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MULTIPLIERS IN THE FOURIER TRANSFORM OF DISTRIBUTIONS OF RAPID GROWTH

DAE HYEON PAHK, BYUNG KEUN SOHN AND SUN WOO IM

ABSTRACT. Let $\mathcal{K}'_M$ be the space of distributions on R^n which grow no faster than $e^{M(kx)}$ for some $k > 0$ and an index function $M(x)$ and K'_M be the Fourier transform of $\mathcal{K}'_M$. We establish the characterizations of the space $\mathcal{O}_M(K'_M; K'_M)$ of multipliers in K'_M.

Let $\mathcal{K}'_M$ be the space of distributions on R^n which grow no faster than $e^{M(kx)}$ for some $k > 0$ and an index function $M(x)$ and K'_M be the Fourier transform of $\mathcal{K}'_M$. In [4], we established the characterizations of the space $\mathcal{O}_M(\mathcal{K}'_M; \mathcal{K}'_M)$ of multipliers in $\mathcal{K}'_M$ and introduced the space $\mathcal{O}_M(K'_M; K'_M)$ of multipliers in K'_M. Therein we also investigated the relation between $\mathcal{O}_M(K'_M; K'_M)$ and the space $\mathcal{O}'_C(\mathcal{K}'_M; \mathcal{K}'_M)$ of convolution operators in $\mathcal{K}'_M$ by the Fourier transform. In this note, we establish the characterizations of $\mathcal{O}_M(K'_M; K'_M)$. This is a continuation of our works in [4].

Before presenting our theorems, we recall the basic facts about the space $\mathcal{K}'_M$ and K'_M. For further details we refer to [3].

The space $\mathcal{K}'_M$. Let $\mu(\xi)(0 \leq \xi \leq \infty)$ denote a continuous increasing function such that $\mu(0) = 0, \mu(\infty) = \infty$. For $x \geq 0$, we define

$$M(x) = \int_0^x \mu(\xi) d\xi.$$

The function $M(x)$ is an increasing, convex and continuous function with $M(0) = 0$, $M(\infty) = \infty$. For $x < 0$, we define $M(x)$ to be $M(-x)$

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and for $x = (x^1, x^2, \cdots, x^n) \in R^n, n \geq 2$, we define $M(x)$ to be $M(x^1) + M(x^2) + \cdots + M(x^n)$. Now we list some properties of $M(x)$ which will be used in the proof,

$$(i) \quad M(x) + M(y) \leq M(x+y) \quad \textit{for all } x, y \geq 0$$

$$(ii) \quad M(x+y) \leq M(2x) + M(2y) \quad \textit{for all } x, y \geq 0$$

Let $\mathcal{K}_M$ be the space of all C^∞-functions ϕ in R^n such that

$$\mathcal{V}_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leq k}} e^{M(kx)} \mid D^\alpha \phi(x) \mid < \infty, \quad k = 1, 2, 3, \cdots$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $D_j = i^{-1}(\frac{\partial}{\partial x_j})$. Provided with the topology defined by the seminorms $\mathcal{V}_k, \mathcal{K}_M$ is a Frechet space. The dual $\mathcal{K}'_M$ of $\mathcal{K}_M$ is the space of all continous linear functionals on $\mathcal{K}_M.\mathcal{K}'_M$ is endowed with the topology of uniform convergence on all bounded sets in $\mathcal{K}_M$.

The space K'_M. For $\phi \in \mathcal{K}_M$, the Fourier transform

$$\widehat{\phi}(\xi) = \int_{R^n} e^{-i<x,\xi>} \phi(x) dx$$

can be continued in C^n as an entire function of $\zeta = \xi + i\eta \in C^n$ such that

$$\omega_k(\widehat{\phi}) = \sup_{\zeta \in C^n} (1 + \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \mid \widehat{\phi}(\zeta) \mid < \infty, \ k = 1, 2, \cdots \tag{1}$$

where $\Omega(y)$ is the dual of $M(x)$ in the sense of Young. $\Omega(y)$ has the same properties with $M(x)$. If K_M is the space of all entire functions with the property (1) and the topology in K_M is defined by the seminorms ω_k, then the Fourier transform is an isomorphism of $\mathcal{K}_M$ onto K_M. The dual K'_M of K_M is the space of the Fourier transform of distributions in $\mathcal{K}'_M$. For $u \in \mathcal{K}'_M$, the Fourier transform $\widehat{u}$ is defined by the Parseval formula. We introduced the space of multipliers in K'_M by the following in [4];

DEFINITION 1. We denote by $\mathcal{O}_M(K'_M; K'_M)$ of all $\psi \in C^\infty$ extendable over C^n as entire functions such that there exist $k_1 \in N$ and $C_1 > 0$ such that

$$|\psi(\zeta)| \leq C_1(1+|\zeta|)^{k_1} e^{-\Omega(\frac{\eta}{k_1})},$$

where $\zeta = \xi + i\eta \in C^n$.

We define on $\mathcal{O}_M(K'_M; K'_M)$ the topology, τ, by the family of seminorms

$$\sigma_{\psi,k}(g) = \sup_{\zeta \in C^n} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})} |\psi \cdot g|$$

where $\psi \in K_M$. For $\psi \in \mathcal{O}_M(K'_M; K'_M)$ and $T \in K'_M$, we define the product ψT by

$$< \psi T, g > = < T, \psi g > \ for \ \ g \in K_M.$$

We present the characterizations of the element of $\mathcal{O}_M(K'_M; K'_M)$ which correspond to that of the element of $\mathcal{O}_M(\mathcal{K}'_M; \mathcal{K}'_M)$ in [1] [4].

THEOREM 2. *Let ψ be an entire function. The following statements are equivalent :*

(a) *There exist an integer k_1 and a positive real number C_1 such that* $|\psi(\zeta)| \leq C_1(1+|\zeta|)^{k_1} e^{-\Omega(\frac{\eta}{k_1})}$.
(b) *The linear mapping $g \to \psi g$ from K_M into K_M is continous.*
(c) *The linear mapping $T \to \psi T$ from K'_M into K'_M is continous.*

PROOF. This will follow the following implication (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). This implication follows Theorem 7 in [4].

(b) $\Rightarrow$ (c). For $\psi \in \mathcal{O}_M(K'_M; K'_M)$, ψT is well-defined as a linear functional on K_M by the definition of ψT. Since $\mathcal{K}_M$ is bornological in [2] and $K_M$is topologically isomorphic to $\mathcal{K}_M, K_M$ is also bornological. Hence to show the continuity of ψT as a map from K_M to C, it suffices to show that it is sequentially continuous. Let $\{g_i\}$ be a sequence in K_M which converges to 0. By hypothesis and the continuity of $T \in K'_M$,

$$< \psi T, g_i > = < T, \psi g_i > \to 0$$

Hence ψT is a continuous linear functional on K_M. And by the hypothesis, the map $T \to \psi T$ is sequentially continuous. But since $\mathcal{K}'_M$ is bornological in [2] and K'_M is topologically isomorphics to $\mathcal{K}'_M$, K'_M is also bornological. Thus the map $T \to \psi T$ from K'_M into K'_M is continuous.

(c) $\Rightarrow$ (b). Define $\Lambda_g : T \to < \psi T, g > = < T, \psi g >$ for every $g \in K_M$. By the hypothesis, Λ_g is continous. Hence $\Lambda_g \in K''_M$. Since $\mathcal{K}_M$ is reflexive in [2], K_M is also reflexive. Hence there exist $\phi \in K_M$ so that $\Lambda_g(T) = < \psi T, g > = < T, \psi g > = < T, \phi >$ for all $T \in K'_M$, i.e., $\psi g = \phi \in K_M$ for all $\phi \in K_M$. Since K_M is metrizable and reflexive and $\psi T \in K'_M$ for all $T \in K'_M$, the map $g \to \psi g$ from K_M into K_M is continuous.

(b) $\Rightarrow$ (a). We remark that the condition (a) is implied by the following condition ; there exist $k_1 \in N$ and $L > 0$ so that for $\zeta = \xi + i\eta \in C^n$ with $| \zeta | > L$

$$| \psi(\zeta) | \leq (1 + | \zeta |)^{k_1} e^{-\Omega(\frac{\eta}{k_1})} \tag{2}$$

Indeed, if (2) holds, (a) is satisfied for $C_1 = \max\{\sup_{\substack{\zeta \in C^n \\ |\zeta| \leq L}} | \psi(\zeta) |, 1\}$. The proof will be by contradiction. Suppose (2) does not hold. Hence by induction, we can find a sequence $\{\zeta_j\}$ in C^n with $| \zeta_{j+1} | \geq | \zeta_j | + 2$ such that

$$| \psi(\zeta_j) | > (1 + | \zeta_j |)^{2j} e^{-\Omega(\frac{\eta_j}{2j})}$$

where $\zeta_j = \xi_j + i\eta_j \in C^n$. Let $\gamma \in C_c^\infty(C^n)$ be such that $0 \leq \gamma \leq 1$, $supp\gamma \subset \bar{B}(0,1)$, the ball centered 0 with radius 1, and $\gamma(0) = 1$. For $\zeta = (\xi^1 + i\eta^1, \xi^2 + i\eta^2, \cdots, \xi^n + i\eta^n)$ and $\zeta_j = (\xi_j^1 + i\eta_j^1,\ \xi_j^2 + i\eta_j^2,\ \cdots,\ \xi_j^n + i\eta_j^n)$, we define

$$\phi_j(\zeta) = \sum_{j=1}^{\infty} \frac{\gamma(\zeta - \zeta_j)}{(1 + | \zeta_j |)^j} \cdot e^{\Omega(\frac{\eta_j}{j})},$$

where

$$\zeta - \zeta_j = ((\xi^1 + i\eta^1) - (\xi_j^1 + i\eta_j^1), \cdots, (\xi^n + i\eta^n) - (\xi_j^n + i\eta_j^n)).$$

The sum is well-defined, since the support of the functions $\gamma(\zeta - \zeta_j)$ are disjoint. Then ϕ_j is an entire function for all $j = 1, 2, 3, \cdots$. Now we will prove that (ϕ_j) converges to 0 in K_M as $j \to \infty$, but $(\phi_j \psi)$ does not converges to 0 in K_M as $j \to \infty$. Here a sequence (ϕ_j) converges to 0 in K_M as $j \to \infty$ means that the norms of ϕ_j in K_M, $\omega_k(\phi_j)$, converges to 0 as $j \to \infty$. Indeed, by the properties of $\Omega(y)$ which is the same as $M(x)$ and the support of γ,it follows that for any $k \in N$,

$$\begin{aligned}
\omega_k(\phi_j) &= \sup_{\zeta}(1+ \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \mid \phi_j(\zeta) \mid \\
&= \sup_{\zeta}(1+ \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \Big(\frac{\gamma(\zeta - \zeta_j)}{(1+ \mid \zeta_j \mid)^j} \cdot e^{\Omega(\frac{\eta_j}{j})} \Big) \\
&\leq \sup_{\zeta}(1+ \mid \zeta - \zeta_j \mid)^k \gamma(\zeta - \zeta_j)(1+ \mid \zeta_j \mid)^{k-j} e^{-\Omega(|\frac{\eta}{k}| - |\frac{\eta_j}{j}|)} \\
&\leq 2^k (1+ \mid \zeta_j \mid)^{k-j} \\
&\to 0 \quad as \quad j \to \infty.
\end{aligned}$$

But since we have

$$\begin{aligned}
&(1+ \mid \zeta_j \mid)^k e^{-\Omega(\frac{\eta_j}{k})} \mid \phi_j(\zeta_j) \psi(\zeta_j) \mid \\
&\geq (1+ \mid \zeta_j \mid)^k e^{-\Omega(\frac{\eta_j}{k})} (1+ \mid \zeta_j \mid)^j e^{-\Omega(\frac{\eta_j}{2j}) + \Omega(\frac{\eta_j}{j})} \\
&\geq (1+ \mid \zeta_j \mid)^{k+j} e^{\Omega((\frac{1}{k} - \frac{1}{2j})|\eta_j|)} \\
&\to \infty \quad as \quad j \to \infty,
\end{aligned}$$

$\omega_k(\phi_j \psi)$ does not bounded. Hence $\phi_j \psi$ does not converges to 0 in K_M as $j \to \infty$. This contradicts the condition (b).

From the above theorem, we can provide $\mathcal{O}_M(K'_M; K'_M)$ with two more topologies. The first one, τ_b, is the topology induced on it by the space of all continuous linear maps from K_M into K_M provided with the topology of uniform convergence on bounded subsets of K_M. The second one, τ'_b, is the topology induced on it by the space of all continuous linear maps from K'_M into K'_M provided with the topology of uniform convergence on bounded subset K_M. Moreover, we can also provided $\mathcal{O}_M(K'_M; K'_M)$ with τ_p, the topology of pointwise convergence on K_M. Since $K_M(resp.K'_M)$ has the same topological properties with $\mathcal{K}_M(resp.\mathcal{K}'_M)$ and $\omega_k(\psi g) = \sigma_{g,k}(\psi)$ for $\psi \in \mathcal{O}_M(K'_M; K'_M)$ and $g \in K_M$, we have the following theorem by the same method in [1].

THEOREM 3. *The topologies τ, τ_p, τ_b and τ_b' on $\mathcal{O}_M(K_M';K_M')$ are equivalent.*

THEOREM 4. *We have the inclusions*

$$K_M \hookrightarrow \mathcal{O}_M(K_M';K_M')$$

with continuous imbeddings. Moreover K_M is dense in $\mathcal{O}_M(K_M';K_M')$.

PROOF. Let $\psi \in \mathcal{O}_M(K_M';K_M')$ and $\phi, g \in K_M$. It is clear that $K_M \subset \mathcal{O}_M(K_M';K_M')$. Since K_M is a bornological space, it suffices to show that the embedding is sequentially continuous. The inequality

$$\begin{aligned}
\sigma_{\phi,k}(g) &= \sup_{\zeta}(1+\mid\zeta\mid)^k e^{-\Omega(\frac{\eta}{k})}\mid \phi(\zeta)g(\zeta)\mid \\
&\leq \left(\sup_{\zeta}(1+\mid\zeta\mid)^{2k} e^{-\Omega(\frac{\eta}{2k})}\mid \phi(\zeta)\mid\right) \\
&\quad \times \left(\sup_{\zeta}(1+\mid\zeta\mid)^{2k} e^{-\Omega(\frac{\eta}{2k})}\mid g(\zeta)\mid\right) \\
&= \omega_{2k}(\phi)\omega_{2k}(g) \\
&\leq C_\phi \omega_{2k}(g)
\end{aligned}$$

implies that the embedding is sequentially continuous. Now, we shall prove the second assertion. For $\psi \in \mathcal{O}_M(K_M';K_M')$, since ψ is also in C^∞ and C_c^∞ is a dense subspace of C^∞, there exist a sequence $\{\psi_n\}$ in C_c^∞ such that $\psi_n \to \psi$ in C^∞. Since for all $n \in N$, $\psi_n \in C_c^\infty$, $k \in N$ and $\phi \in K_M$, there exist $C_{\phi,k,n}$ which is dependent on ϕ, k and n such that

$$\sup_{\zeta}(1+\mid\zeta\mid)^k e^{-\Omega(\frac{\eta}{k})}\mid \phi(\zeta)\psi_n(\zeta)\mid \leq C_{\phi,k,n} \tag{3}$$

This inequality imply that given $\varepsilon > 0$, there exists $M > 0$ such that for all $n \in N$ and ζ with $\mid\zeta\mid > M$,

$$\sup_{\zeta}(1+\mid\zeta\mid)^k e^{-\Omega(\frac{\eta}{k})}\mid \phi(\zeta)\psi_n(\zeta)\mid \leq \varepsilon \tag{4}$$

Indeed, suppose that (4) is false. Then for some two k_0 and n_0 in N, there exist $\varepsilon_0 > 0$ and a sequence $\{\zeta_j\}$ with $\mid\zeta_j\mid \to \infty$ as $j \to \infty$ such that

$$(1+ \mid \zeta_j \mid)^{k_0} e^{-\Omega(\frac{\eta_j}{k_0})} \mid \phi_0(\zeta_j)\psi_{n_0}(\zeta_j) \mid \geq \varepsilon_0,$$

where $\zeta_j = \xi_j + i\eta_j \in C^n$. For $k'_0 \geq k_0$,

$$\begin{aligned}
&(1+ \mid \zeta_j \mid)^{k'_0} e^{-\Omega \frac{\eta_j}{k'_0}} \mid \phi_o(\zeta_j)\psi_{n_0}(\zeta_j) \mid \\
&\geq (1+ \mid \zeta \mid)^{k_0} e^{-\Omega(\frac{\eta_j}{k_0})} \cdot e^{\Omega((\frac{1}{k_0} - \frac{1}{k'_0}) \mid \eta_j \mid)} \mid \phi_0(\zeta_j)\psi_{n_0}(\zeta_j) \mid \\
&\geq \varepsilon_0 e^{\Omega((\frac{1}{k_0} - \frac{1}{k'_0}) \mid \eta_j \mid)} \to \infty \quad \text{as} \;\; j \to \infty,
\end{aligned}$$

which is contradict to (3). Now since $\psi_n \to \psi$ in C^∞,

$$\sup_\zeta (1+ \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \mid \phi(\zeta)\psi(\zeta) \mid \leq C_{\phi,k,n}.$$

Hence by above process, for ζ with $\mid \zeta \mid > M$,

$$\sup_\zeta (1+ \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \mid \phi(\zeta)\psi(\zeta) \mid \leq \varepsilon. \tag{5}$$

On the other hand, since $\psi_n \to \psi$ in C^∞, ψ_n converges uniformly to f on the compact set $\{\zeta \in C^n : \mid \zeta \mid \leq M\}$. This implies that, given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$ and ζ with $\mid \zeta \mid \leq M$,

$$\sup_\zeta (1+ \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \mid \phi(\zeta)(\psi_n - \psi)(\zeta) \mid < \varepsilon. \tag{6}$$

Our last three inequalities (4), (5) and (6) implies that given $\varepsilon > 0$, there exist $n_0 \in N$ such that for all $n \geq n_0$,

$$\sup_{\zeta \in C^n} (1+ \mid \zeta \mid)^k e^{-\Omega(\frac{\eta}{k})} \mid \phi(\zeta)(\psi_n - \psi)(\zeta) \mid < \varepsilon.$$

Thus $\psi_n \to \psi$ in $\mathcal{O}_M(K'_M; K'_M)$.

REMARK. We can show that $\psi \in \mathcal{O}_M(K'_M; K'_M)$ defines an element of K'_M, i.e., $\mathcal{O}_M(K'_M; K'_M) \subset K'_M$.

Indeed, for $\phi \in K_M$ and sufficiently large k,

$$
\begin{aligned}
&|< \psi, \phi >|=| \int \psi(\zeta)\phi(\zeta)d\zeta \mid \\
&\leq \int (1+ \mid \zeta \mid)^{-k} e^{\Omega(\frac{\eta}{k})} \mid \psi(\zeta) \mid (1+ \mid \zeta \mid)^{k} e^{-\Omega(\frac{\eta}{k})} \mid \phi(\zeta) \mid d(\mid \zeta \mid) \\
&\leq \omega_k(\phi) \int (1+ \mid \zeta \mid)^{-k} e^{\Omega(\frac{\eta}{k})} \mid \psi(\zeta) \mid d(\mid \zeta \mid) \\
&\leq C_1 \omega_k(\phi) \int (1+ \mid \zeta \mid)^{-k+k_1} e^{-\Omega(|\frac{\eta}{k_1}|-|\frac{\eta}{k}|)} d(\mid \zeta \mid) \\
&= C'_1 \omega_k(\phi).
\end{aligned}
$$

But we did not succeed to prove the continuity of the imbedding $\mathcal{O}_M(K'_M; K'_M) \hookrightarrow K'_M$.

THEOREM 5. *Let $g \in K_M$. The map Λ_g from $\mathcal{O}_M(K'_M; K'_M)$ into $\mathcal{O}_M(K'_M; K'_M)$ which maps ψ to $g\psi$ is continuous.*

PROOF. Clearly the map Λ_g is well-defined. Since $\mathcal{O}_M(K'_M; K'_M)$ is bornological by the similar method in [2] and [5], it suffices to show that the map Λ_g is sequentially continuous. Since $\omega_k(g \cdot h) \leq \omega_{2k}(g)\omega_{2k}(h)$ for $g,\ h \in K_M,\ g \cdot h \in K_M$. Then the equality $\sigma_{h,k}(g\psi) = \sigma_{gh,k}(\psi),\ h \in K_M$, implies that the map Λ_g is sequentially continuous.

REMARK. Since $\omega_k(\psi g) = \sigma_{g,k}(\psi)$ for $\psi \in \mathcal{O}_M(K'_M; K'_M)$ and $g \in K_M$, the mapping $\psi \rightarrow \psi g$ from $\mathcal{O}_M(K'_M; K'_M)$ into K_M and hence the mapping $\psi \rightarrow \psi T$ from $\mathcal{O}_M(K'_M; K'_M)$ into K'_M for $T \in K'_M$ are sequentially continuous. Hence since $\mathcal{O}_M(K'_M; K'_M)$ is bornological, we have the following by combining with Theorem 2.

The bilinear mapping

$$
\begin{aligned}
\mathcal{O}_M(K'_M; K'_M) \times K_M &\rightarrow K_M \\
(\psi,\ g) &\longmapsto \psi g
\end{aligned}
$$

and

$$\begin{aligned} \mathcal{O}_M(K'_M; K'_M) \times K'_M &\to K'_M \\ (\psi, \ T) &\longmapsto \psi T \end{aligned}$$

are separately continuous.

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Dae Hyeon Pahk
Department of Mathematics
Yonsei University
Seoul 120-749, Korea

Byung Keun Sohn and Sun Woo Im
Department of Mathematics
Inje University
Kimhae 621-749, Korea