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ON 6-DIMENSIONAL QUASI-KAEHLER MANIFOLDS WITH POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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ABSTRACT. In a 6-dimensional quasi-Kaehler manifold M with pointwise constant holomorphic sectional curvature $\mu \neq 0$, we find some conditions for M to be a space form or a complex space form.

1. Introduction

An almost Hermitian manifold (M, J, g) is said to be of *pointwise constant holomorphic sectional curvature* if holomorphic sectional curvature $-R(X, JX, X, JX)$ of M is constant for every unit tangent vector X and depends only on points of M. If the holomorphic sectional curvature is constant whole on M, then M is said to be of *constant holomorphic sectional curvature.*

S. Tanno([5]) proved that *if a 6-dimensional nearly Kaehler manifold M is of constant holomorphic sectional curvature μ, then either M is Kaehlerian, or M is of constant curvature $\mu > 0$.* It is well known that any 6-dimensional nearly Kaehler manifold is an Einstein manifold([7]) and its curvature tensor R satisfies the following identity([4]):

$$(*) \qquad R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$$

for $X, Y, Z, W \in \mathfrak{X}(M)$

The present author proved the following theorem which generalizes the result of Tanno.

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THEOREM A([3]). *Let M be a 6-dimensional connected almost Hermitian manifold with pointwise constant holomorphic sectional curvature μ and with curvature identity $(*)$. If M is Einsteinian or weakly $*$-Einsteinian, then M is one of the following:*

(a) *a manifold of constant sectional curvature μ,*
(b) *a complex space form.*

The purpose of the present paper is to obtain a theorem which asserts that in the case of quasi-Kaehler manifold the assumption "Einsteinian" of theorem A can be relaxed by " a manifold with parallel Ricci tensor".

2. Preliminaries

Let $M = (M, J, g)$ be a 6-dimensional almost Hermitian manifold and let ∇, R, ρ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively. The curvature tensor R is given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. The Ricci operator Q of M is given by $\rho(X,Y) = g(QX, Y)$. In the almost Hermitian manifold M, we have

$$(2.1)\quad \begin{aligned} &(\nabla_X J)JY = -J(\nabla_X J)Y, \qquad g((\nabla_X J)Y, Z) = -g(Y, (\nabla_X J)Z),\\ &g((\nabla_X J)Y, Y) = 0, \qquad g((\nabla_X J)Y, JY) = 0. \end{aligned}$$

We shall recall the definitions of special kinds of almost Hermitian manifolds. An almost Hermitian manifold (M, J, g) is called *Kaehlerian* if $\nabla_X J = 0$ for all $X \in \mathfrak{X}(M)$, M is called *nearly Kaehlerian* if $(\nabla_X J)Y + (\nabla_Y J)X = 0$ for all $X, Y \in \mathfrak{X}(M)$ and M is called *quasi-Kaehlerian* if

$$(2.2)\qquad (\nabla_X J)Y + (\nabla_{JX} J)(JY) = 0$$

for all $X, Y \in \mathfrak{X}(M)$.

A. Gray obatained the following

LEMMA 2.1([1]). *Let M be a quasi-Kaehler manifold. Then*

$$
\begin{aligned}
&G(X,Y,Z,W)+(G(JX,JY,JZ,JW)+G(JX,Y,JZ,W)\\
&+G(X,JY,Z,JW)=-2g((\nabla_{(\nabla_X J)Y-(\nabla_Y J)X}J)Z,W),
\end{aligned}
\tag{2.3}
$$

where $G(X,Y,Z,W)=R(X,Y,Z,W)-R(X,Y,JZ,JW)$.

In a quasi-Kaehler manifold M with the curvature identiy $(*)$, the equation (2.3) is reduced to

$$
G(X,Y,Z,W)=-\frac{1}{2}g((\nabla_{(\nabla_X J)Y-(\nabla_Y J)X}J)Z,W). \tag{2.4}
$$

In an almost Hermitian manifold M with the curvature identity $(*)$, the Ricci tensor ρ satisfies ([6])

$$
\rho(JX,JY)=\rho(X,Y) \qquad \rho(JX,Y)=-\rho(X,JY). \tag{2.5}
$$

Using (2.1) and (2.5) we obtain the following

LEMMA 2.2. *Let M be an almost Hermitian manifold with curvature identity $(*)$. If the Ricci tensor ρ of M is parallel, then we have*

$$
\begin{aligned}
&\rho((\nabla_X J)Y,Z)=-\rho(Y,(\nabla_X J)Z), \qquad \rho((\nabla_X J)Y,Y)=0,\\
&\rho((\nabla_X J)Y,JY)=0.
\end{aligned}
\tag{2.6}
$$

3. Quasi-Kaehler manifolds with parallel Ricci tensor

Let M be a 6-dimensional quasi-Kaeheler manifold with pointwise constant holomorphic sectional curvature $\mu(\neq 0)$ and the curvature tensor R of M satisfies the identity $(*)$. Then μ is globally constant([4])

and the curvature tensor R of M is given by ([3])

$$
\begin{aligned}
R(X,Y)Z =& \frac{\tau+2\mu}{8}[g(X,Z)Y - g(Y,Z)X] - \frac{1}{2}[\rho(X,Z)Y - \rho(Y,Z)X] \\
&- \frac{\tau+10\mu}{24}[2g(JX,Y)JZ + g(JX,Z)JY - g(JY,Z)JX] \\
&+ \frac{1}{6}[2\rho(JX,Y)JZ + \rho(JX,Z)JY - \rho(JY,Z)JX] \\
&- \frac{1}{2}[g(X,Z)QY - g(Y,Z)QX] \\
&+ \frac{1}{6}[2g(JX,Y)QJZ + g(JX,Z)QJY - g(JY,Z)QJX].
\end{aligned} \tag{3.1}
$$

Now we assume that the Ricci tensor of M is parallel. Since $\dim M = 6$ it is possible to choose two unit vectors X and W which define orthogonal holomorphic planes $\{X, JX\}$ and $\{W, JW\}$. Putting $Z = X$ and $Y = JW$ in the second Bianchi identity

$$(\nabla_W R)(X,Y,Z) + (\nabla_X R)(Y,W,Z) + (\nabla_Y R)(W,X,Z) = 0, \tag{3.2}$$

we obtain from (2.5), (2.6) and (3.1)

(3.3)

$$
\begin{aligned}
&\frac{\tau+10\mu}{4}\{g((\nabla_X J)X, JW)JW + g((\nabla_X J)X, W)W + 2(\nabla_X J)X\} \\
&+ 3\rho(X,W)(\nabla_W J)X - \rho(X,W)(\nabla_X J)W - 3\rho(JW,X)J(\nabla_W J)X \\
&- \rho(W,(\nabla_X J)X)W - \rho(W,JX)J(\nabla_X J)W - \rho((\nabla_X J)X, JW)JW \\
&- 2Q(\nabla_X J)X - 2\rho(W,W)(\nabla_X J)X - g((\nabla_X J)X, W)QW \\
&- g((\nabla_X J)X, JW)QJW = 0.
\end{aligned}
$$

Now suppose that M is not nearly Kaehlerian. Then there exists a unit vector field V in a neighborhood $\mathcal{U}$ of some point $p \in M$ such that $(\nabla_V J)V \neq 0$ on $\mathcal{U}$. We put

$$V = e_1, \qquad JV = e_2, \qquad \frac{(\nabla_V J)V}{\|\nabla_V J)V\|} = e_3, \qquad Je_3 = e_4.$$

Then $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are orthogonal holomorphic planes. Putting $X = e_1$ and $W = e_3$ in (3.3), we obtain

$$Q(\nabla_{e_1} J)e_1 = \left\{ \frac{\tau + 10\mu}{4} - \rho(e_3, e_3) \right\} (\nabla_{e_1} J)e_1, \tag{3.4}$$

which implies

$$\rho(e_3, e_3) = \frac{\tau + 10\mu}{8}. \tag{3.5}$$

Hence we get from (3.4) and (3.5)

$$Q(\nabla_{e_1} J)e_1 = \frac{\tau + 10\mu}{8} (\nabla_{e_1} J)e_1. \tag{3.6}$$

Using above argument for arbitrary nonzero vector field Z instead of V we get at p

$$Q(\nabla_Z J)Z = \frac{\tau + 10\mu}{8} (\nabla_Z J)\, Z. \tag{3.7}$$

Notice that (3.7) holds validly if $(\nabla_Z J)Z = 0$.

Next we choose another holomorphic plane $\{e_5, e_6 = Je_5\}$ which is orthogonal to holomorphic planes $\{e_1, e_2\}$ and $\{e_3, e_4\}$ respectively. Using (3.5) and (3.6) we find

$$\begin{aligned} \rho(e_3, e_3) = \rho(e_4, e_4) &= \frac{\tau + 10\mu}{8}, \\ \rho(e_3, e_j) &= 0 \text{ for } j \neq 3. \end{aligned} \tag{3.8}$$

If we put $X = e_1$ and $W = e_5$ in (3.3), then using (3.4) and (3.8) we obtain

$$\begin{aligned} &\left\{ \frac{\tau + 10\mu}{8} - \rho(e_5, e_5) \right\} (\nabla_{e_1} J)e_1 \\ &= \frac{1}{2}\{-3\rho(e_1, e_5)(\nabla_{e_5} J)e_1 + \rho(e_1, e_5)(\nabla_{e_1} J)e_5 \\ &+ 3\rho(e_1, e_6)J(\nabla_{e_5} J)e_1 - \rho(e_1, e_6)J(\nabla_{e_1} J)e_5\}. \end{aligned} \tag{3.9}$$

Since $g((\nabla_{e_1}J)e_5, e_1) = g((\nabla_{e_1}J)e_5, e_2) = 0$, we can put

$$(\nabla_{e_1}J)e_5 = \gamma e_3 + \delta e_4, \tag{3.10}$$
$$(\nabla_{e_5}J)e_1 = \epsilon e_3 + \mu e_4 + \nu e_5 + \omega e_6.$$

Since $g((\nabla_{e_1}J)e_3, e_1) = -\|(\nabla_{e_1}J)e_1\| \equiv -\xi$, $g((\nabla_{e_1}J)e_3, e_5) = -\gamma$ and $g((\nabla_{e_1}J)e_3, e_6) = -\delta$, we get

$$(\nabla_{e_1}J)e_3 = -\xi e_1 - \gamma e_5 - \delta e_6. \tag{3.11}$$

Since $\rho((\nabla_{e_1}J)e_3, e_1) = -\rho(e_3, (\nabla_{e_1}J)e_1)$, we obtain by the help of (3.11),

$$\xi[\rho(e_1,e_1) - \rho(e_3,e_3)] + \gamma\rho(e_1,e_5) + \delta\rho(e_1,e_6) = 0. \tag{3.12}$$

If $\rho(e_1,e_5) = \rho(e_1,e_6) = 0$, then we have, by the help of (3.8), (3.9) and (3.12),

$$\rho(e_1,e_1) = \rho(e_3,e_3) = \rho(e_5,e_5) = \frac{\tau + 10\mu}{8}.$$

Hence to show $Q = \dfrac{\tau+10\mu}{8}I$ all we need is $\rho(e_1,e_5) = \rho(e_1,e_6) = 0$, where I denotes the identity transformation. If we put $Y = JX$ and $Z = X$ in (3.2), then we obtain

(3.13)
$$\begin{aligned}
&\frac{\tau+10\mu}{4}\{3(\nabla_W J)X - (\nabla_X J)W - g((\nabla_X J)X, W)X - g(W, (\nabla_{JX}J)X)JX\}\\
&+ 2\rho(X,W)(\nabla_X J)X + \rho(X,X)(\nabla_X J)W + \rho((\nabla_X J)X, W)X\\
&+ \rho((\nabla_{JX}J)X, W)JX - 3\rho(X,X)(\nabla_W J)X - 2\rho(X,JW)(\nabla_{JX}J)X\\
&- 3Q(\nabla_W J)X + Q(\nabla_X J)W + g((\nabla_X J)X, W)QX\\
&+ g((\nabla_{JX}J)X, W)QJX = 0
\end{aligned}$$

for arbitrary orthogonal holomorphic planes $\{X, JX\}$ and $\{W, JW\}$.

If we put $X = e_1$ and $W = e_5$ in (3.13), then we obtain, by the help of (3.7), (3.8), (3.10) and $QJ = JQ$,

(3.14)
$$\begin{aligned}
&\left\{\rho(e_1,e_1) - \frac{\tau+10\mu}{4}\right\}\{(3\epsilon-\gamma)e_3 + (3\mu-\delta)e_4 + 3\nu e_5 + 3\omega e_6\}\\
&+ \frac{\tau+10\mu}{8}\{(3\epsilon-\gamma)e_3 + (3\mu-\delta)e_4\} + 3\nu Qe_5 + 3\omega Qe_6\\
&- 2\rho(e_1,e_5)(\nabla_{e_1}J)e_1 - 2\rho(e_1,e_6)J(\nabla_{e_1}J)e_1 = 0.
\end{aligned}$$

Using (3.9) and (3.10), we obtain

$$\rho(e_1, e_5)\omega - \rho(e_1, e_6)\nu = 0, \qquad \rho(e_1, e_5)\nu + \rho(e_1, e_6)\omega = 0.$$

If $\nu^2 + \omega^2 \neq 0$, then from these equations we have
$\rho(e_1, e_5) = \rho(e_1, e_6) = 0$.

Now we assume that $\nu = \omega = 0$ at p. Then (3.10) can be rewritten as

$$(\nabla_{e_1} J)e_5 = \gamma e_3 + \delta e_4, \quad (\nabla_{e_5} J)e_1 = \epsilon e_3 + \mu e_4. \tag{3.15}$$

Substituting (3.15) into (3.9), we find

$$\left\{\frac{\tau + 10\mu}{8} - \rho(e_5, e_5)\right\}(\nabla_{e_1} J)e_1 = \frac{1}{2}\{(\gamma - 3\epsilon)\rho(e_1, e_5) + (\delta - 3\mu)\rho(e_1, e_6)\}e_3 + \frac{1}{2}\{(\delta - 3\mu)\rho(e_1, e_5) - (\gamma - 3\epsilon)\rho(e_1, e_6)\}e_4.$$

Since $(\nabla_{e_1} J)e_1$ and e_4 are orthogonal, we have

$$\left\{\frac{\tau + 10\mu}{8} - \rho(e_5, e_5)\right\}(\nabla_{e_1} J)e_1 = \frac{1}{2}\{(\gamma - 3\epsilon)\rho(e_1, e_5) + (\delta - 3\mu)\rho(e_1, e_6)\}e_3, \tag{3.16}$$

$$(\delta - 3\mu)\rho(e_1, e_5) - (\gamma - 3\epsilon)\rho(e_1, e_6) = 0. \tag{3.17}$$

If we put $X = e_3$ and $W = e_5$ in (3.3), then we obtain

(3.18)

$$\frac{\tau + 10\mu}{4}\{g((\nabla_{e_3} J)e_3, e_5)e_5 + g((\nabla_{e_3} J)e_3, e_6)e_6 + 2(\nabla_{e_3} J)e_3\} - \rho((\nabla_{e_3} J)e_3, e_5)e_5 - \rho((\nabla_{e_3} J)e_3, e_6)e_6 - 2\rho(e_5, e_5)(\nabla_{e_3} J)e_3 - g((\nabla_{e_3} J)e_3, e_5)Qe_5 - g((\nabla_{e_3})e_3, e_6)Qe_6 - 2Q(\nabla_{e_3} J)e_3 = 0.$$

From (3.7) we have

$$\rho((\nabla_{e_3} J)e_3, e_5) = \frac{\tau + 10\mu}{8} g((\nabla_{e_3} J)e_3, e_5),$$

which and (3.18) imply

$$\text{(3.19)} \qquad \begin{aligned} &\left\{\frac{\tau + 10\mu}{8} - \rho(e_5, e_5)\right\} g((\nabla_{e_3} J)e_3, e_5) = 0, \\ &\left\{\frac{\tau + 10\mu}{8} - \rho(e_5, e_5)\right\} g((\nabla_{e_3} J)e_3, e_6) = 0. \end{aligned}$$

Putting $X = e_3$ and $W = e_1$ in (3.3) and using (3.7), we obtain

(3.20)
$$\begin{aligned} &\frac{\tau + 10\mu}{8} \{g((\nabla_{e_3} J)e_3, e_1)e_1 + g((\nabla_{e_3} J)e_3, e_2)e_2 + 2(\nabla_{e_3} J)e_3\} \\ &- 2\rho(e_1, e_1)(\nabla_{e_3} J)e_3 - g((\nabla_{e_3} J)e_3, e_1)Qe_1 - g((\nabla_{e_3}, e_2)Qe_2 = 0, \end{aligned}$$

which implies

$$\text{(3.21)} \qquad \begin{aligned} &\left\{\frac{\tau + 10\mu}{8} - \rho(e_1, e_1)\right\} g((\nabla_{e_3} J)e_3, e_1) = 0, \\ &\left\{\frac{\tau + 10\mu}{8} - \rho(e_1, e_1)\right\} g((\nabla_{e_3} J)e_3, e_2) = 0. \end{aligned}$$

From (3.14) we obtain, using (3.8),

$$\text{(3.22)} \qquad \begin{aligned} &\left\{\rho(e_1, e_1) - \frac{\tau + 10\mu}{8}\right\} (3\epsilon - \gamma) = 2\rho(e_1, e_5)\xi, \\ &\left\{\rho(e_1, e_1) - \frac{\tau + 10\mu}{8}\right\} (3\mu - \delta) = 2\rho(e_1, e_6)\xi. \end{aligned}$$

If $(\nabla_{e_3} J)e_3 \neq 0$, then we have $\rho(e_1, e_1) = \frac{\tau + 10\mu}{8}$ or $\rho(e_5, e_5) = \frac{\tau + 10\mu}{8}$ by the help of (3.19) and (3.21). If $\rho(e_1, e_1) = \frac{\tau + 10\mu}{8}$, then we obtain from (3.22) $\rho(e_1, e_5) = \rho(e_1, e_6) = 0$. If $\rho(e_5, e_5) = \dfrac{\tau + 10\mu}{8}$, then we obtain, from (3.16) and (3.17)

$$\begin{aligned} &\left\{(\delta - 3\mu)^2 + (\gamma - 3\epsilon)^2\right\} \rho(e_1, e_5) = 0, \\ &\left\{(\delta - 3\mu)^2 + (\gamma - 3\epsilon)^2\right\} \rho(e_1, e_6) = 0, \end{aligned} \tag{3.23}$$

which imply $\rho(e_1, e_5) = \rho(e_1, e_6) = 0$ provided that $(\delta-3\mu)^2+(\gamma-3\epsilon)^2 \neq 0$. If $(\delta-3\mu)^2+(\gamma-3\epsilon)^2 = 0$, then also we can find $\rho(e_1, e_5) = \rho(e_1, e_6) = 0$ from (3.22).

Now suppose that $(\nabla_{e_3} J)e_3 = 0$. Then we can write

$$(\nabla_{e_3})e_1 = \alpha e_5 + \beta e_6. \tag{3.24}$$

Since $\rho((\nabla_{e_3} J)e_1, e_1) = \rho((\nabla_{e_3} J)e_1, Je_1) = 0$, we find from (3.24)

$$\alpha\rho(e_1, e_5) + \beta\rho(e_1, e_6) = 0, \qquad \beta\rho(e_1, e_5) - \alpha\rho(e_1, e_6) = 0,$$

which imply $(\alpha^2 + \beta^2)\rho(e_1, e_5) = (\alpha^2 + \beta^2)\rho(e_1, e_6) = 0$. If $\alpha^2 + \beta^2 \neq 0$, then we get $\rho(e_1, e_5) = \rho(e_1, e_6) = 0$. If $\alpha^2 + \beta^2 = 0$, then $(\nabla_{e_3} J)e_1 = 0$. If we put $W = e_1$ and $X = e_3$ in (3.13) then we have, by the help of (3.8),

$$Q(\nabla_{e_1} J)e_3 = \frac{\tau + 10\mu}{8}(\nabla_{e_1} J)e_3,$$

which and (3.11) give

$$\begin{aligned} &\xi\rho(e_1, e_1) + \gamma\rho(e_1, e_5) + \delta\rho(e_1, e_6) = \frac{\tau + 10\mu}{8}\xi, \\ &\xi\rho(e_1, e_5) + \gamma\rho(e_5, e_5) = \frac{\tau + 10\mu}{8}\gamma, \\ &\xi\rho(e_1, e_6) + \delta\rho(e_6, e_6) = \frac{\tau + 10\mu}{8}\delta, \\ &\delta\rho(e_1, e_5) = \gamma\rho(e_1, e_6). \end{aligned} \tag{3.25}$$

If we put $X = e_1$ and $W = e_3$ in (3.13) and take account of (3.11), we obtain

$$\begin{aligned} &\frac{\tau + 10\mu}{4}(\gamma e_5 + \delta e_6) - \rho(e_1, e_1)(\xi e_1 + \gamma e_5 + \delta e_6) \\ &+ \xi\rho(e_3, e_3)e_1 - Q(\gamma e_5 + \delta e_6) = 0, \end{aligned}$$

which implies

$$\gamma\{\tau + 10\mu - 4\rho(e_1, e_1) - 4\rho(e_5, e_5)\} = 0,$$
$$\delta\{\tau + 10\mu - 4\rho(e_1, e_1) - 4\rho(e_5, e_5)\} = 0. \tag{3.26}$$

If $\gamma^2 + \delta^2 \neq 0$, then we obtain, by the help of (3.26),

$$\rho(e_1, e_1) + \rho(e_5, e_5) = \frac{\tau + 10\mu}{4}. \tag{3.27}$$

From (3.25) we obtain

$$\gamma\xi\rho(e_1, e_1) + \gamma^2\rho(e_1, e_5) + \gamma\delta\rho(e_1, e_6) = \frac{\tau + 10\mu}{8}\gamma\xi,$$
$$\gamma\xi\rho(e_5, e_5) + \xi^2\rho(e_1, e_5) = \frac{\tau + 10\mu}{8}\gamma\xi,$$

from which and (3.25) and (3.27), we have $(\xi^2 + \gamma^2 + \delta^2)\rho(e_1, e_5) = 0$. Similarly, we have $(\xi^2 + \gamma^2 + \delta^2)\rho(e_1, e_6) = 0$. Hence we get $\rho(e_1, e_5) = \rho(e_1, e_6) = 0$.

If $\gamma^2 + \delta^2 = 0$, then we have from (3.25) $\rho(e_1, e_5) = \rho(e_1, e_6) = 0$.

Summing up, we can conclude that $Q = \dfrac{\tau + 10\mu}{8} I$ at p and hence

$$\tau = 30\mu \tag{3.28}$$

holds at p. Since τ and μ are constants on M, the relation $\tau = 30\mu$ holds whole on M. Thus we have $Q = \dfrac{\tau}{6} I$ at such a point p.

Now suppose that $(\nabla_Z J)Z = 0$ for any vector field Z at some point $q \in M$. We take arbitrary orthogonal holomorpinc planes $\{X, JX\}$ and $\{W, JW\}$. Then from (3.3) and $(\nabla_W J)X + (\nabla_X J)W = 0$ we obtain

$$\rho(X, W)(\nabla_W J)X - \rho(X, JW)J(\nabla_W J)X = 0,$$

which implies

$$\rho(X, W)(\nabla_W J)X = 0. \tag{3.29}$$

Assume that $\rho(X, W) \neq 0$ at q. Then we have $(\nabla_W J)X = (\nabla_X J)W = 0$ from (3.29) and hence we obtain, by the help of (2.4),

$$R(X, W, Y, Z) - R(X, W, JY, JZ) = 0 \tag{3.30}$$

for any vector fields Y and Z at q. If we put $Y = X$ and $Z = W$ in (3.30) and use (3.1), then we find

$$\rho(X, X) + \rho(W, W) = \frac{\tau}{4} + \mu. \tag{3.31}$$

If we take another holomorphic plane $\{Y, JY\}$ which is orthogonal to $\{X, JX\}$ and $\{X, JW\}$ respectively, then we find from (3.30) and (3.1)

$$\begin{aligned} &\rho(X, JY)g(W, JZ) - \rho(X, Y)g(W, Z) \\ &\qquad - g(X, JZ)\rho(W, JY) + g(X, Z)\rho(W, Y) = 0 \end{aligned} \tag{3.32}$$

for every vector field Z at q. If we put $Z = X$, W in (3.32) respectively, we get

$$\rho(W, Y) = \rho(X, Y) = 0. \tag{3.33}$$

For the orthogonal holomorphic planes $\left\{\frac{X+Y}{\sqrt{2}}, J\frac{X+Y}{\sqrt{2}}\right\}$ and $\left\{\frac{X-Y}{\sqrt{2}}, J\frac{X-Y}{\sqrt{2}}\right\}$, we obtain from (3.29)

$$\{\rho(X, X) - \rho(Y, Y)\}\{(\nabla_Y J)X - (\nabla_X J)Y\} = 0.$$

If $\rho(X, X) \neq \rho(Y, Y)$ at q, then we have $(\nabla_Y J)X = (\nabla_X J)Y$ at q. Since $(\nabla_Y J)X + (\nabla_X J)Y = 0$ at q, we have $(\nabla_Y J)X = (\nabla_X J)Y = 0$ at q. By the same arguments as the preceding paragraph, we have $\rho(X, W) = \rho(Y, W) = 0$. This contradicts to the assumption $\rho(X, W) \neq 0$. Hence $\rho(X, X) = \rho(Y, Y)$. Similarly, for the orthgonal holomorphic planes $\left\{\frac{W+Y}{\sqrt{2}}, J\frac{W+Y}{\sqrt{2}}\right\}$ and $\left\{\frac{W-Y}{\sqrt{2}}, J\frac{W-Y}{\sqrt{2}}\right\}$, we have $\rho(W, W) = \rho(Y, Y)$. Therefore we find, by the help of (3.31), $\tau = 12\mu$ and hence we get $\mu = 0$ from (3.28). This contradicts to our hypothesis $\mu \neq 0$. Hence $\rho(X,W) = 0$ for any orthoghonal holomorphic planes $\{X, JX\}$ and $\{W, JW\}$. Therefore we get $\rho(X,W) = \rho(X,Y) = \rho(W,Y) = \cdots = \rho(JW,JY) = 0$ for the orthogonal holomorphic planes $\{X,JX\},\{W,JW\}$ and $\{Y,JY\}$. For the orthogonal holomorphic planes $\left\{\frac{X+W}{\sqrt{2}}, J\frac{X+W}{\sqrt{2}}\right\}$ and $\left\{\frac{X-W}{\sqrt{2}}, J\frac{X-W}{\sqrt{2}}\right\}$ we have $\rho\left(\frac{X+W}{\sqrt{2}}, \frac{X-W}{\sqrt{2}}\right) = 0$. Hence we have $\rho(X,X) = \rho(W,W)$. Similarly, we obtain $\rho(X,X) = \rho(Y,Y)$. Hence we get $\rho(X,X) = \rho(Y,Y) = \rho(W,W) = \rho(JX,JX) = \rho(JY,JY) = \rho(JW,JW) = \lambda$. Therefore, we have $Q = \lambda I$ at q. Since M is connected we obatin $\lambda = \frac{\tau}{6}$. Summing up, we have $Q = \frac{\tau}{6}I$ whole on M and hence M is Einsteinian. From Theorem A and the hypothesis that M is not nearly Kaehlerian, we can conclude that M is of constant sectional curvature μ. Thus we have the following

THEROEM 1. *Let M be a 6-dimensional connected quasi-Kaehler manifold with pointwise constant holomorphic sectional curvature $\mu \neq 0$ and with curvature identity*

$$(*) \qquad R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$$

for all vector fields X, Y, Z and W. If M is not nearly Kaehlerian and the Ricci tensor of M is parallel, then M is of constant sectional curvature μ.

On the other hand, if M is nearly Kaehlerian and is of constant holomrophic sectional cuvature, then M is a manifold of constant sectional curvature or a complex space form ([5]). Combining this fact and theorem 1, we have the following

THEOREM 2. *Let M be a 6-dimensional connected quasi-Kaehler manifold with pointwise constant holomorphic sectional curvature $\mu \neq 0$ and with curvature identity*

$$(*) \qquad R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$$

for all vector fields X, Y, Z and W. If the Ricci tensor of M is parallel, then M is one of the following:

(a) *a manifold of constant sectional curvature μ,*

(b) *a complex space form.*

REMARK. In [2], A. Gray and L. Vanhecke announced as a theorem with false proof that any dimensional quasi-Kaehler manifold with pointwise constant holomorphic sectional curvature $\mu(\neq 0)$ and with curvature identity $(*)$ is nearly Kaehlerian. They used lemma 3.3 ([2]) as an essential tool to assert the above result, but it is erroneous.

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