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ELLIPTIC MODULES OF RANK ONE AND CLASS FIELDS OF FUNCTION FIELDS

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ABSTRACT. We obtained some class fields associated to an order R of a function field and evaluated the valuation of the invariant $\xi(\mathfrak{c})$ for an invertible ideal $\mathfrak{c}$ of R.

0. Introduction

Let K be a global function field over a finite field $\mathbb{F}_q$, ∞ be a fixed place of degree δ, and A be the subringof K consisting of those elements which are regular outside ∞. For an order R of A Hayes[2] introduced elliptic R-module and using this generated some class fields of K explicitly. He also obtained some other class fields using sgn-normalized elliptic A-modules [3]. In this note we generalized the notion of sgn-normalized to invertible elliptic R-modules and obtained some class fields associated to R. We also evaluated the valuation of the invariant $\xi(\mathfrak{c})$ for an invertible ideal $\mathfrak{c}$ of R using the value of partial zeta function associated to $\mathfrak{c}$ in the case that the field of constants of R is equal to that of A.

1. Invertible Elliptic Modules on Orders

A subring R of A which contains 1 and has K as its field of fractions is called an *order* in A. Let $\mathfrak{f} = \{x \in K : xA \subset R\}$ be the conductor of R, and $\mathbb{F}_{q'}$ the field of constants in R. Then $\mathbb{F}_q$ is a finite extension of $\mathbb{F}_{q'}$. Let H_R be the *Hilbert class field* of R as defined in [2]. Then H_R corresponds to the subgroup $J_R = K^* \cdot \pi^{\mathbb{Z}} \cdot U_R$ of the group J_K of ideles

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of K, where π is a uniformizer at ∞ and $U_R = \prod_{v\ finite} R_v^* \times A_\infty^*$. Here R_v is the completion of R at v. Let K_∞ be the completion of K at ∞ and C the completion of the algebraic closureof K_∞. For the elementary theory of elliptic R-modules we refer to [2]. We say that an elliptic R-module (or, Drinfeld R-module) ρ of rank 1 over C is an *invertible elliptic R-module* if ρ is isomorphic to the elliptic R-module $\rho^{\mathfrak{a}}$ associated to an invertible ideal $\mathfrak{a}$ of R. Then H_R is the smallest extension field of K with the property that every invertible elliptic R-module is isomorphic to an elliptic R-module defined over H_R. From now on we mean by an elliptic R-module an invertible elliptic R-module unless otherwise stated. We denote by $Pic\,R$ the group of all the isomorphism classes of invertible ideals of R and h_R its order. Let $\mathcal{I}(\mathfrak{f})$ be the group of all the fractional ideals of A prime to $\mathfrak{f}$ and $\mathcal{P}(\mathfrak{f})$ be the subgroup of all the principal ideals xA with $x \in R$ and prime to $\mathfrak{f}$. Denote by $\mathcal{R}_\mathfrak{f}$ the qutient $\mathcal{I}(\mathfrak{f})/\mathcal{P}(\mathfrak{f})$. Then $\mathcal{R}_\mathfrak{f}$ is isomorphic to $Pic\,R$ via the map induced by $\mathfrak{a} \mapsto \mathfrak{a} \cap R$. Let h_K be the class number of the field K.Then ([2], Theorem 1.5)

$$h_R = h_K \delta \frac{q'-1}{q-1} \frac{|(A/\mathfrak{f})^*|}{|(R/\mathfrak{f})^*|}.$$

PROPOSITION 1.1. *([2] Theorem 8.10) i) $Gal(H_R/K)$ is isomorphic to $Pic\,R$.*

ii) *H_R/K is the class field to the group $\mathcal{P}(\mathfrak{f})$.*
iii) *The only places dividing $\mathfrak{f}$ can ramify in H_R/K.*
iv) *The field of constants of H_R has degree δ over $\mathbb{F}_q$.*

Denote by $\kappa(\infty)$ the residue field at ∞. Let σ be an $\mathbb{F}_{q'}$-automorphism of $\kappa(\infty)$. Then, for a sign function sgn of K_∞^*, the composite $\sigma \circ sgn$ is called a *twisting* of sgn by σ, or a *twisted sign function* which generalizes the notion of twisting in [3] Let ρ be an invertible elliptic R-module. We say that ρ is *normalized* if the leading coefficient $s_\rho(x)$ of ρ_x belongs to $\kappa(\infty)$ for any $x \in R \setminus \{0\}$. For a normalized elliptic R-module ρ, the leading coefficient map s_ρ can be extended to a twisted sign function as in the case $R = A$ (see [3]). Now fix a sign function sgn. We say that an invertible elliptic R-module ρ is *sgn-normalized* if ρ is normalized and s_ρ is equal to a twisting of sgn. Then as in the case of $R = A$ every invertible elliptic R-module is isomorphic to a sgn-normalized elliptic R-module.

LEMMA 1.2. *δ is the greatest common divisor of $deg\, x$'s, $x \in R$.*

PROOF. It is well known that δ is the greatest common divisor of $deg\, x$'s, $x \in A$. Choose $x_1, x_2, \ldots, x_r$ in A and $m_1, m_2, \ldots, m_r$ in $\mathbb{Z}$ such that

$$\delta = \sum m_i \, deg\, x_i.$$

Pick an element $y \in \mathfrak{f}$ of degree m. Let $d_i = g.c.d.(deg\, x_i, m)$. Then$D_i = \frac{deg\, x_i}{d_i} + \frac{m}{d_i} n_i$ is a large prime number for some integer n_i by the Dirichlet theorem on arithmetic progression. We can choose n_i so that D_i's are all distinct and prime to m. Then $g.c.d.\{deg\, x_i y^{n_i}\} = g.c.d.\{d_i\}$. But since $\delta \mid m$ and $\delta = g.c.d.\{deg\, x_i\}$, $\delta = g.c.d.\{d_i\}$. Now the result follows from the fact that $x_i y^n$ lies in R.

Let Γ be an R-lattice in C homothetic to some invertible ideal of R. We call such a lattice *invertible R-lattice.* We say that an invertible R-lattice Γ is *special* if its associated elliptic R-module ρ^Γ is sgn-normalized. For an invertible R-lattice Γ in C define $\xi(\Gamma)$ to be an element of C^* so that $\xi(\Gamma)\Gamma$ is special. Then $\xi(\Gamma)$ is determined up to multiplication by elements of $\kappa(\infty)^*$. For an integral ideal $\mathfrak{a}$ of R, let $\rho_\mathfrak{a}$ be the monic generator of the ideal generated by $\rho_a, a \in \mathfrak{a}$. Then the elliptic module $\mathfrak{a} * \rho$ is defined to be the unique elliptic module satisfying $(\mathfrak{a} * \rho)_x \cdot \rho_\mathfrak{a} = \rho_\mathfrak{a} \cdot \rho_x$. Then we have the following lemma whose proof is straightforward.

LEMMA 1.3. *i) For $x \in R$, we have* $\qquad (x) * \rho = s_\rho(x)^{-1} \rho s_\rho(x)$.

ii) $(\omega^{-1}\rho\omega)_\mathfrak{a} = \omega^{-q^{deg\, \mathfrak{a}}} \rho_\mathfrak{a} \omega$, *for any $\omega \in C$ and any integral ideal $\mathfrak{a}$ of R.*

iii) $s_{\mathfrak{a}*\rho} = \sigma^{deg\, \mathfrak{a}} \circ s_\rho$, *where σ is the qth power map and $\mathfrak{a}$ is an ideal of R.*

LEMMA 1.4. *Let ρ_1 and ρ_2 be two isomrphic sgn-normalized elliptic R-modules. Then*

$$s_{\rho_1} = s_{\rho_2}.$$

PROOF. Pick $c \in C$ such that $\rho_2 = c^{-1}\rho c$. Then $c^{q^\delta - 1} \in \kappa(\infty)^*$. Write $a = c^{q^\delta - 1}$. Then $s_{\rho_2}(x) = a^{deg\, x/\delta} s_{\rho_1}(x)$. Since their corresponding sign functions are the same, a must be 1 from Lemma 4.2 of [3].

LEMMA 1.5. *For each invertible elliptic R-module ρ there exist exactly, $\frac{q^{\delta}-1}{q'-1}$ distinct sgn-normalized elliptic R-modules which are isomorphic to ρ.*

PROOF. Let ρ be a sgn-normalized elliptic R-module. For each $\alpha \in \kappa(\infty)^*$, $\alpha^{-1}\rho\alpha$ is sgn-normalized. From the proof of the above lemma any sgn-normalized elliptic R-module isomorphic to ρ is of this form. Now the result follows from the fact that $\alpha^{-1}\rho\alpha = \beta^{-1}\rho\beta$ if and only if $\alpha/\beta \in \mathbb{F}_{q'}^*$.

We have the following important property of invertible R-modules. This property will be used throughout this section.

LEMMA 1.6. *Let ρ be an invertible R-module and f_R be the dimension of $\mathbb{F}_{q'}$ over $\mathbb{F}_p$. Then f_R is the greatest common divisor of the exponents m of all those monomials X^{p^m} which appear in some ρ_a, $a \in R$, with nonzero coefficient.*

PROOF. Let d be the greatest common divisor of the exponents m of all those monomials X^{p^m} which appear in some ρ_a, $a \in R$, with nonzero coefficient. Then it is known that $f_R \mid d$ ([1], Corollary 3.9). Since any invertible$R$-modules are patterned alike ([2], Proposition 8.7), we may assume that $\rho = \rho^{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of R prime to the conductor $\mathfrak{f}$. Note that $Aut(\rho) = \mathbb{F}_{p^d}^*$. Since $Aut(\rho)$ is isomorphicto $\{\omega \in C \mid \omega\mathfrak{p} = \mathfrak{p}\}$, we have that $\omega\mathfrak{p} = \mathfrak{p}$ for all $\omega \in \mathbb{F}_{p^d}^*$. Choose $x \in \mathfrak{p}$, $y \in \mathfrak{f}$ so that $x + y = 1$. Then

$$\omega = \omega x + \omega y.$$

Hence $\omega \in R$, and so $\omega \in \mathbb{F}_{q'}$.

Let ρ be a sgn-normalized elliptic R-module. Then there exists $w \in C^*$ such that$\rho' = w\rho w^{-1}$ is defined over H_R. By Lemma 1.2 $w^{q^{\delta}-1} \in H_R$. Let $w_0 = w^{q'-1}$. Put $\tilde{H}_R = H_R(\omega_0)$. Let $\tilde{Pic}\, R$ be the quotient group of the group of invertible ideals modulo the subgroup of principal ideals generated by an element $x \in R$ with $sgn(x) = 1$.

LEMMA 1.7. *Let $\mathfrak{P}$ be a prime divisor of H_R which does not lie over the conductor $\mathfrak{f}$ of R and let $Norm(\mathfrak{P}) = xA$ with $x \in R$. Then $s_\rho(x)$ belongs to $\mathbb{F}_{q'}$ modulo $\mathfrak{P}$.*

PROOF. Except using Lemma 1.6 the proof is almost the same as that of [2], Lemma 9.4.

Using Lemma 1.7 and following the proof of [3], Proposition 4.7, we get

PROPOSITION 1.8. *Let $\mathfrak{P}$ be a finite place of H_R which does not lie over $\mathfrak{f}$ and does not ramify in $H_R(\omega)/H_R$. Let $\tau_{\mathfrak{P}}$ be the Frobenius automorphism of $H_R(\omega)$ over H_R associated to $\mathfrak{P}$. Let $x_{\mathfrak{P}}$ be a generator of the ideal $Norm(\mathfrak{P})$ in A. Then we have*

i) $\omega^{1-\tau_{\mathfrak{P}}} s_\rho(x_{\mathfrak{P}}) \in \mathbb{F}_{q'}$
ii) $\tau_{\mathfrak{P}}\rho = s_\rho(x_{\mathfrak{P}})^{-1} \cdot \rho \cdot s_\rho(x_{\mathfrak{P}})$.

THEOREM 1.9. *(cf; [3] §4) i) $Gal(\tilde{H}_R/K)$ is isomorphic to $\tilde{Pic}\, R$, and*

$$[\tilde{H}_R : K] = \frac{q^\delta - 1}{q' - 1} \cdot h_R.$$

ii) *$\tilde{H}_R/K$ is unramified at the finite places prime to $\mathfrak{f}$.*
iii) *$\tilde{H}_R/H_R$ is totally ramified at ∞.*
iv) *A finite place $\mathfrak{p}$ prime to $\mathfrak{f}$ splits completely in $\tilde{H}_R/K$ if and only if $\mathfrak{p} = xA$ with $x \in R$ and $sgn(x) \in \mathbb{F}_{q'}$.*
v) *Let $\tilde{B}$ be the integral closure of A in $\tilde{H}_R$. Then for a sgn-normalized elliptic R-module ρ and an ideal $\mathfrak{a}$ of R prime to $\mathfrak{f}$, the extended ideal $\mathfrak{a}\tilde{B}$ is a principal ideal and generated by the constant term $D(\rho_{\mathfrak{a}})$ of $\rho_{\mathfrak{a}}$.*
vi) *For a given sgn-normalized R-module ρ and an ideal $\mathfrak{a}$ of A prime to $\mathfrak{f}$, we have $\tau_{\mathfrak{a}}\rho = \mathfrak{a} * \rho$.*

PROOF. Since the proof is mostly the same as in [3] except vi), we only prove vi) in the case that $\mathfrak{a} = \mathfrak{p}$, a prime ideal. We know from [2], Theorem 8.5 that $\tau_{\mathfrak{p}}\rho$ and $\mathfrak{p} * \rho$ are isomorphic, so that $\tau_{\mathfrak{p}}\rho = a^{-1}(\mathfrak{p} * \rho)a$ for some $a \in C$. Since $s_{\mathfrak{p}*\rho}(x) = s_\rho(x)^{N\mathfrak{p}} = s_\rho(x)^{\tau_{\mathfrak{p}}} = s_{\tau_{\mathfrak{p}}\rho}(x)$, we have $a \in \kappa(\infty)^*$ by Lemma 1.2. We have to show that $a \in \mathbb{F}_{q'}$. Pick $y \in R$ so that the coefficient α of $X^{q'}$ of ρ_y is nonzero(cf: Lemma 1.6). One can choose $\mathfrak{p}$ so that $\mathfrak{p}$ is prime to $\mathfrak{f}$ and α. Then for a place $\tilde{\mathfrak{P}}$ of $\tilde{H}_R$ which divide $\mathfrak{p}$, $\tau_{\mathfrak{p}}$ and $\mathfrak{p} * \rho$ have equal reduction. Then a is an automorphism of this reduction, and so a must be in $\mathbb{F}_{q'}$ by our choice of $\mathfrak{p}$. Since we can find representatives of $\tilde{Pic R}$ by such prime ideals $\mathfrak{p}$, we are done.

We call the field $\tilde{H}_R$ the *normalizing field* of (R, sgn, ∞). Then $H_R(w_0)$ is the class field associated to the subgroup

$$J'_R = K^* \pi^{\mathbb{Z}} U'_R,$$

where $U'_R = \{(u_v) \in U_R : sgn(u_\infty) = 1\}$ (cf [2]). Let $\mathfrak{m}$ be an ideal of R which is prime to $\mathfrak{f}$ and ρ a sgn-normalized module. Let $\Lambda_{\mathfrak{m}}$ be the set of $\mathfrak{m}$-torsion points of ρ. Then

$$\Lambda_{\mathfrak{m}} \simeq R/\mathfrak{m} \simeq A/\mathfrak{m}.$$

Put $\tilde{K}_{\mathfrak{m}} = \tilde{H}_R(\Lambda_{\mathfrak{m}})$ be the field generated by $\mathfrak{m}$-torsion points of ρ over $\tilde{H}_R$. Exactly the same proof as in the case $R = A$ would give the following theorem.

THEOREM 1.10. *i) $\tilde{K}_{\mathfrak{m}}$ is abelian over K.*

ii) $Gal(\tilde{K}_{\mathfrak{m}}/\tilde{H}_R) \simeq (A/\mathfrak{m})^*$.

iii) *Let $\lambda \in \Lambda_{\mathfrak{m}}$ and $\sigma_{\mathfrak{a}}$ be the Artin automorphism of $Gal(\tilde{K}_{\mathfrak{m}}/K)$ associated to the ideal $\mathfrak{a}$. Then*

$$\lambda^{\sigma_{\mathfrak{a}}} = \rho_{\mathfrak{a}}(\lambda).$$

iv) *Let G_∞ be the decomposition group of $\tilde{K}_{\mathfrak{m}}/K$ at ∞. Then G_∞ is the inertia group at ∞ and isomorphic to $\kappa(\infty)^*$.*

v) *Let $H_{\mathfrak{m}}$ be the fixed field of $\tilde{K}_{\mathfrak{m}}$ under G_∞ and $N^-_{\mathfrak{m}} : \tilde{K}_{\mathfrak{m}} \longrightarrow H_{\mathfrak{m}}$ be the corresponding norm map. Then $N^-_{\mathfrak{m}}(\tilde{K}^*_{\mathfrak{m}})$ consists of totally positive elements. Here an element x is said to be totally positive if $sgn(\sigma(x)) = 1$, for any automorphiosm σ over K.*

vi) *For $\lambda \in \Lambda_{\mathfrak{m}}$ and $\sigma \in Gal(\tilde{K}_{\mathfrak{m}}/K)$, $\lambda^{\sigma-1}$is a unit in the ring of integers of $\tilde{H}_{\mathfrak{m}} = \tilde{H}_R H_{\mathfrak{m}}$, the fixed field of $\mathbb{F}_{q'} \subset Gal(\tilde{K}_{\mathfrak{m}}/\tilde{H}_R)$.*

2. $v_\infty(\xi(\mathfrak{c}))$

In this section we assume that $q' = q$. For an integral ideal $\mathfrak{c}$ of R define the partial zeta function

$$\zeta_{\mathfrak{c}}(s) = \sum_{x \in \mathfrak{c}} |x|_\infty^{-s}.$$

Put $S = q^{-s}$. Then

$$\zeta_{\mathfrak{c}}(s) = Z_{\mathfrak{c}}(S) = \sum_{x \in \mathfrak{c}} S^{deg\, x}.$$

In the case of $R = A$ it is shown in ([1], (4.10)) that

$$v_\infty(\xi(\mathfrak{c})) = -Z_{\mathfrak{c}}'(1)/\delta.$$

In fact, this holds for any order R of A and the proof is exactly the same. Now we are going to evaluate $Z_{\mathfrak{c}}'(1)$ for any invertible integral ideal $\mathfrak{c}$ of R. For each integer i we define

$$i^* = inf\{n : n \geq i, n \equiv 0 \quad (\delta)\}$$

and

$$i_* = sup\{n : n \leq i, n \equiv 0 \quad (\delta)\}.$$

For an invertible integral ideal $\mathfrak{c}$ of R of degree c, let

$$T_t(\mathfrak{c}) = \{x \in \mathfrak{c} : deg\, x \leq t\delta + c_*\} \quad \text{and} \quad u(t) = u_{\mathfrak{c}}(t) = dim_{\mathbb{F}_q} T_t(\mathfrak{c}).$$

Take an element $f \in \mathfrak{f}$ of degree r. We usually take $f = 1$ in case $R = A$. Define

$$m = m_{\mathfrak{c},f} = (c + 2g - 1)^* - c_* + r \quad \text{and} \quad n = n_{\mathfrak{c},f} = u(\frac{m}{\delta}),$$

where g is the genus of the smooth curve associated to K.

LEMMA 3.1. *If $t \geq \frac{m}{\delta}$, then*

$$u(t) = n + t\delta - m.$$

PROOF. Let $\mathfrak{c}' = \mathfrak{c}A$ and $T_t(\mathfrak{c}') = \{x \in \mathfrak{c}' : deg\, x \leq t\delta + c_*\}$. By the Riemann-Roch theorem, $dim_{\mathbb{F}_q} T_t(\mathfrak{c}')$ increases by δ if $t \geq \frac{(c+2g-1)^* - c_*}{\delta}$. Since $fT_t(\mathfrak{c}') \subset T_{t+r}(\mathfrak{c})$ and $dim_{\mathbb{F}_q} T_t(\mathfrak{c})$ increases at most by δ, $dim_{\mathbb{F}_q} T_t(\mathfrak{c})$ increases by δ for $t \geq m$. Hence we get the result.

Let

$$F_t(\mathfrak{c}) = \{x \in \mathfrak{c} : deg\, x = t\delta + c_*\}$$

and

$$\ell_f(\mathfrak{c}) = -\sum_{t=0}^{\frac{m}{\delta}} t\delta |F_t(\mathfrak{c})|.$$

Then by Lemma 3.1

$$\begin{aligned} Z_{\mathfrak{c}}(S) &= \sum_{t=1}^{\frac{m}{\delta}} |F_t(\mathfrak{c})| S^{t\delta + c_*} + \sum_{t=1+\frac{m}{\delta}}^{\infty} (q^{n+t\delta-m} - q^{n+(t-1)\delta-m}) S^{t\delta+c_*} \\ &= \sum_{t=1}^{\frac{m}{\delta}} |F_t(\mathfrak{c})| S^{t\delta+c_*} + q^n(q^\delta - 1)\frac{S^{m+\delta+c_*}}{1-q^\delta S^\delta}. \end{aligned}$$

Thus

$$\begin{aligned} Z'_{\mathfrak{c}}(1) &= -\ell_{\mathfrak{f}}(\mathfrak{c}) + c_* \sum_{t=1}^{\frac{m}{\delta}} |F_t(\mathfrak{c})| - q^n(m+\delta+c_*) + \frac{\delta q^{n+\delta}}{q^\delta - 1} \\ &= -\ell_{\mathfrak{f}}(\mathfrak{c}) + c_*(q^n - 1) - q^n(m+\delta+c_*) + \frac{\delta q^{n+\delta}}{q^\delta - 1} \\ &= -\ell_f(\mathfrak{c}) - c_* - mq^n + \frac{\delta q^n}{q^\delta - 1}. \end{aligned}$$

Therfore we get

PROPOSITION 3.2. *We have*

$$\delta v_\infty(\xi(\mathfrak{c})) = \ell_f(\mathfrak{c}) + c_* + m_{\mathfrak{c},f} q^{n_{\mathfrak{c},f}} - \frac{\delta q^{n_{\mathfrak{c},f}}}{q^\delta - 1}.$$

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