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2-TYPE SURFACES WITH 1-TYPE GAUSS MAP

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ABSTRACT. It is well-known that a null 2-type surface in 3-dimensional Euclidean space E^3 is an open portion of circular cylinder. In this article we prove that a surface with 2-type and 1-type Gauss map in E^3 is in fact of null 2-type and thus it is an open portion of circular cylinder.

1. Introduction

The study of submanifolds of finite type began in the late seventies through the B.- Y. Chen's attempts to find the best possible estimate of the total mean curvature of compact submanifolds of Euclidean space and to find a notion of degree for submanifolds of Euclidean space.

Since then many works were done to characterize or classify submanifolds in terms of finite type. However, the class of submanifolds of finite type is very large. Furthermore, there are still many unknown classes of finite type submanifolds of Euclidean space or pseudo-Euclidean space. One of such classes is the finite type surfaces in Euclidean 3-space.

On the other hand, the study of finite type submanifolds provides a natural way to combine spectral theory with the geometry of submanifolds and also with the geometry of smooth maps, in particlar, Gauss map.

In this article, we give a partial solution for that matter, that is, we prove that 2-type surfaces with 1-type Gauss map in Euclidean 3-space is circular cylinders.

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2. 2-type surfaces with 1-type Gauss map

Let M be a 2-type surface of E^3 with 1-type Gauss map. We denote by h, A, H, ∇ and D the second fundamental form, the shape operator, the mean curvature vector, the Riemannian connection and the normal connection of M which are induced from the geometric structure in E^3. Let $\tilde{\nabla}$ be the Levi-Civita connection of E^3 compatible with the natural inner product $<,>$. Let G be the Gauss map of M into $G(2,3)$ which is the Grassmannian manifold of the oriented 2-planes in E^3. Also, $G(2,3)$ can be identified with the decomposable 2-vectors of norm 1 in 3-dimensional Euclidean space $\wedge^2 E^3 \cong E^3$. Let $\{e_1, e_2, e_3\}$ be a moving frame over M such that e_1 and e_2 are principle directions of M and e_3 is the unit normal vector field to M satisfying $H = \alpha e_3$, where α is the mean curvature of M, that is, $\alpha = \frac{\mu_1+\mu_2}{2}$ and $Ae_i = \mu_i e_i \quad (i = 1, 2)$. Then, $G: M \to G(2,3)$ can be given by $G(p) = (e_1 \wedge e_2)(p)$, $p \in M$. Since M has 1-type Gauss map, there exist a constant λ and a constant vector a such that

$$\Delta G - \lambda(G - a) = 0, \tag{1}$$

where Δ is the Laplace operator on M. As is well-known, the Beltrami equation for the Laplacian is given by

$$\Delta x = -2H. \tag{2}$$

Suppose that a surface M is of nonnull 2-type in E^3. Then the position vector field x of M has the spectral decomposition : $x = x_0 + x_p + x_q$ for a constant vector field x_0 and non-constant vector fields x_p and x_q satisfying $\Delta x_p = \lambda_p x_p$ and $\Delta x_q = \lambda_q x_q$ for some constants λ_p and λ_q. Then, the Beltrami equation (2) implies

$$-2H = \lambda_p x_p + \lambda_q x_q. \tag{3}$$

We now put

$$x_p = f_1 e_1 + f_2 e_2 + f_3 e_3 \quad \text{and} \quad x_q = g_1 e_1 + g_2 e_2 + g_3 e_3 \tag{4}$$

for some smooth functions f_A and g_A $(A = 1,2,3)$. The last two equations together with the equation (3) yield

$$(5) \qquad \lambda_p f_i + \lambda_q g_i = 0 \quad (i = 1,2) \quad \text{and} \quad -2\alpha = \lambda_p f_3 + \lambda_q g_3.$$

Since M is of 2-type, it is well known that there exist constants b and c satisfying

$$(6) \qquad \Delta H + bH + c(x - x_0) = 0,$$

where $b = \lambda_p + \lambda_q$ and $c = \frac{\lambda_p \lambda_q}{2}$ (For detail, see [4]). Let $\omega^1, \omega^2, \omega^3$ be the dual 1-forms to e_1, e_2 and e_3 and ω_A^B the connection forms associated with $\omega^1, \omega^2, \omega^3$ satisfying $\omega_A^B + \omega_B^A = 0$ $(A, B = 1,2,3)$ and

$$(7) \qquad \tilde{\nabla}_{e_i} e_j = \sum_k \omega_j^k(e_i) e_k + h_{ji} e_3, \nabla_{e_i} e_j = \sum_k \omega_j^k(e_i) e_k,$$

$$(8) \qquad \tilde{\nabla}_{e_i} e_3 = \sum_k \omega_3^k(e_i) e_k,$$

$$(9) \qquad \mu_1 = \omega_1^3(e_1) = h_{11}, \mu_2 = \omega_2^3(e_2) = h_{22}, h_{12} = h_{21} = 0.$$

From now on, the indices A, B and C run over the range $\{1,2,3\}$ and i, j over $\{1,2\}$ unless stated otherwise. We now prove the following lemma for later use.

LEMMA. *Let M be a surface in E^3. Then, we have*

$$(10) \qquad e_i \mu_j = (\mu_i - \mu_j) w_i^j(e_j) \quad (i \neq j).$$

PROOF. Differentiating $Ae_i = \mu_i e_i$ covariantly with respect to e_j, we get

$$(\nabla_{e_j} A) e_i = (e_j \mu_i) e_i + \sum_k (\mu_i - \mu_k) w_i^k(e_j) e_k.$$

Making use of the Codazzi equation, we obtain

$$(e_i\mu_j)e_j - (e_j\mu_i)e_i = \sum_k (\mu_i - \mu_k)w_i^k(e_j)e_k - \sum_k (\mu_j - \mu_k)w_j^k(e_i)e_k,$$

which implies

$$e_i\mu_j = (\mu_i - \mu_j)w_i^j(e_j) \quad (i \neq j).$$

This completes the proof of lemma. ||||

If we compute the Laplacian of H and take account of Lemma, we have

$$\Delta H = (\Delta\alpha + \alpha \parallel h \parallel^2)e_3 + \sum_{i=1}^{2}\{2(e_i\alpha)\mu_i e_i + \alpha(e_i\mu_i)e_i + \alpha\sum_{i\neq j}(e_j\mu_i)e_j\},$$

where $\parallel h \parallel^2 = \mu_1^2 + \mu_2^2$. Substituting the last equation into (6), we may have

(11) $$\Delta\alpha + (\parallel h \parallel^2 + b)\alpha + c(f_3 + g_3) = 0,$$

(12) $$(e_i\alpha)(\mu_i + \alpha) + c(f_i + g_i) = 0.$$

Using (5), (11) and (12), we get

$$2\Delta\alpha + 2(\parallel h \parallel^2 + b - \lambda_p)\alpha + \lambda_p(\lambda_q - \lambda_p)f_3 = 0,$$

$$2\Delta\alpha + 2(\parallel h \parallel^2 + b - \lambda_q)\alpha + \lambda_q(\lambda_p - \lambda_q)g_3 = 0,$$

$$(e_i\alpha)(\mu_i + \alpha) + \lambda_p(\lambda_q - \lambda_p)f_i = 0,$$

$$(e_i\alpha)(\mu_i + \alpha) + \lambda_q(\lambda_p - \lambda_q)g_i = 0.$$

Then, these four equations give

(13) $$f_i = -\frac{(e_i\alpha)(\mu_i + \alpha)}{m}, \quad g_i = -\frac{(e_i\alpha)(\mu_i + \alpha)}{n},$$

$$(14)\quad f_3 = -\frac{2\Delta\alpha + 2(\| h \|^2 + \lambda_q)\alpha}{m}, \quad g_3 = -\frac{2\Delta\alpha + 2(\| h \|^2 + \lambda_q)\alpha}{n},$$

where $m = \lambda_p(\lambda_q - \lambda_p)$ and $n = \lambda_q(\lambda_p - \lambda_q)$. Since $\tilde{\nabla}_{e_i} x = e_i$, the equations (4), (7) and (9) imply

$$(15)\qquad e_1(f_1 + g_1) + (f_2 + g_2)\omega_2^1(e_1) - \mu_1(f_3 + g_3) = 1,$$

$$(16)\qquad e_2(f_1 + g_1) + (f_2 + g_2)\omega_2^1(e_2) = 0,$$

$$(17)\qquad (f_1 + g_1)\omega_1^2(e_1) + e_1(f_2 + g_2) = 0,$$

$$(18)\qquad (f_1 + g_1)\omega_1^2(e_2) + e_2(f_2 + g_2) - \mu_2(f_3 + g_3) = 1,$$

$$(19)\qquad \mu_1(f_1 + g_1) + e_1(f_3 + g_3) = 0.$$

Since M has 1-type Gauss map, (1) implies

$$(e_2\mu_1 + e_2\mu_2)e_1 \wedge e_3 + (e_1\mu_1 + e_1\mu_2)e_3 \wedge e_2 - (\mu_1^2 + \mu_2^2 - \lambda)e_1 \wedge e_2 = \lambda a.$$

Taking the covariant differentiation of the equation above with respect to e_1 and using (9), we have

$$(20)\qquad e_1e_2(\mu_1 + \mu_2) + e_1(\mu_1 + \mu_2)\omega_1^2(e_1) = 0,$$

$$(21)\qquad e_2(\mu_1 + \mu_2)\omega_1^2(e_1) - e_1e_1(\mu_1 + \mu_2) + \mu_1(\| h \|^2 - \lambda) = 0,$$

$$(22)\qquad \mu_1e_1(\mu_1 + \mu_2) + e_1 \| h \|^2 = 0.$$

Similarly, if we take the covariant differentiation with respect to e_2, we get

$$(23)\qquad e_2e_2(\mu_1 + \mu_2) + e_1(\mu_1 + \mu_2)\omega_1^2(e_2) - \mu_2(\| h \|^2 - \lambda) = 0,$$

$$e_2(\mu_1+\mu_2)\omega_1^2(e_2) - e_2e_1(\mu_1+\mu_2) = 0, \tag{24}$$

$$\mu_2 e_2(\mu_1+\mu_2) + e_2 \parallel h \parallel^2 = 0. \tag{25}$$

Taking account of Lemma, we obtain from (22) and (25)

$$3\mu_1 e_1\mu_1 + (\mu_1+2\mu_2)e_1\mu_2 = 0, \quad (2\mu_1+\mu_2)e_2\mu_1 + 3\mu_2 e_2\mu_2 = 0. \tag{26}$$

Using the structure equation : $d\omega_i^3 = \sum_j \omega_i^j \wedge \omega_j^3$, we have

$$e_2\mu_1 = (\mu_2-\mu_1)\omega_2^1(e_1), \tag{27}$$

$$e_1\mu_2 = (\mu_1-\mu_2)\omega_1^2(e_2), \tag{28}$$

since $\omega_1^3 = \mu_1\omega^1, \omega_2^3 = \mu_2\omega^2$ and $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$.

We now consider a subset $M_0 = \{p \in M \mid \mu_1(p) \neq 0, \mu_2(p) \neq 0, \mu_1(p) \neq \mu_2(p)\}$. Suppose $M_0 \neq \phi$. On M_0, (25) - (28) imply

$$e_1\mu_1 = -\frac{(\mu_1+2\mu_2)(\mu_1-\mu_2)}{3\mu_1}\omega_1^2(e_2), \tag{29}$$

$$e_2\mu_2 = \frac{(\mu_2+2\mu_1)(\mu_1-\mu_2)}{3\mu_2}\omega_2^1(e_1). \tag{30}$$

If we substitute (27) and (30) into (20) and make use of (28) and (29), a direct computation implies

$$e_1(\omega_2^1(e_1)) = \frac{3\mu_1^2+5\mu_1\mu_2+7\mu_2^2}{3\mu_1\mu_2}\omega_2^1(e_1)\omega_1^2(e_2). \tag{31}$$

Putting (27) and (30) into (23), we obtain

(32)

$$e_2(\omega_2^1(e_1)) = -\frac{\mu_2}{\mu_1}(\omega_1^2(e_2))^2 + \frac{2\mu_1^2+3\mu_1\mu_2+7\mu_2^2}{3\mu_2^2}(\omega_2^1(e_1))^2 - \frac{3\mu_2^2}{2(\mu_1-\mu_2)^2}(\lambda-\mu_1^2-\mu_2^2).$$

Similarly, (21) and (27) - (30) produce

(33)
$$e_1(\omega_1^2(e_2)) = -\frac{\mu_1}{\mu_2}(\omega_2^1(e_1))^2 + \frac{7\mu_1^2+3\mu_1\mu_2+2\mu_2^2}{3\mu_1^2}(\omega_1^2(e_2))^2 - \frac{3\mu_1^2}{2(\mu_1-\mu_2)^2}(\lambda-\mu_1^2-\mu_2^2).$$

On the other hand, (24) together with (27) - (30) yields

$$e_2(\omega_1^2(e_2)) = \frac{7\mu_1^2+5\mu_1\mu_2+3\mu_2^2}{3\mu_1\mu_2}\omega_2^1(e_1)\omega_1^2(e_2). \tag{34}$$

Substituting (13) into (17) and making use of (27) - (31), we have after a long computation

$$-\frac{4(\mu_1-\mu_2)^4}{9\mu_1\mu_2}\omega_2^1(e_1)\omega_1^2(e_2)(\frac{1}{m}+\frac{1}{n}) = 0.$$

It follows that
$$\omega_2^1(e_1)\omega_1^2(e_2) = 0$$
on M_0. Let M_1 be the subset of M defined by $\{p \in M_0 \mid \omega_1^2(e_2) \neq 0\}$. Suppose $M_1 \neq \phi$. Then $\omega_2^1(e_1) = 0$ on M_1. Thus, $e_2\mu_1 = e_2\mu_2 = 0$ on the open set M_1. On M_1, (21) and (22) become

$$e_1e_1(\mu_1+\mu_2) = \mu_1(\mu_1^2+\mu_2^2-\lambda), \tag{35}$$

$$\mu_1e_1(\mu_1+\mu_2) + e_1(\mu_1^2+\mu_2^2) = 0. \tag{36}$$

Since $0 = e_2(\omega_2^1(e_1))$, it follows that

$$(\omega_2^1(e_2))^2 = \frac{3\mu_1\mu_2}{2(\mu_1-\mu_2)^2}(\mu_1^2+\mu_2^2-\lambda) \tag{37}$$

on M_1. Differentiating (35) covariantly with respect to e_1 and making use of (23) with $e_2\mu_1 = e_2\mu_2 = 0$, (33), (35) and (37), we get

$$\frac{\mu_2(\mu_1 + 2\mu_2)}{3\mu_1(\mu_1 - \mu_2)}(\lambda - \| h \|^2)(2\mu_1 + \mu_2)(2\mu_1 - 1) = 0.$$

It follows that $(\mu_1 + 2\mu_2)(2\mu_1 + \mu_2) = 0$ on M_1. Let $M_2 = \{p \in M_1 \mid (2\mu_1 + \mu_2)(p) \neq 0\}$. If $M_2 \neq \phi$, then $\mu_1 + 2\mu_2 = 0$ on M_2. Differentiating this covariantly with respect to e_1 and using (28) and (29), we get $\mu_1 = \mu_2$, which is a contradiction. Hence M_2 must be empty and $2\mu_1 + \mu_2 = 0$ on M_1. If we take the covariant differentiation of $2\mu_1 + \mu_2 = 0$ in the direction of e_1, then we can get $\mu_1 = 0$ and $\mu_2 = 0$ on M_1. It contradicts the poperty of μ_1 and μ_2 on M_0. Therefore, the subset M_1 is empty, that is, $\omega_1^2(e_2) = 0$ on M_0. By developing the same argument as we did before, we get $\omega_1^2(e_1) = 0$ on M_0. Thus, μ_1 and μ_2 are constant on each component of M_0 and we can prove that μ_1 and μ_2 are globally constant by the same argument in the proof of Theorem 3.2 in [5]. Consequently, the possible cases are that M is an open portion of planes, spheres and circular cylinders in E^3. Since M is of nonnull 2-type, it is impossible. Hence, M must be of null 2-type. In the sequel, 2-type surfaces of E^3 with 1-type Gauss map are null 2-type. Thus, we have

THEOREM A. *Circular cylinders are the only connected 2-type surfaces with 1-type Gauss map in E^3.*

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