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A GENERALIZATION OF A LATTICE FUZZY TOPOLOGY

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ABSTRACT. In this paper we introduce a new definition of a lattice fuzzy topology which is a generalization of Lowen's fuzzy topology and show that the category of Lowen's fuzzy topological spaces is a bireflective full subcategory of the category of lattice fuzzy topological spaces.

Introduction

As a generalization of a set, the concept of fuzzy set was introduced by Zadeh. Chang[1] introduced a concept of fuzzy topology and Lowen[6] introduced a more natural definition of fuzzy topology.

Some authors[2,4,8] introduced new definitions of fuzzy topology such as smooth topology. But their definitions are generalizations of Chang's fuzzy topology.

We have introduced a concept of lattice fuzzy topology in [5]. In this paper we will introduce a new definition of the lattice fuzzy topology, which is a generalization of Lowen's fuzzy topology. Also we will study subspaces of lattice fuzzy topological spaces and fuzzy continuous maps. Moreover, we show that the category of Lowen's fuzzy topological spaces is a bireflective full subcategory of the category of lattice fuzzy topological spaces.

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1. Preliminaries

Chang introduced the concept of fuzzy topology on a non-empty set X as follows.

DEFINITION. [1] A *fuzzy topology* on X is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0},\ \tilde{1} \in \mathcal{T}$;
(2) If $\mu,\ \nu \in \mathcal{T}$ then $\mu \wedge \nu \in \mathcal{T}$;
(3) If $\mu_i \in \mathcal{T}$ for each $i \in \Gamma$, then $\bigvee_{i\in\Gamma} \mu_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

Hence a fuzzy topology on X can be regarded as a map $\mathcal{T} : I^X \to \{0,1\}$(where $I = [0,1]$) which satisfies following three conditions:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$;
(2) If $\mathcal{T}(\mu) = \mathcal{T}(\nu) = 1$ then $\mathcal{T}(\mu \wedge \nu) = 1$;
(3) If $\mathcal{T}(\mu_i) = 1$ for each $i \in \Gamma$, then $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) = 1$.

It is easily seen that with Chang's definition a constant function between fuzzy topological spaces is not necessarily continuous. This can be true only if one uses the alternative Lowen's definition.

DEFINITION. [6] A *fuzzy topology* on X is a family $\mathcal{T} \subset I^X$ which satisfies the following properties:

(1) For all $\alpha \in [0,1]$, $\tilde{\alpha} \in \mathcal{T}$, where $\tilde{\alpha}$ is a constant map with value α ;
(2) If $\mu,\ \nu \in \mathcal{T}$ then $\mu \wedge \nu \in \mathcal{T}$;
(3) If $\mu_i \in \mathcal{T}$ for each $i \in \Gamma$, then $\bigvee_{i\in\Gamma} \mu_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

Also, the above definition can be translated as a map $\mathcal{T} : I^X \to \{0,1\}$ which satisfies corresponding three conditions. Hence it can be generalized to a map $\mathcal{T} : L^X \to \{0,1\}$ (where L is a lattice) which satisfies corresponding three conditions. But fuzziness in the concept of openness of a fuzzy subset is absent in the above two definitions. So for fuzzifying the openness of a fuzzy subset, some authors[2,8] gave new definitions of fuzzy topology.

DEFINITION. [2] A *gradation of openness* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$;
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$;
(3) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i)$.

DEFINITION. [8] A *smooth topology* on X is a map $\mathcal{T} : L^X \to L'$(where L and L' are copies of $[0,1]$ and $\{0,1\}$) which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$;
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$;
(3) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i)$.

These definitions are generalizations of Chang's fuzzy topology. Now we introduce a new definition of the lattice fuzzy topology which is a generalization of all the above notions of fuzzy topology.

2. Lattice fuzzy topology

Let L be a fixed fuzzy lattice (a complete, completely distributive lattice with an order reversing involution $a \to a'$, and with least and greatest elements denoted by 0 and 1 respectively) and X a non-empty set. An L-fuzzy subset of X is a function from X into L (See [3]). L^X denotes the set of all L-fuzzy subsets of X. Let Y be a subset of X and $\mu \in L^X$; the restriction of μ on Y is denoted by $\mu|_Y$. For each $\mu \in L^Y$ an extension of μ on X, denoted by μ_X, is defined by

$$\mu_X = \begin{cases} \mu & \text{on } Y, \\ 0 & \text{on } X - Y. \end{cases}$$

Now we introduce a new definition of a lattice fuzzy topological space as follows.

DEFINITION. A *lattice fuzzy topology* on X is a map $\mathcal{T} : L^X \to L$ satisfying the following properties :

(1) $\mathcal{T}(\tilde{\alpha}) = 1$ for any constant L-fuzzy set $\tilde{\alpha}$;
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$;
(3) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i)$.

Then we call $(X, \mathcal{T})$ a *lattice fuzzy topological space* on X.

If we choose $L = [0, 1]$, then this lattice fuzzy topology axioms imply the fuzzy topology axioms of Chattopadhyay *et al.* [2] and axioms of Ramadan[8]. If $L = [0, 1]$ and $\mathcal{T}$ has only two values 0 and 1, then it is identical to Lowen's fuzzy topology. If we choose $L = \{0, 1\}$, then it is identical to a topology on X. That is, this new definition is an extension of all the above definitions.

In fact, we can define a lattice fuzzy topology as a map $\mathcal{T} : L^X \to M$ for different fuzzy lattices L and M.

DEFINITION. A *family of closed subsets* of a set X is a map $\mathcal{F} : L^X \to L$ satisfying the following properties:

(1) $\mathcal{F}(\tilde{\alpha}) = 1$ for any constant L-fuzzy set $\tilde{\alpha}$,
(2) $\mathcal{F}(\mu_1 \vee \mu_2) \geq \mathcal{F}(\mu_1) \wedge \mathcal{F}(\mu_2)$,
(3) $\mathcal{F}(\bigwedge_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{F}(\mu_i)$.

Since L has an order reversing involution $a \to a'$, for each $\mu \in L^X$ we can define $\mu' \in L^X$ by $\mu'(x) = (\mu(x))'$ for all $x \in X$. We note that $\mu \to \mu'$ is also an order reversing involution in L^X.

LEMMA. *For any family $\{\mu_i\}_{i\in\Gamma}$ of L-fuzzy sets in X the De Morgan's Laws hold:*

(1) $(\bigwedge \mu_i)' = \bigvee \mu_i'$.
(2) $(\bigvee \mu_i)' = \bigwedge \mu_i'$.

PROOF. Since $\bigwedge \mu_i \leq \mu_i$, $(\bigwedge \mu_i)' \geq \mu_i'$ by the order reversing involution. So $(\bigwedge \mu_i)' \geq \bigvee \mu_i'$. Similarly, from the fact $\bigvee \mu_i \geq \mu_i$, we get $(\bigvee \mu_i)' \leq \bigwedge \mu_i'$. If we substitute μ_i' for μ_i, we have $(\bigvee \mu_i')' \leq \bigwedge (\mu_i')'$, and hence $(\bigvee \mu_i')' \leq \bigwedge \mu_i$. So $\bigvee \mu_i' \geq (\bigwedge \mu_i)'$. Thus $(\bigwedge \mu_i)' = \bigvee \mu_i'$. Similarly $(\bigvee \mu_i)' \leq \bigwedge \mu_i'$. Hence $(\bigvee \mu_i)' = \bigwedge \mu_i'$.

By the above lemma we have following basic properties of the lattice fuzzy topology and the family of closed subsets.

PROPOSITION 2.1. *Let $\mathcal{T}$ be a lattice fuzzy topology on X and $\mathcal{F}_\mathcal{T} : L^X \to L$ a map defined by $\mathcal{F}_\mathcal{T}(\mu) = \mathcal{T}(\mu')$. Then $\mathcal{F}_\mathcal{T}$ is a family of closed subsets of X.*

PROPOSITION 2.2. *Let $\mathcal{F}$ be a family of closed subsets of X and $\mathcal{T}_{\mathcal{F}} : L^X \to L$ a map defined by $\mathcal{T}_{\mathcal{F}}(\mu) = \mathcal{F}(\mu')$. Then $\mathcal{T}_{\mathcal{F}}$ is a lattice fuzzy topology on X.*

PROPOSITION 2.3. *(1) If $\mathcal{F}$ is a family of closed subsets of X, then*

$$\mathcal{T}_{\mathcal{F}_{\mathcal{T}}} = \mathcal{T}.$$

(2) If $\mathcal{T}$ is a lattice fuzzy topology on X, then

$$\mathcal{F}_{\mathcal{T}_{\mathcal{F}}} = \mathcal{F}.$$

DEFINITION. Let $(X, \mathcal{T})$ be a lattice fuzzy topological space and $\mu \in L^X$. Let $e \in L$ be a fixed element which is neither 0 nor 1. Then *$\mathcal{T}$-closure* of μ, denoted by $\overline{\mu}^e$ or simply $\overline{\mu}$, is defined by

$$\overline{\mu}^e = \bigwedge\{\eta \in L^X | \mathcal{F}_{\mathcal{T}}(\eta) \geq e,\ \eta \geq \mu\}.$$

Clearly $\mu \geq \eta$ implies that $\overline{\mu} \geq \overline{\eta}$ for all $\mu, \eta \in L^X$. Also we have $\mathcal{F}_{\mathcal{T}}(\overline{\mu}) \geq e$.

THEOREM 2.4. *Let $(X, \mathcal{T})$ be a lattice fuzzy topological space. Then*

(1) $\overline{\tilde{0}} = \tilde{0}$,
(2) $\overline{\mu} \geq \mu$,
(3) $\overline{\mu_1 \vee \mu_2} = \overline{\mu_1} \vee \overline{\mu_2}$,
(4) $\overline{\overline{\mu}} = \overline{\mu}$.

PROOF. (1) and (2) are obvious. For (3), it is clear that $\mu_1 \vee \mu_2 \leq \overline{\mu_1} \vee \overline{\mu_2}$. Also $\mathcal{F}_{\mathcal{T}}(\overline{\mu_1} \vee \overline{\mu_2}) \geq e$ since $\mathcal{F}_{\mathcal{T}}(\overline{\mu_i}) \geq e$ for each $i = 1, 2$. So, by the definition of closure, $\overline{\mu_1 \vee \mu_2} \leq \overline{\mu_1} \vee \overline{\mu_2}$. Conversely, since $\mu_i \leq \mu_1 \vee \mu_2$ for each $i = 1, 2$, $\overline{\mu_i} \leq \overline{\mu_1 \vee \mu_2}$. Thus $\overline{\mu_1} \vee \overline{\mu_2} \leq \overline{\mu_1 \vee \mu_2}$. Hence $\overline{\mu_1} \vee \overline{\mu_2} = \overline{\mu_1 \vee \mu_2}$. The proof of (4) is straightforward.

PROPOSITION 2.5. *Let $(X, \mathcal{T})$ be a lattice fuzzy topology. Then for each $\mu \in L^X$,*

$$\mathcal{F}_{\mathcal{T}}(\mu) \geq e \Longleftrightarrow \mu = \overline{\mu}^e.$$

PROOF. It is obvious from the definition of closure.

Thus we may have many kinds of closedness in a lattice fuzzy topology depending on level.

Let X be a non-empty set. Define $\mathcal{I} : L^X \to L$ by $\mathcal{I}(\tilde{\alpha}) = 1$ for any constant L-fuzzy set $\tilde{\alpha}$, $\mathcal{I}(\mu) = 0$ for any non-constant $\mu \in L^X$. Define $\mathcal{D} : L^X \to L$ by $\mathcal{D}(\mu) = 1$ for any $\mu \in L^X$. Then $\mathcal{I}$ and $\mathcal{D}$ are both lattice fuzzy topologies on X such that for any lattice fuzzy topology $\mathcal{T}$ on X, $\mathcal{I} \leq \mathcal{T} \leq \mathcal{D}$. That is, $\mathcal{I}(\mu) \leq \mathcal{T}(\mu) \leq \mathcal{D}(\mu)$ for any $\mu \in L^X$.

PROPOSITION 2.6. *Let $\{\mathcal{T}_k : k \in \Gamma\}$ be an arbitrary family of lattice fuzzy topologies on X. Then $\mathcal{T} = \bigwedge_{k\in\Gamma} \mathcal{T}_k$ is also a lattice fuzzy topology on X.*

PROOF. Obviously $\mathcal{T}(\mu) = (\bigwedge_{k\in\Gamma} \mathcal{T}_k)(\mu) = \bigwedge_{k\in\Gamma} \mathcal{T}_k(\mu)$ is a map from L^X to L.

(1) $\mathcal{T}(\tilde{\alpha}) = \bigwedge \mathcal{T}_k(\tilde{\alpha}) = 1$ for all constant $\tilde{\alpha}$.

(2) Let $\mu_1, \mu_2 \in L^X$. Then

$$\begin{aligned}
\mathcal{T}(\mu_1 \wedge \mu_2) &= \bigwedge_{k\in\Gamma} \mathcal{T}_k(\mu_1 \wedge \mu_2) \\
&\geq \bigwedge_{k\in\Gamma} \{\mathcal{T}_k(\mu_1) \wedge \mathcal{T}_k(\mu_2)\} \text{(by axiom 2)} \\
&= (\bigwedge_{k\in\Gamma} \mathcal{T}_k(\mu_1)) \wedge (\bigwedge_{k\in\Gamma} \mathcal{T}_k(\mu_2)) \\
&= \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2).
\end{aligned}$$

(3) Let $\{\mu_i\}_{i\in J}$ be a family of L-fuzzy subsets of X. Then

$$\begin{aligned}
\mathcal{T}(\bigvee_i \mu_i) &= \bigwedge_k \{\mathcal{T}_k(\bigvee_i \mu_i)\} \\
&\geq \bigwedge_k \{\bigwedge_i \mathcal{T}_k(\mu_i)\} \text{(by axiom 3)} \\
&= \bigwedge_i \{\bigwedge_k \mathcal{T}_k(\mu_i)\} \\
&= \bigwedge_i \mathcal{T}(\mu_i).
\end{aligned}$$

So $\mathcal{T}$ is a lattice fuzzy topology on X.

Thus we have the greatest lower bound of any family $\{\mathcal{T}_k : k \in \Gamma\}$. Similarly we have the least upper bound of any family. Let $\mathcal{M}_F(X)$ be the set of all lattice fuzzy topologies on X. Then we have:

THEOREM 2.7. *$(\mathcal{M}_F(X), \leq)$ is a complete lattice with the least element $\mathcal{I}$ and the greatest element $\mathcal{D}$.*

Now we will study the relations between Lowen's fuzzy topology and lattice fuzzy topology. From now on, a fuzzy topology means the Lowen's fuzzy topology generalized to a fuzzy lattice L. Let $\mathcal{T}_a = \{\mu \in L^X : \mathcal{T}(\mu) \geq a\}$ be a a-cut of a lattice fuzzy topology $\mathcal{T}$.

PROPOSITION 2.8. *Let $(X, \mathcal{T})$ be a lattice fuzzy topological space. Then for each $a \in L$, the a-cut $\mathcal{T}_a$ is a fuzzy topology on X. Moreover $\mathcal{T}_a \subset \mathcal{T}_b$ if $a \geq b$.*

PROOF. (1) For any $\tilde{\alpha}$, $\mathcal{T}(\tilde{\alpha}) = 1$. So $\tilde{\alpha} \in \mathcal{T}_a$.

(2) Let $\mu_1, \mu_2 \in \mathcal{T}_a$. Then $\mathcal{T}(\mu_1) \geq a$ and $\mathcal{T}(\mu_2) \geq a$. So $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2) \geq a$. Hence $\mu_1 \wedge \mu_2 \in \mathcal{T}_a$.

(3) Let $\mu_i \in \mathcal{T}_a$ for all $i \in \Gamma$. Then $\mathcal{T}(\mu_i) \geq a$ for all $i \in \Gamma$. So $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i) \geq a$. Hence $\bigvee \mu_i \in \mathcal{T}_a$.

By the above result, $\mathcal{I}_a$ and $\mathcal{D}_a$ become the indiscrete and discrete fuzzy topology, respectively.

PROPOSITION 2.9. *Let $(X, \mathcal{T})$ be a lattice fuzzy topological space and $\mathcal{T}_a$ be the a-cut. Then $\mathcal{T}(\mu) = \bigvee\{a | \mu \in \mathcal{T}_a\}$*

PROOF. $\bigvee\{a | \mu \in \mathcal{T}_a\} = \bigvee\{a | \mathcal{T}(\mu) \geq a\} = \mathcal{T}(\mu)$.

COROLLARY. *Two lattice fuzzy topologies $\mathcal{T}$ and $\mathcal{U}$ on X are equal if and only if $\mathcal{T}_a = \mathcal{U}_a$ for any $a \in L$.*

PROPOSITION 2.10. *Let (X, T) be a fuzzy topological space. Define for each $a \in L$, a map $T^a : L^X \to L$ by the rule:*

$$T^a(\mu) = \begin{cases} 1 & \text{for any constant } L\text{-fuzzy set } \tilde{\alpha}, \\ a & \text{if } \mu \in T \text{ and } \mu \text{ is non-constant}, \\ 0 & \text{otherwise}. \end{cases}$$

Then T^a is a lattice fuzzy topology on X such that $(T^a)_a = T$.

PROOF. (1) $T^a(\tilde{\alpha}) = 1$ for any constant L-fuzzy set $\tilde{\alpha}$.

(2) Note that $T^a(\mu_1) \wedge T^a(\mu_2) = 0, a$ or 1. If $T^a(\mu_1) \wedge T^a(\mu_2) = 0$, it is obvious. If $T^a(\mu_1) \wedge T^a(\mu_2) = 1$, then μ_1 and μ_2 are constant, and hence $\mu_1 \wedge \mu_2$ is also constant. Thus $T^a(\mu_1 \wedge \mu_2) = 1 \geq T^a(\mu_1) \wedge T^a(\mu_2)$. If $T^a(\mu_1) \wedge T^a(\mu_2) = a$, then $T^a(\mu_1) \geq a$ and $T^a(\mu_2) \geq a$. Thus $\mu_1 \in T$ and $\mu_2 \in T$, and hence $\mu_1 \wedge \mu_2 \in T$. So $T^a(\mu_1 \wedge \mu_2) \geq a = T^a(\mu_1) \wedge T^a(\mu_2)$.

(3) If $\bigwedge_{i \in \Gamma} T^a(\mu_i) = 0$, then it is obvious. If $\bigwedge_{i \in \Gamma} T^a(\mu_i) = 1$, then $T^a(\mu_i) = 1$ for all $i \in \Gamma$. Thus μ_i is constant for all $i \in \Gamma$. Hence $\bigvee \mu_i$ is constant. So $T^a(\bigvee_{i \in \Gamma} \mu_i) = 1 \geq \bigwedge_{i \in \Gamma} T^a(\mu_i)$. If $\bigwedge_{i \in \Gamma} T^a(\mu_i) = a$, then $T^a(\mu_i) \geq a$ for all $i \in \Gamma$. Thus $\mu_i \in T$, and hence $\bigvee \mu_i \in T$. So $T^a(\bigvee \mu_i) \geq a = \bigwedge T^a(\mu_i)$. In all $T^a(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} T^a(\mu_i)$.

Moreover $\mu \in (T^a)_a$ if and only if $T^a(\mu) \geq a$ if and only if $\mu \in T$. Hence $(T^a)_a = T$.

DEFINITION. Let T be a fuzzy topology on X. Then the lattice fuzzy topology T^a on X is said to be *a-th graded.*

3. Fuzzy continuous maps and subspaces

DEFINITION. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two lattice fuzzy topological spaces and $f : X \to Y$ a map. Then the map f is called a *fuzzy continuous* map if for each $\mu \in L^Y$, $\mathcal{U}(\mu) \leq \mathcal{T}(f^{-1}(\mu))$.

PROPOSITION 3.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two lattice fuzzy topological spaces and $f : X \to Y$ a map. Then the map f is fuzzy continuous if and only if, for all $a \in L$, $f : (X, \mathcal{T}_a) \to (Y, \mathcal{U}_a)$ is fuzzy continuous.*

PROOF. Suppose $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a fuzzy continuous map and $a \in L$ is fixed. Take $\mu \in \mathcal{U}_a$, then $a \leq \mathcal{U}(\mu) \leq \mathcal{T}(f^{-1}(\mu))$. Hence $f^{-1}(\mu) \in \mathcal{T}_a$. That is $f : (X, \mathcal{T}_a) \to (Y, \mathcal{U}_a)$ is fuzzy continuous. Conversely, suppose $f : (X, \mathcal{T}_a) \to (Y, \mathcal{U}_a)$ is fuzzy continuous for all $a \in L$. Let $\mu \in L^Y$. If $\mathcal{U}(\mu) = 0$, then obviously, $\mathcal{U}(\mu) \leq \mathcal{T}(f^{-1}(\mu))$. If $\mathcal{U}(\mu) \neq 0$, let $\mathcal{U}(\mu) = r$, then $\mu \in \mathcal{U}_r$. So $f^{-1}(\mu) \in \mathcal{T}_r$, by the fuzzy continuity of $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$. Hence $\mathcal{T}(f^{-1}(\mu)) \geq r = \mathcal{U}(\mu)$. Thus $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is fuzzy continuous.

PROPOSITION 3.2. *Let* (X,T) *and* (Y,U) *be two fuzzy topological spaces and* $f : X \to Y$ *a map. Then* $f : (X,T) \to (Y,U)$ *is fuzzy continuous if and only if* $f : (X,T^a) \to (Y,U^a)$ *is fuzzy continuous for each* $a \in L$.

PROOF. Suppose $f : (X,T) \to (Y,U)$ is fuzzy continuous. Take $\mu \in L^Y$. Then either $\mu \in U$ or $\mu \notin U$. If $\mu \in U$ and $\mu = \tilde{\alpha}$ for some constant L-fuzzy set $\tilde{\alpha}$, then $f^{-1}(\tilde{\alpha}) = \tilde{\alpha}$. So $U^a(\mu) = U^a(\tilde{\alpha}) = 1 \leq 1 = T^a(\tilde{\alpha}) = T^a(f^{-1}(\tilde{\alpha})) = T^a(f^{-1}(\mu))$. Let $\mu \in U$ and $\mu \neq \tilde{\alpha}$. Then $U^a(\mu) = a$. By the fuzzy continuity of f, $f^{-1}(\mu) \in T$. So $U^a(\mu) \leq T^a(f^{-1}(\mu))$. In case $\mu \notin U$, $U^a(\mu) = 0$. So $U^a(\mu) \leq T^a(f^{-1}(\mu))$. Hence $f : (X,T^a) \to (Y,U^a)$ is a fuzzy continuous map, for each $a \in L$. Conversely, suppose $f : (X,T^a) \to (Y,U^a)$ is a fuzzy continuous map. Then $f : (X,(T^a)_a) \to (Y,(U^a)_a)$ is fuzzy continuous by the above proposition. Since $(T^a)_a = T$ and $(U^a)_a = U$, $f : (X,T) \to (Y,U)$ is fuzzy continuous.

PROPOSITION 3.3. *Let* $(X,\mathcal{T})$, $(Y,\mathcal{U})$ *and* $(Z,\mathcal{V})$ *be three lattice fuzzy topological spaces. If* $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ *and* $g : (Y,\mathcal{U}) \to (Z,\mathcal{V})$ *are fuzzy continuous maps, then so is* $g \circ f : (X,\mathcal{T}) \to (Z,\mathcal{V})$.

PROOF. Straightforward.

DEFINITION. A map $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called a *fuzzy homeomorphism* if f is bijective and f and f^{-1} are fuzzy continuous. A map $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ is said to be *fuzzy open* if $\mathcal{T}(\mu) \leq \mathcal{U}(f(\mu))$ for all $\mu \in L^X$. A map $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ is said to be *fuzzy closed* if $\mathcal{F}_{\mathcal{T}}(\mu) \leq \mathcal{F}_{\mathcal{U}}(f(\mu))$ for all $\mu \in L^X$.

THEOREM 3.4. *Let* $(X,\mathcal{T})$ *and* $(Y,\mathcal{U})$ *be two lattice fuzzy topologies and* $f : X \to Y$ *a bijection. Then the following statements are equivalent.*

(1) *f is a fuzzy homeomorphism.*
(2) *f is fuzzy continuous and fuzzy open.*
(3) *f is fuzzy continuous and fuzzy closed.*
(4) $\mathcal{T}(\lambda) = \mathcal{U}(f(\lambda))$ *for all* $\lambda \in L^X$.
(5) $\mathcal{U}(\mu) = \mathcal{T}(f^{-1}(\mu))$ *for all* $\mu \in L^Y$.

PROOF. Straightforward.

THEOREM 3.5. *Let $(X, \mathcal{T})$ be a lattice fuzzy topological space and $i : Y \hookrightarrow X$ an inclusion map. Define a map $\mathcal{T}_Y : L^Y \to L$ by*

$$\mathcal{T}_Y(\mu) = \bigvee\{\mathcal{T}(\lambda) : \lambda \in L^X, i^{-1}(\lambda) = \mu\}.$$

Then $\mathcal{T}_Y$ is the smallest lattice fuzzy topology on Y such that the map i is fuzzy continuous.

PROOF. Clearly $\mathcal{T}_Y$ is a map from L^Y to L. Let $\hat{\alpha}$ be a constant map from Y to L with the value α. Then

$$\begin{aligned}
&\mathcal{T}_Y(\tilde{\alpha}) \\
&= \bigvee\{\mathcal{T}(\lambda) : \lambda \in L^X, i^{-1}(\lambda) = \tilde{\alpha}\} \\
&\geq \mathcal{T}(\tilde{\alpha}_X), \text{ where } \tilde{\alpha}_X : X \to L \text{ is a constant map with the value } \alpha \\
&= 1.
\end{aligned}$$

Thus $\mathcal{T}_Y(\tilde{\alpha}) = 1$. For any fuzzy sets μ and $\nu \in L^Y$,

$$\begin{aligned}
&\mathcal{T}_Y(\mu \wedge \nu) \\
&= \bigvee\{\mathcal{T}(\lambda) : \lambda \in L^X, i^{-1}(\lambda) = \mu \wedge \nu\} \\
&= \bigvee\{\mathcal{T}(\tau \wedge \sigma) : \tau, \sigma \in L^X, i^{-1}(\tau) \wedge i^{-1}(\sigma) = i^{-1}(\tau \wedge \sigma) = \mu \wedge \nu\} \\
&\geq \bigvee\{\mathcal{T}(\tau \wedge \sigma) : \tau, \sigma \in L^X, i^{-1}(\tau) = \mu \text{ and } i^{-1}(\sigma) = \nu\} \\
&\geq \bigvee\{\mathcal{T}(\tau) \wedge \mathcal{T}(\sigma) : \tau, \sigma \in L^X, i^{-1}(\tau) = \mu \text{ and } i^{-1}(\sigma) = \nu\} \\
&= [\bigvee\{\mathcal{T}(\tau) : \tau \in L^X, i^{-1}(\tau) = \mu\}] \wedge \\
&\qquad [\bigvee\{\mathcal{T}(\sigma) : \sigma \in L^X, i^{-1}(\sigma) = \nu\}] \\
&\quad \text{(because } L \text{ is completely distributive)} \\
&= \mathcal{T}_Y(\mu) \wedge \mathcal{T}_Y(\nu).
\end{aligned}$$

For any family $\{\mu_k\}_{k\in K}$ of fuzzy sets in L^Y

$$\begin{aligned}\mathcal{T}_Y(\bigvee_{k\in K}\mu_k) &= \bigvee\{\mathcal{T}(\lambda):\lambda\in L^X, i^{-1}(\lambda)=\bigvee_{k\in K}\mu_k\}\\ &\geq \bigvee\{\mathcal{T}(\bigvee_{k\in K}\lambda_k):\lambda_k\in L^X \text{ and } i^{-1}(\lambda_k)=\mu_k\}\\ &\geq \bigvee\{\bigwedge_{k\in K}\mathcal{T}(\lambda_k):\lambda_k\in L^X \text{ and } i^{-1}(\lambda_k)=\mu_k\}\\ &= \bigwedge_{k\in K}[\bigvee\{\mathcal{T}(\lambda_k):\lambda_k\in L^X, i^{-1}(\lambda_k)=\mu_k\}]\\ &\text{(because } L \text{ is completely distributive)}\\ &= \bigwedge_{k\in K}\mathcal{T}_Y(\mu_k).\end{aligned}$$

Hence $\mathcal{T}$ is a lattice fuzzy topology on Y. For any $\sigma\in L^X$,

$$\begin{aligned}\mathcal{T}_Y(i^{-1}(\sigma)) &= \bigvee\{\mathcal{T}(\lambda):\lambda\in L^X, i^{-1}(\lambda)=i^{-1}(\sigma)\}\\ &\geq \mathcal{T}(\sigma).\end{aligned}$$

Hence $i: Y\hookrightarrow X$ is continuous. Let $\mathcal{T}^*$ be another lattice fuzzy topology on Y such that $i:(Y,\mathcal{T}^*)\to(X,\mathcal{T})$ is continuous. Since $i:(Y,\mathcal{T}^*)\to(X,\mathcal{T})$ is continuous, $\mathcal{T}^*(i^{-1}(\lambda))\geq\mathcal{T}(\lambda)$ for any $\lambda\in L^X$. Hence $\mathcal{T}^*(\mu)=\bigvee\{\mathcal{T}^*(\sigma):\sigma=\mu\}=\bigvee\{\mathcal{T}^*(i^{-1}(\lambda)):\lambda\in L^X, i^{-1}(\lambda)=\mu\}\geq\bigvee\{\mathcal{T}(\lambda):\lambda\in L^X, i^{-1}(\lambda)=\mu\}=\mathcal{T}_Y(\mu)$. Therefore $\mathcal{T}_Y$ is the smallest lattice fuzzy topology on Y such that i is continuous.

DEFINITION. The lattice fuzzy topology $\mathcal{T}_Y$ determined in Theorem 3.5 is called the *induced lattice fuzzy topology* on Y from $(X,\mathcal{T})$ and the pair $(Y,\mathcal{T}_Y)$ is called a *subspace* of the lattice fuzzy topological space $(X,\mathcal{T})$.

PROPOSITION 3.6. *Let $(Y,\mathcal{T}_Y)$ be a subspace of $(X,\mathcal{T})$ and $\mu\in L^Y$. Then*

(1) $\mathcal{F}_{\mathcal{T}_Y}(\mu)=\bigvee\{\mathcal{F}_{\mathcal{T}}(\eta):\eta\in L^X, i^{-1}(\eta)=\mu\}$,
(2) *If $Z\subset Y\subset X$ then $\mathcal{T}_Z=(\mathcal{T}_Y)_Z$.*

PROOF. (1) $\mathcal{F}_{\mathcal{T}_Y}(\mu) = \mathcal{T}_Y(\mu') = \bigvee\{\mathcal{T}(\lambda) : \lambda \in L^X, i^{-1}(\lambda) = \mu'\} = \bigvee\{\mathcal{T}(\lambda) : \lambda' \in L^X, i^{-1}(\lambda') = \mu\} = \bigvee\{\mathcal{F}_{\mathcal{T}}(\lambda') : \lambda' \in L^X, i^{-1}(\lambda') = \mu\} = \bigvee\{\mathcal{F}_{\mathcal{T}}(\eta) : \eta \in L^X, i^{-1}(\eta) = \mu\}$.

(2) Let $\mu \in L^Z$. Then $(\mathcal{T}_Y)_Z(\mu) = \bigvee\{\mathcal{T}_Y(\lambda) : \lambda \in L^Y, i^{-1}(\lambda) = \mu\} = \bigvee\{\bigvee\{\mathcal{T}(\eta) : \eta \in L^X, i^{-1}(\eta) = \lambda\} : \lambda \in L^Y, i^{-1}(\lambda) = \mu\} = \bigvee\{\mathcal{T}(\eta) : \eta \in L^X, i^{-1}(\eta) = \mu\} = \mathcal{T}_Z(\mu)$.

4. Category of lattice fuzzy topological spaces

Let **FTop** denote the category of all fuzzy topological spaces and fuzzy continuous maps and **L-FTop** denote the category of all lattice fuzzy topological spaces and fuzzy continuous maps, and for each $a \in L$, $\mathbf{L}_a$**-FTop** denote the category of a-th graded fuzzy topological spaces and fuzzy continuous maps.

PROPOSITION 4.1.

(1) *Define* F : **FTop** $\to$ **L-FTop** *by* $F(X, T) = (X, T^a)$ *and* $F(f) = f$. *Then* F *is a functor.*

(2) *Define* G : **L-FTop** $\to$ **FTop** *by* $G(X, \mathcal{T}) = (X, \mathcal{T}_a)$ *and* $G(f) = f$. *Then* G *is a functor.*

PROOF. Proposition 3.1 and Proposition 3.2.

THEOREM 4.2. *For each* $a \in L$, **FTop** *and* $\mathbf{L}_a$**-FTop** *are isomorphic.*

PROOF. Define F : **FTop** $\to$ $\mathbf{L}_a$**-FTop** by $F(X, T) = (X, T^a)$ and $F(f) = f$. Define G : $\mathbf{L}_a$**-FTop** $\to$ **FTop** by $G(X, \mathcal{T}) = (X, \mathcal{T}_a)$ and $G(f) = f$. Then F and G are functors. Clearly $G \circ F(X, T) = (X, T)$. Enough to show that $F \circ G(X, \mathcal{T}) = (X, \mathcal{T})$. Take $(X, \mathcal{T}) \in \mathbf{L}_a$**-FTop**, then there exists $(X, T) \in$ **FTop** such that $\mathcal{T} = T^a$. So $F \circ G(X, \mathcal{T}) = F(X, \mathcal{T}_a) = (X, (\mathcal{T}_a)^a) = (X, ((T^a)_a)^a) = (X, T^a) = (X, \mathcal{T})$. Thus they are isomorphic.

THEOREM 4.3. $\mathbf{L}_a$**-FTop** *is a bireflective full subcategory of* **L-FTop** *for all* $a \in L$.

PROOF. Clearly $\mathbf{L}_a$-**FTop** is a full subcategory of **L-FTop**. Let us take a member $(X, \mathcal{T})$ of **L-FTop**. Then $(X, (\mathcal{T}_a)^a)$ is a member of $\mathbf{L}_a$-**FTop** and also $1_X : (X, \mathcal{T}) \to (X, (\mathcal{T}_a)^a)$ is a fuzzy continuous map. Let $(Y, \mathcal{U})$ be a member of $\mathbf{L}_a$-**FTop** and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a fuzzy continuous map. To complete the proof we need to check that $f : (X, (\mathcal{T}_a)^a) \to (Y, \mathcal{U})$ is a fuzzy continuous map. Note that for $\mu \in L^Y$, $\mathcal{U}(\mu) = 1$ or a or 0. If $\mathcal{U}(\mu) = 1$, then μ is constant and hence $f^{-1}(\mu)$ is also constant. Thus $(\mathcal{T}_a)^a(f^{-1}(\mu)) = 1 \geq \mathcal{U}(\mu)$. If $\mathcal{U}(\mu) = 0$, then $\mathcal{U}(\mu) \leq (\mathcal{T}_a)^a(f^{-1}(\mu))$. If $\mathcal{U}(\mu) = a$, then $a = \mathcal{U}(\mu) \leq \mathcal{T}(f^{-1}(\mu))$ from that $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a fuzzy continuous map. Thus $a \leq \mathcal{T}(f^{-1}(\mu))$. Hence $f^{-1}(\mu) \in \mathcal{T}_a$, and hence $(\mathcal{T}_a)^a(f^{-1}(\mu)) \geq a = \mathcal{U}(\mu)$. Thus $f : (X, (\mathcal{T}_a)^a) \to (Y, \mathcal{U})$ is a fuzzy continuous map.

Therefore we have following result from the above two theorems.

THEOREM 4.4. **FTop** *is a bireflective full subcategory of* **L-FTop**.

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