaces much investigated. But the properties of spaces of discontinuous functions are not well-known. A.V. Skorokhod has defined new convergent topologies[1]. We investigate the relationship between them. And we show that among these new topologies, the M_1-convergent topology coincides with the topology of pointwise convergence (Theorem 3.3), the almost convergent topology is coarser than the M_2-convergent topology (Theorem 3.5), and the topology of compact convergence coincides with the graph topology (Proposition 3.2).

2. New topologies defined by A.V. Skorokhod

The four topologies on $J(X,Y)$ defined by A.V. Skorokhod are presented. A general analysis of these topologies is probably of independent interest.

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Let (Y, d^*) be a complete separable metric space. We denote by $J(X, Y)$ the space of all functions defined on the interval $X = [0, 1]$, with values in Y, and which at every point have a limit on the left and are continuous on the right (and on the left at $x = 1$). And let $d(x_1, x_2)$ denote the usual metric on X.

Let us consider certain properties of the functions which belong to $J(X, Y)$. If f is not continuous at x_0, f will be said to have the discontinuity $d^*(f(x_0 - 0), f(x_0 + 0))$ at the point x_0.

PROPOSITION 2.1. *If $f \in J(X, Y)$, then for any positive ϵ there exists only a finite number of values of x at which the discontinuity of f at x is greater than ϵ.*

PROOF. This follows from the fact that if there exists a sequence (x_k) such that

$$d^*(f(x_k + 0), f(x_k - 0)) > \epsilon \quad (\text{all } k)$$

and $x_k \to x_0$ then at x_0 the function $f(x)$ would have no limit either on the right or on the left.

PROPOSITION 2.2. *Let $x_1, x_2, x_3, \cdots, x_k$ be all the points at which f has discontinuities no less than ϵ. Then there exists a $\delta > 0$ such that if $|x' - x''| < \delta$ and if x' and x'' both belong to one of the intervals $(0, x_1), (x_1, x_2), \cdots, (x_k, 1)$, then $d^*(f(x'), f(x'')) < \epsilon$.*

PROOF. Assume the contrary. Then there would exist sequences (x'_n) and (x''_n) which both converge to some point x_0 and belong to the same one of the intervals $(0, x_1), \cdots, (x_k, 1)$, and the sequences would have the property that $d^*(f(x'_n), f(x''_n)) \geq \epsilon$, for every n.

Now the points x'_n and x''_n lie on opposite sides of x_0 (otherwise $d^*(f(x'_n), f(x''_n)) \geq \epsilon$ would be impossible), so that $d^*(f(x_0 + 0), f(x_0 - 0)) \geq \epsilon$. Therefore x_0 is one of the points $x_1, x_2, \cdots, x_k$, which contradicts the statement that x'_n and x''_n belong to the same one of the intervals $(0, x_1), (x_1, x_2), \cdots, (x_k, 1)$.

DEFINITION 2.3. The sequence of functions (f_n) converges uniformly to f at the point x_0 if for all $\epsilon > 0$ there exists an integer N such that

$$d^*(f_n(x), f(x)) < \epsilon \text{ for all } n > N \text{ and all } x \in X.$$

Obviously if f_n converges uniformly to f at every point of some closed set, then f_n converges uniformly to f on this whole set.

DEFINITION 2.4. The sequence (f_n) is called *J_1-convergent to f* if there exists a sequence of continuous one-to-one mappings λ_n of the interval $X = [0,1]$ onto itself such that

$$\lim_{n\to\infty} \sup_s |\lambda_n(s) - s| = 0; \quad \lim_{n\to\infty} \sup_s d^*(f_n(s), f(\lambda_n(s))) = 0.$$

The uniform convergent topology U and the J_1-convergent topology J_1 take the form of a single jump at a discontinuity point x_0. In both of these topologies, for values of x close to x_0, the function $f_n(x)$ can take on values which are either close to $f(x_0 - 0)$ or $f(x_0 + 0)$. If we wish to keep this last property, but do not require that the transition be in the form of a single jump, that is that a function $f_n(x)$ may jump back and forth between the values $f(x_0 - 0)$ and $f(x_0 + 0)$ several times in the neighbourhood of a point x_0, then we obtain the topology J_1.

DEFINITION 2.5. A sequence (f_n) is said to be *J_2-convergent to f* if there exists a sequence of one-to-one mappings λ_n of the interval $X = [0,1]$ onto itself such that

$$\sup_s |\lambda_n(s) - s| \to 0 \text{ and } \sup_s d^*(f_n(s), f(\lambda_n(s))) \to 0 \text{ as } n \to \infty.$$

DEFINITION 2.6. The pair of functions (x, f) gives a parametric representation of f if those and only those pairs (x, y) belong to the graph $G(f)$ of f for which an s can be found such that $x = x(s)$, $y = f(s)$, where $f(s)$ is continuous, and $x(s)$ is continuous and monotonically increasing (the functions $f(s)$ and $x(s)$ are defined on the segment $[0,1]$). We note that if $(x_1(s), f_1(s))$ and $(x_2(s), f_2(s))$ are parametric representations of (x, y), there exists a monotonically increasing function $\lambda(s)$ such that

$$f_1(s) = f_2(\lambda(s)) \quad \text{and} \quad x_1(s) = x_2(\lambda(s)).$$

The metric R by $R((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d^*(y_1, y_2)$ in the product space $X \times Y$ has been defined by A.V. Skorokhod.

DEFINITION 2.7. The sequence (f_n) is called *M_1-convergent to f* if there exist parametric representations $(x(s), y(s))$ of $G(f)$ and $(x_n(s), y_n(s))$ of $G(f_n)$ such that

$$\lim_{n\to\infty} \sup_s R((x_n(s), y_n(s)), (x(s), y(s))) = 0.$$

We can characterize the topology M_1 in the following way from the point of view of the behaviour at a point of discontinuity x_0 of the function f.

The transition from $f(x_0 - 0)$ to $f(x_0 + 0)$ is such that first $f_n(x)$ is arbitrarily close to the segment $[f(x_0 - 0), f(x_0)]$ and second $f_n(x)$ moves from $f(x_0 - 0)$ to $f(x_0)$ almost always advancing.

DEFINITION 2.8. The sequence (f_n) is called *M_2-convergent to f* if

$$\lim_{n\to\infty} \sup_{(x,y)\in G(f)} \inf_{(x_n,y_n)\in G(f_n)} R((x,y),(x_n,y_n)) = 0.$$

Let us consider the relation between our topologies. It is clear that U is stronger than J_1, and that J_1 in turn is stronger than J_2. It is also clear that M_1 is stronger than M_2. We recall that a topology T_1 is stronger than T_2 if convergence in T_1 implies convergence in T_2. If X is a linear space, we can use any of our topologies. It is easily seen that convergence in J_1 implies convergence in any of the other topologies except the ordinary uniform convergent topology U.

EXAMPLE. Let $X = Y = [0,1]$. We set

$$f(x) = \begin{cases} 0 & : \quad x < \frac{1}{2}, \\ 1 & : \quad x \geq \frac{1}{2}, \end{cases}$$

$$f_n(x) = \begin{cases} 0 & : \quad 0 \leq x < \frac{1}{2} - 1/n, \\ \frac{1}{2}(nx - \frac{n}{2} + 1) & : \quad \frac{1}{2} - 1/n \leq x < \frac{1}{2} + 1/n, \\ 1 & : \quad x \geq \frac{1}{2} + 1/n, \end{cases}$$

$$f'_n(x) = \begin{cases} 0 & : \quad 0 \le x < \frac{1}{2} - 1/n, \\ 1 & : \quad \frac{1}{2} - 1/n \le x < \frac{1}{2}, \\ 0 & : \quad \frac{1}{2} \le x < \frac{1}{2} + 1/n, \\ 1 & : \quad x \ge \frac{1}{2} + 1/n. \end{cases} \qquad (n \ge 2).$$

Examples can be found to show that the topologies M_1 and J_2 cannot be compared. That is, $f_n \xrightarrow{M_1} f$ although this sequence does not converge to f in J_2, while $f'_n \xrightarrow{J_2} f$ although this sequence does not converge to f in M_1.

3. Comparison of topologies on function spaces

DEFINITION 3.1. Let (T, L) and (Z, S) be topological spaces. For each $U \in L \times S$, let $G_U = \{f \in Z^T : G(f) \subset U\}$.

The topology G on Z^T generated by $\{G_U : U \in L \times S\}$ is called graph topology [8].

PROPOSITION 3.2. *Let X and Y be complete separable metric spaces. Then the topology K of compact convergence coincides with the graph topology G* [4].

THEOREM 3.3. *Let X and Y be $[0,1]$ with Euclidean topology $\mathcal{U}$. Then the topology of pointwise convergence P coincides with the M_1-convergent topology M_1 on Y^X.*

PROOF. Given a point x of X, let $P(x, V) = \{f \in Y^X : f(x) \in V \in \mathcal{U}\}$. Then the $P(x, V)$ is a subbasic open set in the topology of pointwise convergence. And let $(x_n(s), y_n(s))$ and $(x(s), y(s))$ be the parametric representations of $G(f_n)$ and $G(f)$ respectively. Define

$$\begin{aligned} S(f,\epsilon) =& \{f_n \in Y^X : R(f, f_n) = R((x(s), y(s)), (x_n(s), y_n(s))) < \epsilon \\ & \text{for each } \epsilon > 0\} \text{ and } P(x, S(f,\epsilon)) = \{f_n \in Y^X : f_n \in S(f,\epsilon)\}. \end{aligned}$$

Then the open sphere $S(f, \epsilon)$ is a basic open set in M_1-topology. If $f_n \in S(f, \epsilon)$, then $f_n \in P(x, S(f, \epsilon))$. On the other hand, if f_n converges to f in M_1, then y_n converges to y and x_n converges to x. Thus, since

$d(x_n(s), x(s)) = 0$ and $d^*(y_n(s), y(s)) = 0$, $x_n = x$ and $y_n = y$. If $f_n \in P(x, S(f, \epsilon))$, then $f_n \in S(f, \epsilon)$. Consequently, $P(x, S(f, \epsilon)) = S(f, \epsilon)$. Accordingly P coincide with M_1.

DEFINITION 3.4. Let (T, L) and (Z, S) be topological spaces. For each pair of open sets $U \in L$ and $V \in S$, let

$$A(U, V) = \{f \in Z^T : f(U) \cap V \neq \emptyset\}.$$

The almost convergent topology A on Z^T is the topology which has as a subbasis all sets of the form $A(U, V)$.

THEOREM 3.5. *The almost convergent topology A is coarser than the M_2-convergent topology M_2.*

PROOF. Since the M_2-metric is

$$R((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d^*(y_1, y_2) \text{ for } (x_1, y_1), (x_2, y_2) \in X \times Y,$$

if we take

$$\epsilon_0 = \frac{1}{2} d^*(y_1, y_2), \quad \delta_0 = \frac{1}{2} d(x_1, x_2), \quad x_0 \in X \quad \text{and} \quad y_0 \in Y$$

such that $x_0 = \inf\{x : d(x_1, x) = d(x, x_2)\}$ and $y_0 = \inf\{y : d^*(y_1, y) = d^*(y, y_2)\}$, we can write

$$U_0 = S_d(x_0, \delta_0) = \{z : d(x_0, z) < \delta_0\} \text{ and}$$
$$V_0 = S_{d^*}(y_0, \epsilon_0) = \{z : d^*(y_0, z) < \epsilon_0\}.$$

And so, we can find an open sphere

$$S(f, \gamma_0) = \{g \in Z^T : R(f, g) < \gamma_0\}$$

where $\gamma_0 = \min\{\delta_0, \epsilon_0\}$, $x \in U_0$ and $g(x) \in V_0$, and then $A(U, V) = \cup\{S(f(x_0), \gamma_0) : x_0 \in U\}$.
Therefore, A is coarser than M_2.

COROLLARY 3.6. *The almost convergent topology is strictly smaller than the pointwise convergent topology except that they coincide when (T, L) is the discrete space.*

PROOF. If (T, L) is a discrete space, then

$$P(x, V) = \{f \in Z^T : f(x) \in V \in L\}$$

and $\{x\} \in L$, which implies $P(x, V) = A(\{x\}, V)$.

CONCLUSION. Denoting the statement "The topology T_1 is stronger than T_2." by $T_1 \to T_2$, all the above theorems can be summarized by the following diagram:

$$M_1 = P$$

$$U = K = G \longrightarrow J_1 \qquad\qquad M_2 \longrightarrow A$$

$$J_2$$

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