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POISSON INTEGRALS CONTAINED IN HARMONIC BERGMAN SPACES ON UPPER HALF-SPACE

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ABSTRACT. On the setting of the upper half-space, H of the euclidean n-space, we consider the question of when the Poisson integral of a function on the boundary of H is a harmonic Bergman function and here we give a partial answer.

1. Introduction

The upper half-space $H = H_n$ is the open subset of $\mathbf{R}^n (n \geq 2)$ given by

$$H = \{(x, y) \in \mathbf{R}^n : \; y > 0\},$$

where we have written a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, with $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. We also identify $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$; with this convention we have $\partial H = \mathbf{R}^{n-1}$. For $z = (x, y) \in H$ and $t \in \partial H$, set

$$P(z, t) = \frac{2}{n\sigma_n} \frac{y}{(|x - t|^2 + y^2)^{n/2}},$$

where σ_n denotes the volume of the unit ball in $\mathbf{R}^n$. The function P is called the Poisson kernel for the upper half-space. Then we can check easily that $P(\cdot, t)$ is positive and harmonic on H for each $t \in \partial H$, and that

$$\int_{\partial H} P(z, t)\, dt = 1$$

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for each $z \in H$.

For $1 \leq p < \infty$, $L^p(\partial H)$ denotes the space of Borel measurable functions on ∂H for which

$$\|f\|_p = \left(\int_{\partial H} |f(t)|^p \, dt \right)^{1/p} < \infty;$$

$L^\infty(\partial H)$ consists of the Borel measurable functions f on ∂H for which $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the essential supremum norm on ∂H with respect to Lebesgue measure. The Poisson integral of $f \in L^p(\partial H)$, for any $p \in [1, \infty]$, is the function $P[f]$ on H defined by

$$P[f](z) = \int_{\partial H} P(z,t) f(t) \, dt.$$

Because $P(z, \cdot)$ belongs to $L^q(\partial H)$ for every range of $q \in [1, \infty]$, the integral above is well-defined for any $z \in H$ and by passing the Laplacian thorough the integral above, we can show that $P[f]$ is harmonic on H.

For $1 \leq p < \infty$, we write b^p for the harmonic Bergman space consisting of all harmonic functions u on H such that

$$\|u\|_p = \left(\int_H |u|^p \, dV \right)^{1/p} < \infty$$

where dV denotes the volume measure on H, which we may write dz, dw, etc. The space b^p turns out to be a closed subspace of L^p, the Lebesgue space on H, and thus b^p is a Banach space (In particular, b^2 is a Hilbert space).

From a standard Hardy space theory, we know that for $f \in L^p(\partial H)$, $P[f]$ is a harmonic Hardy function (See [1] for details). In this paper, we consider the question of when the Poisson integral of a function $f \in L^p(\partial H)$ (with some restrictions) belongs to b^p. The case $p = 2$ is the simplest and here we give a complete answer in terms of the Fourier transform (Theorem 3.1); perhaps surprisingly, the Poisson integral of a function in $L^1(\partial H) \cap L^2(\partial H)$ always belongs to b^2 when $n > 2$, but hardly ever when $n = 2$.

2. Preliminary

In this section, we review some preliminary results from [1], [2]. By the mean value property and Jensen's inequality, one can easily verify that

$$|u(x,y)|^p \le \sigma_n^{-1} y^{-1} \|u\|_p^p \tag{2.1}$$

holds for every range of $p \in [1, \infty)$ and for all $u \in b^p$ and $(x, y) \in H$. It follows from inequality (2.1) that norm convergence in b^p implies uniform convergence on compact subsets of H. Thus, b^p is a Banach space and in particular b^2 is a Hilbert space.

For a function u on H and $\delta > 0$, let $\tau_\delta u$ denote the function on H defined by

$$\tau_\delta u(z) = u(z + (0, \delta)).$$

Now we can show easily that if $u \in b^p$, then $\tau_\delta u \to u$ in the norm of b^p as $\delta \to 0$.

Equation (2.1) also shows that if $u \in b^p$, then u is bounded on each proper half-space contained in H, hence is the Poisson integral of its boundary values on each such half-space. In other words,

$$\tau_\delta u = P[u(\cdot, \delta)] \tag{2.2}$$

on H for each $\delta > 0$ (see [1] for details). From (2.2), we can show that if $u \in b^p$, then the integrals $\int_{\partial H} |u(x,y)|^p \, dx$ increases as y decreases and hence $\tau_\delta u \in h^p$ for every $\delta > 0$, where h^p is the Hardy L^p-space of functions v harmonic on H such that

$$\|v\|_{h^p} = \sup_{y>0} \left(\int_{\partial H} |v(x,y)|^p \, dx \right)^{1/p} < \infty.$$

The Poisson integral gives us a nice way to derive an important property of b^1 Bergman functions, called the b^1 cancellation property. The proof of the following theorem can be founded in [2] but here we also give a proof of it for the reader's convenience.

THEOREM 2.1. *If $u \in b^1$, then*

$$\int_{\partial H} u(x,y)\,dx = 0$$

for every $y > 0$.

PROOF. First note that $u(\cdot,y) \in L^1(\partial H)$ for every $y > 0$, because $u \in b^1$ and $\int_{\partial H} |u(x,y)|dx$ increases as y decreases. Note also that $\tau_\delta u = P[u(\cdot,\delta)]$ for each fixed $\delta > 0$ and so

$$\begin{aligned}\int_H \tau_\delta u(z)\,dz &= \int_0^\infty \int_{\partial H} P[u(\cdot,\delta)](x,y)\,dx\,dy \\ &= \int_0^\infty \int_{\partial H} \int_{\partial H} P((x,y),s)u(s,\delta)\,ds\,dx\,dy \\ &= \int_0^\infty \int_{\partial H} u(s,\delta)\,ds\,dy,\end{aligned} \tag{2.3}$$

where we have used Fubini's theorem and the fact that

$$\int_{\partial H} P((x,y),s)dx = 1.$$

Since the inner integral in (2.3) is independent of y, we must have

$$\int_{\partial H} u(s,\delta)\,ds = 0$$

and this completes the proof. □

As an easy consequence of Theorem 2.1, we have $\int_H u(w)dw = 0$.

3. Poisson Integrals Contained in b^p

Finding functions in b^p that display specified boundary behavior is not as simple as for the Hardy-space h^p. In the latter setting, we simply design an appropriate L^p-function on ∂H and then take its Poisson integral. This suggests the following question: Given $f \in L^p(\partial H)$, when does $P[f]$ lie in b^p?

In the case $p = 2$ we give a complete answer (Theorem 3.1 below) in terms of the Fourier transform. For any $f \in L^2$, we let $\hat{f}$ denote the Fourier transform of f. Here the expression $A(f) \approx B(f)$ means that there are two positive constants c and C such that the nonnegative quantities $A(f)$ and $B(f)$ satisfy

$$cA(f) \leq B(f) \leq CA(f)$$

for all f under consideration.

THEOREM 3.1. *For* $f \in L^2(\partial H)$,

$$\|P[f]\|_2 \approx \left(\int_{\partial H} |\hat{f}(x)|^2 |x|^{-1}\, dx\right)^{1/2}. \tag{3.1}$$

PROOF. Letting $f \in L^2(\partial H)$, we can view $P[f](x, y)$ as a convolution over ∂H:

$$P[f](x,y) = P_y * f(x) = \int_{\partial H} P_y(x-s) f(s)\, ds,$$

where the definition of P_y should be clear from context. We thus have

$$\begin{aligned}\int_H |P[f](w)|^2\, dw &= \int_0^\infty \int_{\partial H} |(P_y * f)(x)|^2\, dx\, dy \\ &= \int_0^\infty \int_{\partial H} |(P_y * f)\hat{}\,(x)|^2\, dx\, dy \\ &= \int_0^\infty \int_{\partial H} |\hat{P}_y(x)\hat{f}(x)|^2\, dx\, dy.\end{aligned}$$

Now, modulo some constants (depending only on n and the normalization of the Fourier transform), we have $\hat{P}_y(x) = e^{-y|x|}$ (see [3], page 16). Reversing the order of integration in the last integral above now gives the desired result. $\square$

Theorem 3.1 shows that an interesting dichotomy occurs between the cases $n = 2$ and $n > 2$. Let $f \in L^1 \cap L^2(\partial H)$. Then $\hat{f}$ is continuous on ∂H. Integrating in polar coordinates, we see for such an f that the right side of (3.1) is always finite when $n > 2$, but is finite when $n = 2$ only if $\hat{f}(0) = 0$, i.e., only if $\int_{\partial H} f(x)\,dx = 0$.

We have not obtained necessary and sufficient conditions for Poisson integrals of L^p-functions to lie in b^p for any $p \neq 2$. However, the following result handles the case of $P[f]$ when f has compact support, at least for $1 < p < \infty$.

THEOREM 3.2. *Let $p \in (1, \infty)$, let $f \in L^p(\partial H)$, and assume that f has compact support.* (a) *If $p > n/(n-1)$, then $P[f] \in b^p$.* (b) *If $1 < p \leq n/(n-1)$, then $P[f] \in b^p$ if and only if $\int_{\partial H} f(x)\,dx = 0$.*

PROOF. Note that if $f \in L^p(\partial H)$, then a standard Hardy-space theory shows that

$$\int_0^R \int_{\partial H} |P[f](x,y)|^p\,dx\,dy \leq R \int_{\partial H} |f(x)|^p\,dx < \infty$$

for any $R > 0$. (True even if $p = 1$.) Thus we only need to worry about whether $P[f] \in L^p(\{|z| > R\} \cap H)$ for R large.

Let $\lambda = \int_{\partial H} f(x)\,dx$, set $c = 2/(n\sigma_n)$, and let K denote the support of f. For large $z = (x, y)$, we have

$$\begin{aligned} \left| P[f](z) - c\lambda \frac{y}{|z|^n} \right| &= \left| c \int_K f(s) \left(\frac{y}{|z-s|^n} - \frac{y}{|z|^n} \right) ds \right| \\ &\leq C \frac{y}{|z|^{n+1}} \int_K |f(s)|\,ds, \end{aligned}$$

where C is a constant depending on n and K. For large $R > 0$, the function $y/|z|^{n+1}$ belongs to $L^p(\{|z| > R\} \cap H)$ for all $p > 1$, while $y/|z|^n \in L^p(\{|z| > R\} \cap H)$ if and only if $p > n/(n-1)$. Both parts of the theorem now follow easily. □

The case $p = 1$ seems difficult. The following proposition indicates that the cancellation condition $\int_{\partial H} f(x)\,dx = 0$ is far from being sufficient to guarantee that $P[f] \in b^1$, even if f is smooth and has compact support.

PROPOSITION 3.3. $(n = 2)$: *If* $f \in L^1(\mathbf{R})$, f *is odd,* f *is not identically* 0, *and* $f \geq 0$ *on* $(0, \infty)$, *then* $P[f]$ *is not in* b^1.

PROOF. Let $x > 0$. Then

$$
\begin{aligned}
P[f](x,x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\frac{x}{(x-s)^2+x^2}\,ds \\
&= \frac{1}{\pi}\int_{0}^{\infty} f(s)\left(\frac{x}{(x-s)^2+x^2} - \frac{x}{(x+s)^2+x^2}\right)ds \qquad (3.2)\\
&= \frac{1}{\pi}\int_{0}^{\infty} f(s)\frac{4x^2 s}{((x-s)^2+x^2)((x+s)^2+x^2)}\,ds.
\end{aligned}
$$

By replacing s by xs in (3.2), we have

$$
\begin{aligned}
P[f](x,x) &= \frac{4}{\pi}\int_{0}^{\infty} f(xs)\frac{s}{((1-s)^2+1)((1+s)^2+1^2)}\,ds \\
&\geq \frac{4}{10\pi}\int_{0}^{1} f(xs)s\,ds \\
&= \frac{4}{10\pi x^2}\int_{0}^{x} f(s)s\,ds.
\end{aligned}
$$

Because f is not identically 0, we have $P[f](x,x) \geq C/x^2$ for large x. A similar estimate holds for rays from the origin close to the ray $y = x$. Thus for large z lying in a sector of H, we have

$$|P[f](z)| \geq \frac{C}{|z|^2},$$

which implies $P[f]$ is not in b^1. □

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