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FLOER HOMOLOGY AS THE STABLE MORSE HOMOLOGY

DARKO MILINKOVIĆ AND YONG-GEUN OH

ABSTRACT. We prove that there exists a canonical level-preserving isomorphism between the stable Morse homology (or the Morse homology of generating functions) and the Floer homology on the cotangent bundle T^*M for any closed submanifold $N \subset M$ for any compact manifold M.

1. Introduction

Morse theory studies topology of manifolds by studying the change of topology of sub-level sets of functions. Change of topology could occur when the level passes by the critical values of the given function. Critical points of function f can be identified with intersections of the zero section o_M with Graph(df) in the cotangent bundle T^*M. Then Morse theory can be considered as the intersection theory of the graphs of functions f with the zero section. An invariant one gets out of this is so called *Morse homology* of M (see [F2, Sc]). By now, most of the *homological* characteristics of the manifold M can be recovered by this setting (see [Fl2, 1, Sc, Fu, FO]). Noting that Graph(df) is a special example of *exact Lagrangian submanifolds* of T^*M, which is Hamiltonian isotopic to the zero section, one might enlarge *the space of functions on M* to the space of *Hamiltonian deformations of the zero section on T^*M*, and seek for some invariants that are defined in this enlarged setting. Large parts of symplectic topology are closely tied to this attempt.

In hindsight, many results of the topology and geometry of these Lagrangian submanifolds are based on the crucial fact that there is a

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naturally associated function to every Lagrangian submanifold Hamiltonian isotopic to the zero section, called *generating function.* When the "history" $t \mapsto \phi_t^H(o_M)$ of the Lagrangian submanifold $L = \phi_1^H(o_M)$ is known, there exists a canonically defined generating function of L, the *classical Hamilton's action functional*

$$\mathcal{A}_H(\gamma) = \int_\gamma pdq - H dt$$

on the space of paths with zero as "initial momentum"

$$\Omega := \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M\}.$$

This seemingly natural viewpoint that the classical action functional is a "generating function" of $L = \phi_1^H(o_M)$, was first observed by Weinstein. He also recognized (see details in [W]) that Chaperon [2] and Laudenbach-Sikorav construction [LS] in the proof of Arnold's conjecture in T^*M could be put in the calculus of Lagrangian submanifolds and their generating functions, which had been extensively used by Hörmander [Hö] in his calculus of Fourier integral operators. Partly due to the apparent analytical difficulties which had been well-known for sometime in relation to the critical point theory of the action functional on general cotangent bundle T^*M, finite dimensional approach to generating functions has been developed extensively by Sikorav [Si1], Chekanov [3], Viterbo [V1], Eliashberg-Gromov [EG] and others. Viterbo in his ICM talk in Zürich [V2] first indicated that the direct approach, using the Floer theory, is possible in the study of topology of the action functional. On the other hand, motivated by Weinstein's observation above and by some non-degeneracy questions in Hofer's geometry, the second author [O1,2] independently developed a direct infinite dimensional approach to the study of geometry of the action functional, using the Floer theory. Subsequently, most of the constructions that were previously done by the finite dimensional approach have been recovered in this setting (see [O1,O2,MO,M] and [FO] in somewhat different form). Furthermore, this direct approach has enabled Kasturirangan and the second author to define the Floer homology of open subsets of M, which yields to the refined (localized) estimate in Arnold's conjecture (see [KO] and [O3]).

The main purpose of the present paper is to establish the level-preserving isomorphism between the Morse homology of generating functions (or GF-homology [Tr]) defined by the finite dimensional approach and the Floer homology defined in [O1] for any closed submanifold $N \subset M$. The proof of this fact was implicit in our previous paper [MO] where we proved the symplectic invariants constructed in [V1] and [O1] coincide. Since the existence of level-preserving isomorphism is of independent interest in the point of view that Floer theory developed in [O1] is the limit of stable Morse theory of generating functions, we give a complete proof of the isomorphism theorem which justifies the view-point at least in the homology level.

Existence of such isomorphism and an indication of the proof, for the case $N \equiv M$, was first given by Viterbo in [V2], and proven later in [V3]. His idea [V2] is to interpolate the two theories using a functional defined on $\Omega \times E$ which is, roughly, *a fiber sum* of the action functional and the (finite dimensional) generating function, and to do a Floer-type theory for the functional.

There are two non-trivial problems to solve to realize the above idea: *one is the problem of natural boundary condition for the relevant Floer theory, and the other is is to find a family of functionals the set of whose critical values is unchanged during the interpolation.* The first problem was solved by the second author in [O1] using the conormal bundle boundary condition which also enabled us to solve the second problem for all submanifold N.

In [MO], partly motivated by the ideas in [V2] and [O1,2], we solved the second problem by giving a construction of such functionals which are defined on a path space of T^*E with Graph(dS) as the initial condition and with the conormal bundle $\nu^*(N \times \mathbb{R}^m)$ as the final condition of the paths. This gives the interpolation of the two approaches in a realm of the Floer theory developed in [O1,2]. In [V3], Viterbo gave a different construction of such functionals for the case $N = M$ in his interpolation that uses a version of Floer's "area functional" [F1] instead of the action functional $\mathcal{A}_H$. This way he also bypassed the problem of natural boundary condition mentioned above. We would like to emphasize that this kind of boundary condition must be imposed in view of elliptic boundary problem when the classical action functional $\mathcal{A}_H$ is involved in the Floer theory.

The setting we found in [MO] has enabled us to prove the isomorphism between the two homologies *for any closed submanifold* $N \subset M$ (and for any coefficient ring R).

More precisely, we prove the following theorem. We refer readers to Section 2 and 5 for more precise meaning of the statements in the theorem.

MAIN THEOREM *Let $L \subset T^*M$ be a Lagrangian submanifold with $L = \phi_1^H(o_M)$ and S its generating function quadratic at infinity. Then for any $\lambda \notin \mathrm{Spec}(L : N)$ there exists an isomorphism*

$$H_*^\lambda(S, N : E) \cong HF_*^\lambda(J, H, N : M)$$

such that the following diagram commutes

$$\begin{array}{ccc} H_*^\lambda(S, N : E) & \xrightarrow{\cong} & HF_*^\lambda(J, H, N : M) \\ \downarrow & & \downarrow \\ H_*(S, N : E) & \xrightarrow{\cong} & HF_*(J, H, N : M) \end{array}$$

Although we state the theorem only in homology, it is obvious from the proof that the same holds for cohomology. This theorem partially verifies that *the Floer theory on the cotangent bundle T^*M* in [O1] is *the stable Morse theory (of generating functions) on M* at least in the homological level.

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2. Morse homology of generating functions

In this section, we briefly summarize the Morse homology of generating functions which has been a folklore among the symplectic topologists. Some parts can be found in [Mi], [Sc] and [M]. A treatement in view of singular homology was given by Traynor [Tr]. Recall that function $S : E \to \mathbb{R}$ on a vector bundle E is called *the generating function* of the Lagrangian submanifold L if L can be represented as

$$L = L_S := \left\{ (x, \frac{\partial S}{\partial x}(x, y)) \in T^*M \mid (x, y) \in \Sigma_S \right\}$$

where $\Sigma_S = \{(x, y) \in E \mid \frac{\partial S}{\partial y}(x, y) = 0\}$. In that case, the critical points of S have one-to-one correspondence with the intersections $L \cap o_M$ in T^*M. Every Hamiltonian deformation of zero section has essentially unique *generating function quadratic at infinity* [LS,V1,Th], which is commonly abbreviated as GFQI. Without loss of any generality, we may assume that E is trivial. In general, one might have to increase the rank of the vector bundle E as structure of the *caustics* of the given Lagrangian submanifold becomes more complicated. We denote by $\mathcal{L}_{ham}(M)$ the set of all Hamiltonian deformations of the zero section in T^*M and by $\mathcal{S}_{ham}(M)$ the set of GFQI's generating them. For each $m \in \mathbb{Z}_+$, we define

$$\mathcal{L}^m_{ham}(M) := \{L \in \mathcal{L}_{ham}(M) \quad | \quad L = L_S \text{ for some GFQI, } S \text{ defined on } E = M \times \mathbb{R}^k \text{ with } k \leq m\}.$$

It is proven in [LS] that

$$\mathcal{L}_{ham}(M) = \bigcup_{m=0}^{\infty} \mathcal{L}^m_{ham}(M)$$

and in [V1,Th] that the assignment $S \mapsto L_S$ from $\mathcal{S}_{ham}(M)$ to $\mathcal{L}_{ham}(M)$ defines a Serre fibration. When the "history" $t \mapsto \phi^H_t(o_M)$ is known, there is the canonically defined classical action functional

$$\mathcal{A}_H(\gamma) := \int_\gamma p\,dq - H(\gamma(t), t)\, dt$$

defined on the (infinite dimensional) path space

$$\Omega = \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M\}.$$

The construction in [LS] of GFQI S for $L = \phi^H_1(o_M)$ was given by discretizing this action functional over the piecewise Hamiltonian paths. Weinstein [W] observed that $\mathcal{A}_H$ itself is a generating function of $L = \phi^H_1(o_M)$ with respect to the fibration $p : \Omega \to M$; $\quad p(\gamma) = \pi(\gamma(1))$ where $\pi : T^*M \to M$ is the canonical projection.

For a non-degenerate fiberwise quadratic form $Q : E \to \mathbb{R}$ we denote by $\mathcal{S}_{(Q:E)}$ the set of functions on E such that $S \equiv Q$ outside a compact subset of E. We can do Morse theory on non-compact set E for functions in class $\mathcal{S}_{(Q:E)}$. Let $N \overset{i}{\hookrightarrow} M$ be a closed submanifold and $E_N = i^*E$ a pullback bundle over N. For a Morse function $S \in \mathcal{S}_{(Q:E)}$ on E we denote by $\mathrm{Crit}_p(S, N)$ the set of critical points of $S_N := S|_{E_N} : E_N \to \mathbb{R}$ of index p and define

$$C_p(S, N : E) := \text{ free abelian group generated by } \mathrm{Crit}_p(S, N : E).$$

For given metric g on E, we consider the (negative) gradient flow generated by the equation

$$\frac{d\gamma}{d\tau} = -\text{grad}_g S_N, \tag{1}$$

where $\text{grad}_g S_N$ is defined with respect to the Riemannian metric on E_N induced from g. Denote by $\mathcal{M}_g(S, N : E)$ the set of all $\gamma : \mathbb{R} \to E_N$ satisfying (1) such that

$$\int_{-\infty}^{+\infty} \left|\frac{d\gamma}{d\tau}\right|^2 d\tau < \infty.$$

For a generic choice of a metric, the spaces

$$\mathcal{M}_{g,S,N}(x^-, x^+) = \{\gamma \in \mathcal{M}_{g,S,N}(M) \mid \gamma(\tau) \to x^\pm \text{ as } \tau \to \pm\infty\}$$

are smooth manifolds of dimension $m(x^+) - m(x^-)$, where $m(x)$ denotes Morse index of a critical point x. The group $\mathbb{R}$ acts on $\mathcal{M}_{g,S,N}(x, y)$ by $\gamma \mapsto \gamma(. + \tau_0)$; we denote

$$\widehat{\mathcal{M}}_{g,S,N}(x, y) := \mathcal{M}_{g,S,N}(x, y)/\mathbb{R}$$

and give the manifolds $\widehat{\mathcal{M}}_{g,S,N}$ an orientation as in [Mi,Sc]. Define $\partial = \sum_p \partial_p$,

$$\partial_p : C_p(S, N : E) \to C_{p-1}(S, N : E)$$
$$\partial_p x := \sum_{y \in \text{Crit}_p(S)} n(x, y) y,$$

where $n(x, y)$ is the algebraic number of points in the zero dimensional manifold $\mathcal{M}_{g,S,N}(x, y)$. The proof of $\partial \circ \partial = 0$ is based on standard gluing and cobordism arguments [F3]. Now Morse homology groups are defined by

$$H_p^{\text{Morse}}(S, N : E) := \text{Ker}(\partial)/\text{Im}(\partial).$$

We will omit the superscript *Morse* in the above notation, wherever there is no possibility of confusion. For generic choice of Morse function S the group $H_*(S, N : E)$ is defined and is isomorphic to singular homology of N.

Note that S is decreasing along the trajectories solving the gradient equation (1). Therefore, the boundary operator ∂ preserves the downward filtration given by level sets of S. In other words, if we denote

$$\text{Crit}_p^\lambda(S, N : E) := \text{Crit}_p(S, N : E) \cap S^{-1}((-\infty, \lambda])$$

and

$$C_p^\lambda(S, N : E) := \text{ free abelian group generated by } \mathrm{Crit}_p^\lambda(S, N : E)$$

then the boundary operator ∂_p restricts to

$$\partial_p^\lambda : C_p^\lambda(S, N : E) \to C_{p-1}^\lambda(S, N : E).$$

Obviously, $\partial^\lambda \circ \partial^\lambda = 0$ and hence we can define the relative Morse homology groups

$$H_p^\lambda(S, N : E) := \mathrm{Ker}(\partial_p^\lambda)/\mathrm{Im}(\partial_{p+1}^\lambda).$$

An obvious inclusion $j^\lambda : \mathrm{Crit}_p^\lambda(S, N : E) \to \mathrm{Crit}_p(S, N : E)$ generates the homomorphism $j_\sharp^\lambda : C_p^\lambda(S, N : E) \to C_p(S, N : E)$ which commutes with ∂. Hence, we have an inclusion homomorphism

$$j_*^\lambda : H_*^\lambda(S, N : E) \to H_*(S, N : E).$$

Definition of Morse homology (and cohomology) with coefficients in arbitrary ring R is straightforward by the standard algebraic procedure.

3. Floer theory and the action functional

In this section, we summarize the second author's construction [O1] of the Floer homology on the cotangent bundle with respect to the co-normal space $\nu^* N$. A straightforward computation yields to the first variation formula of the action functional

$$\mathcal{A}_H = \int_\gamma pdq - Hdt,$$

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \omega(\dot\gamma - X_H(\gamma), \xi)dt + \langle \xi(1), \theta(\gamma(1))\rangle - \langle \xi(0), \theta(\gamma(0))\rangle.$$

If we restrict $\mathcal{A}_H$ to Ω, this becomes

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \omega(\dot\gamma - X_H(\gamma), \xi)dt + \langle \xi(1), \theta(\gamma(1))\rangle.$$

It follows from this that $\mathcal{A}_H|_\Omega$ is a generating function of $\phi_1^H(o_M)$ (see [W,O1]).

For a submanifold $N \subset M$ and its co-normal bundle $\nu^* N \subset T^* M$ we consider the path space

$$\Omega(N) := \{\gamma : [0,1] \to T^* M \mid \gamma(0) \in o_M,\ \gamma(1) \in \nu^* N\}.$$

If we further restrict $\mathcal{A}_H$ to $\Omega(N)$, its first derivative becomes

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \omega(\dot{\gamma} - X_H(\gamma), \xi)dt.$$

For a family J_t of ω-compatible almost complex structures we define an L^2-type inner product

$$\langle\langle \xi, \eta \rangle\rangle_J := \int_0^1 \omega(\xi(t), J_t\eta(t))dt.$$

The negative gradient flow $u : \mathbb{R} \times [0,1] \to T^*M$ of $\mathcal{A}_H|_{\Omega(N)}$ in terms of metric $\langle\langle .,. \rangle\rangle_J$ satisfies the perturbed Cauchy-Riemann equation with Lagrangian boundary condition,

$$(2) \qquad \begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0 \\ u(\tau, 0) \in o_M, \; u(\tau, 1) \in \nu^*N. \end{cases}$$

DEFINITION 1. Denote by $\mathcal{M}(J, H, N : M)$ the set of the solutions of (2) which satisfy the additional condition

$$(3) \qquad -\infty < \inf_{\tau\in\mathbb{R}} \mathcal{A}_H(u(\tau)) < \sup_{\tau\in\mathbb{R}} \mathcal{A}_H(u(\tau)) < +\infty.$$

The solutions of (2) and (3) have the property that there exist $z, w : [0,1] \to T^*M$ such that

$$z(t) = \lim_{\tau\to+\infty} u(\tau, t),$$

$$w(t) = \lim_{\tau\to-\infty} u(\tau, t),$$

and both z and w are the solutions of the Hamilton's equations

$$(4) \qquad \begin{cases} \dot{x} = X_H(x) \\ x(0) \in o_M, \quad x(1) \in \nu^*N, \end{cases}$$

i.e z and w are the critical points of $\mathcal{A}_H|_{\Omega(N)}$. In [O1], it is shown that there exists a canonically defined Maslov index for each solution of (4), which induces a canonical grading below.

DEFINITION 2. 1. Denote by $CF_p(H, N : M)$ the free abelian group generated by the solutions of (4) with grading p.

2. For two solutions z^-, z^+ of (4) define

$$\mathcal{M}_{(J,H)}(z^-, z^+) := \{u \in \mathcal{M}(J, H, N : E) \mid \lim_{\tau\to\pm\infty} u(\tau, t) = z^\pm\}.$$

For the generic choice of J and H the sets $\mathcal{M}_{(J,H)}(z^-, z^+)$ are finite dimensional manifolds [F1,O1]. It has been proved in [O1] that they can be given a coherent orientation in the sense of [FH]. As a consequence, one can define the homomorphism $\partial = \sum_p \partial_p$,

$$\partial_p : CF_p(H, N : M) \to CF_{p-1}(H, N : M)$$

by

$$\partial(x^-) := \sum_{x^-} n(x^-, x^+)x^+,$$

where $n(x^+, x^-)$ is the algebraic number of points in the (zero dimensional) manifold

$$\widehat{\mathcal{M}}_{(J,H)}(x^-, x^+) := \mathcal{M}_{(J,H)}(x^-, x^+)/\mathbb{R}.$$

Again, by the standard cobordism argument one can prove that $\partial \circ \partial = 0$.

Since $\mathcal{A}_H$ is non-increasing along the trajectories of (2), homomorphism ∂ preserves the real filtration of $CF_*(H, N : M)$ given by the level sets of $\mathcal{A}_H$. More precisely, we can give the following

DEFINITION 3. 1. We will denote by $CF_*^\lambda(H, N : M)$ the subgroup of
$CF_*(H, N : M)$ generated by the solutions z of (4) which satisfy

$$\mathcal{A}_H(z) < \lambda.$$

2. We denote by ∂^λ the restriction of ∂ to $CH_*^\lambda(H, N : M)$ and define the relative Floer homology groups by

$$HF_*^\lambda(J, H, N : M) := \mathrm{Ker}(\partial^\lambda)/\mathrm{Im}(\partial^\lambda).$$

In particular, for $\lambda = \infty$ we write

$$HF_*(J, H, N : M) := HF_*^\infty(J, H, N : M).$$

When $\lambda = \infty$, Floer homology groups $HF_*(J, N, H : M)$ are isomorphic to singular homology groups $H_*(N)$ (see [Pz,O1]).

Applying the standard algebraic procedure we define relative Floer homology with coefficients in ring R and relative Floer cohomology (see [O2]).

4. Interpolation

We first find a functional on T^*E which interpolates the two functional used in the previous sections. *In this section, the letter H exclusively denotes a Hamiltonian defined on T^*E, not on T^*M.* The set of such Hamiltonians will be denoted by $\mathcal{H}(E)$.

For generic $S \in \mathcal{S}_{(Q:E)}$, we define the space of paths

$$\mathcal{P}(S:E) := \{\Gamma : [0,1] \to T^*E \mid \Gamma(0) \in \text{ Graph } (dS)\}.$$

Straightforward computation yields:

$$\begin{aligned} d\mathcal{A}_H(\Gamma)\eta &= \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta]dt \\ &\quad + \langle \Gamma(1), T\pi(\eta(1))\rangle - \langle \Gamma(0), T\pi(\eta(0))\rangle \\ &= \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta]dt \\ &\quad + \langle \Gamma(1), T\pi(\eta(1))\rangle - dS(\pi(\Gamma(0))T\pi(\eta(0)). \end{aligned}$$

Therefore to get a good variational problem, we study

$$\mathcal{A}_{(H,S)}(\Gamma) = \mathcal{A}_H(\Gamma) + S(\pi(\Gamma(0)))$$

as in [O1]. (In [MO], we denoted this functional by $F^{(H,S)}$ borrowing the notation used in [V2]. It is, however, a different functional from the one that was suggested and denoted by the same notation in [V2].)

The derivative of $\mathcal{A}_{(H,S)}$ on $\mathcal{P}(S:E)$ becomes

$$(5) \quad d\mathcal{A}_{(H,S)}(\Gamma)\eta = \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta]dt + \langle \Gamma(1), T\pi(\eta(1))\rangle.$$

From this, it is easy to check that $\mathcal{A}_{(H,S)}$ is a generating function of the Lagrangian submanifold $\phi_H(L_S)$ with respect to the fibration

$$p : \mathcal{P}(S:E) \to M;, \quad p(\Gamma) := \pi_{T^*E} \circ \pi_E(\Gamma(1))$$

when H is the obvious lift from a Hamiltonian on T^*M, i.e. when $H = H \oplus 0$ (see Section 5).

As in Section 3, for each closed submanifold $N \subset M$, we define the space

$$\mathcal{P}(S, N : E) = \{\Gamma : [0,1] \to T^*E \mid \Gamma(0) \in \text{ Graph } (dS),\ \Gamma(1) \in \nu^*(E|_N)\}$$

and, restricting $\mathcal{A}_{(H,S)}$ to $\mathcal{P}(S, N : E)$, we have

$$(6) \qquad d\mathcal{A}_{(H,S)}(\Gamma)\eta = \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta] dt$$

Hence, the critical points of $\mathcal{A}_{(H,S)}$ on $\Omega(S; N)$ are the solutions of

$$(7) \qquad \begin{cases} \dot{\Gamma} = X_H(\Gamma) \\ \Gamma(0) \in \text{ Graph } (dS) \\ \Gamma(1) \in \nu^*(E|_N) = \nu^*(N \times \mathbb{R}^m) = \nu^* N \times o_{\mathbb{R}^m} \end{cases}$$

For a given Riemannian metric g on M, we denote by J_g the canonical almost complex structure. Denote by $j^c_\omega(M)$ the set of ω-compatible almost complex structures which coincide with J_g outside a compact set in T^*M, and by $\mathcal{J}^c_\omega(M)$ the set of smooth paths $J_t : [0,1] \to j^c_\omega(M)$.

For a path $\{J_t\} \in \mathcal{J}^c_\omega(M)$, the family of product almost complex structures

$$J \oplus i := \{J_t \oplus i\}_{0 \le t \le 1}$$

is compatible with the product symplectic structure $\omega \oplus \omega_0$ on $T^*E = T^*M \times \mathbb{C}^m$. Denote by $j^c_\omega(E)$ and $\mathcal{J}^c_\omega(E)$ the set of almost complex structures on T^*E which coincide with product structure $J_g \oplus i$ outside a compact set and the space of paths there in respectively. These almost complex structures induce the family of metrics on T^*E

$$\langle \eta_1, \eta_2 \rangle_{J_t} := \omega \oplus \omega_0(\eta_1, J_t \eta_2)$$

and hence a L^2-type metric

$$\langle\langle \eta_1, \eta_2 \rangle\rangle_J := \int_0^1 \langle \eta_1(t), \eta_2(t) \rangle_{J_t} dt$$

on $\mathcal{P}(S, N : E)$. In terms of metric $\langle\langle .,. \rangle\rangle_J$ the gradient flow $U := (u, v)$ of $\mathcal{A}_{(H,S)}$ restricted to $\mathcal{P}(S, N : E)$ satisfies

$$(8) \qquad \begin{cases} \bar{\partial}_{J,H} U := \frac{\partial U}{\partial \tau} + J(\frac{\partial U}{\partial t} - X_H(U)) = 0 \\ U(\tau, 0) \in \text{ Graph } (dS), \quad U(\tau, 1) \in \nu^*(N \times \mathbb{R}^m) \end{cases}$$

Denote by $\text{Crit}(H, S, N : E)$ the set of critical points of $\mathcal{A}_{(H,S)}|_{\mathcal{P}(S,N:E)}$, i.e., the set of solutions of (7). The set of critical values of $\mathcal{A}_{(H,S)}$ in $\mathbb{R}$

$$\text{Spec}(H, S, N : E) := \mathcal{A}_{(H,S)}(\text{Crit}(H, S, N : E))$$

is called *the action spectrum* of $(H, S, N : E)$ and depends only on the Lagrangian submanifold $\phi_H(L_S)$ but independent of the choice of (H, S)

(up to a universal normalization for all $N \subset M$ (see [O1])). In the construction of Floer homology we always assume $\mathrm{Graph}(dS)(\phi_1^H)^{-1}(\nu^* N \times o_{\mathbb{R}^m})$. In that case, the sets $\mathrm{Crit}(H, S, N : E)$ and $\mathrm{Spec}(H, S, N : E)$ are finite. In the general case, we have the following lemma, which describes the size of the set $\mathrm{Spec}(H, S, N : E)$. Similar results were established in [Si2,O1].

LEMMA 4. *The action spectrum* $\mathrm{Spec}(H, S, N : E)$ *is a compact nowhere dense subset of* $\mathbb{R}$.

Proof. Since H is compactly supported, the set

$$\mathrm{Spec}(H, S : E) := \mathcal{A}_{(H,S)}\Big(\{\Gamma : [0,1] \to T^*E \mid \Gamma(0) \in o_E,\ \dot{\Gamma} = X_H\}\Big) \subset \mathbb{R}$$

is bounded. For it is nothing but the projection to $\mathbb{R}$ of the *wave front set* in $M \times \mathbb{R}$ of the compact Lagrangian submanifold $\phi_H(L_S)$ (see [O1]). Hence, its closed subset $\mathrm{Spec}(H, S, N : E)$ is compact. Furthermore, for the smooth function f defined by

$$f : \nu^*(N \times \mathbb{R}^m) \to \mathbb{R}; \quad f(x) = \mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x))$$

we have, by (5)

$$df(x) = -\theta_E((\phi_1^H)^{-1}(x))T(\phi_1^H)^{-1}(x) + dS(\pi(\phi_1^H)^{-1}(x))T\pi T(\phi_1^H)^{-1}(x)$$

and thus the set $\mathrm{Spec}(H, S, N : E)$ is contained in the set of critical values of f. The latter is nowhere dense in $\mathbb{R}$ by the classical Sard's theorem. □

Denote by $\mathcal{M}_{(J,H,S)}(N : E)$ the set of solutions U of (8) with finite energy, i.e. of those U which satisfy the condition:

$$(9) \qquad E(U) := \int_{-\infty}^{+\infty}\int_0^1 \left(\left|\frac{\partial U}{\partial \tau}\right|_J^2 + \left|\frac{\partial U}{\partial t} - X_H(U)\right|_J^2 \right) dt d\tau < \infty.$$

More generally, consider the τ-dependent families

$$S^{\alpha\beta} := \{S_\tau^{\alpha\beta}\} \subset \mathcal{S}_{(Q:E)},\ H^{\alpha\beta} := \{H_\tau^{\alpha\beta}\} \subset \mathcal{H}(E),\ J^{\alpha\beta} := \{J_\tau^{\alpha\beta}\} \subset \mathcal{J}_\omega^c(E),$$

such that for some $R > 0$ and $\tau < -R$

$$S_\tau^{\alpha\beta} \equiv S^\alpha,\ H_\tau^{\alpha\beta} \equiv H^\alpha,\ J_\tau^{\alpha\beta} \equiv J^\alpha,$$

for some fixed S^α, H^α, J^α and, similarly,

$$S_\tau^{\alpha\beta} \equiv S^\beta,\ H_\tau^{\alpha\beta} \equiv H^\beta,\ J_\tau^{\alpha\beta} \equiv J^\beta,$$

for $\tau > R$ and S^β, H^β, J^β fixed. We denote the sets of all such homotopies by

$$\overline{\mathcal{H}}(E),\ \overline{\mathcal{S}}_{(Q:E)},\ \overline{\mathcal{J}}^c_\omega.$$

We define $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(N:E)$ as the set of solutions of

$$(10)\qquad \begin{cases} \overline{\partial}_{J^{\alpha\beta},H_\tau}U := \frac{\partial U}{\partial \tau} + J^{\alpha\beta}(\frac{\partial U}{\partial t} - X_{H_{\alpha\beta}}(U)) = 0 \\ U(\tau,0) \in \text{ Graph }(dS_{\alpha\beta}), \quad U(\tau,1) \in \nu^*N \times o_{\mathbb{R}^m} \end{cases}$$

with finite energy $E(U) < \infty$.

It is a standard result in the elliptic regularity theory that the solutions of (10) are smooth. Finally, for two solutions x, y of (7) we denote by $\mathcal{M}_{(J,H,S)}(x,y)$ the set of solutions U of (8) such that

$$\lim_{\tau\to\infty} U(\tau,t) = x(t),$$

$$\lim_{\tau\to-\infty} U(\tau,t) = y(t).$$

In an analogous way, we define $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$ to be the set of solutions U of equation (10) such that

$$\lim_{\tau\to-\infty} U(\tau,t) = x^\alpha(t)$$

$$\lim_{\tau\to\infty} U(\tau,t) = x^\beta(t),$$

where

$$(11)\qquad \begin{cases} \dot{x}^\alpha = X_{H^\alpha}(x^\alpha) \\ x^\alpha(0) \in \text{ Graph }(dS) \\ x^\alpha \in \nu^*N \times o_{\mathbb{R}^m} \end{cases} \begin{cases} \dot{x}^\beta = X_{H^\beta}(x^\beta) \\ x^\beta(0) \in \text{ Graph }(dS) \\ x^\beta(1) \in \nu^*N \times o_{\mathbb{R}^m} \end{cases}$$

In order to define Floer homology for arbitrary coefficients we need orientations of manifolds $\mathcal{M}_{(J,H,S)}$ and $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}$. Contrary to the case of holomorphic spheres or cylinders (see [F3,FH]), manifolds of holomorphic discs with Lagrangian boundary conditions need not to be orientable in general. However, it has been proven in [O1] that for the co-normal boundary conditions in the cotangent bundle the Floer cells can always be oriented in a coherent way:

PROPOSITION 5. [O1] *For each* $(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta}) \in (\overline{\mathcal{J}}^c_\omega \times \overline{\mathcal{H}} \times \overline{\mathcal{S}^0})_{reg}$ *and each* x^α, x^β *the determinant bundle*

$$\mathbf{Det} \to \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$$

is trivial. Hence, the manifold $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha, x^\beta)$ *is oriented. Moreover, there exists a coherent orientation in the sense of* Definition 11 *in* [FH] *of all* $\mathcal{M}_{(J,H,S)}$ *and* $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}$ *in each isotopy class of* (J, H, S).

As a consequence, for $x, y \in \text{Crit}\ (H, S, N : E)$, we define the algebraic number $n(x, y)$ of points in the (zero dimensional) manifold

$$\widehat{\mathcal{M}}_{(J,H,S)}(N : E) := \mathcal{M}_{(J,H,S)}(N : E)/\mathbb{R}$$

Here $\mathbb{R}$ acts on $\mathcal{M}_{(J,H,S)}(N : E)$ in a standard way, by the translation in τ-variable. As in Section 3 or [O1], we can provide a grading to Crit $(H, S, N : E)$. We denote by $\text{Crit}_p(H, S, N : E)$ the subset of elements of degree p and by $CF_p(H, S, N : E)$ the free abelian group generated by it.

The following proposition is a reformulation of the results proved in [F1,3] and [O1].

THEOREM 6. **(1):** *For* $(J, H, S) \in (\mathcal{J}^c_\omega \times \mathcal{H} \times \mathcal{S}_{(Q:E)})_{\text{reg}}$ *the homomorphisms* $\partial = \sum_p \partial_p$,

$$\partial_p : CF_p(H, S, N : E) \to CF_{p-1}(H, S, N : E)$$

$$\partial x = \sum_y n(x, y)y$$

satisfy

$$\partial \circ \partial = 0.$$

We define

$$HF_p(J, H, S, N : E) := \text{Ker } \partial_p / \text{Im } \partial_{p+1}.$$

(2): *For two given parameters*

$$(J^\alpha, H^\alpha, S^\alpha), (J^\beta, H^\beta, S^\beta) \in (\mathcal{J}^c_\omega \times \mathcal{H} \times \mathcal{S}_{(Q:E)})_{reg},$$

there exist canonical isomorphisms

$$h_{\alpha\beta} : HF_*(J^\alpha, H^\alpha, S^\alpha, N : E) \to HF_*(J^\beta, H^\beta, S^\beta, N : E)$$

which preserve the grading and satisfy

(i): $h_{\alpha\alpha} = id$

(ii): $h_{\gamma\beta} \circ h_{\beta\alpha} = h_{\gamma\alpha}$.

Since the equation (8) is the negative gradient flow of $\mathcal{A}_{(H,S)}$, the boundary operator ∂ preserves the sub-level sets of $\mathcal{A}_{(H,S)}$. As in Section 3, we define by $CF_p^\lambda(H,S,N:E)$ the free abelian group generated by

$$\mathrm{Crit}_p^\lambda(H,S,N:E) := \{x \in \mathrm{Crit}_p(H,S,N:E) \mid \mathcal{A}_{(H,S)}(x) < \lambda\}.$$

Then, the boundary map $\partial_p : CF_p(H,S,N:E) \to CF_{p-1}(H,S,N:E)$ induces the relative boundary map

$$\partial_p^\lambda : CF_p^\lambda(H,S,N:E) \to CF_{p-1}^\lambda(H,S,N:E)$$

which satisfy the obvious identity $\partial_p^\lambda \circ \partial_{p+1}^\lambda = 0$. Therefore, we can define the relative Floer homology groups

$$HF_p^\lambda(J,H,S,N:E) := \mathrm{Ker}(\partial_p^\lambda)/\mathrm{Im}(\partial_{p+1}^\lambda).$$

The natural inclusion j^λ : Crit $^\lambda(H,S,N:E) \to$ Crit $(H,S,N:E)$ induces the group homomorphism

$$j_\sharp^\lambda : CF_*^\lambda(H,S,N:E) \to CF_*(H,S,N:E)$$

which commutes with ∂, i.e. $\partial \circ j_\sharp^\lambda = j_\sharp^\lambda \circ \partial^\lambda$. Hence, $j_\sharp^\lambda$ induces the natural homomorphism

$$j_*^\lambda : HF_*^\lambda(J,H,S,N:E) \to HF_*(J,H,S,N:E).$$

THEOREM 7. *For regular parameters the diagram*

$$\begin{array}{ccc} HF_*^{\lambda+\epsilon_{(H,S)}^{\alpha\beta}}(J,H^\beta,S^\beta,N:E) & \xrightarrow{j_*^{\lambda+\epsilon_{(H,S)}^{\alpha\beta}}} & HF_*(J,H^\beta,S^\beta,N:E) \\ \uparrow h_{\alpha\beta} & & \uparrow h_{\alpha\beta} \\ HF_*^\lambda(J,H^\alpha,S^\alpha,N:E) & \xrightarrow{j_*^\lambda} & HF_*(J,H^\alpha,S^\alpha,N:E) \end{array}$$

commutes, where

$$\epsilon_{(H,S)}^{\alpha\beta} := -\int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha)$$

Proof. We fix regular parameters (H^α, S^α) and (H^β, S^β) and choose the C^∞ function

$$\rho : \mathbb{R} \to \mathbb{R}$$

such that

$$\rho(\tau) = 1 \quad \text{for } \tau \geq 1$$

$$\rho(\tau) = 0 \quad \text{for } \tau \leq 0.$$

Denote by $(\overline{H}_\tau, \overline{S}_\tau)$ a regular homotopy connecting (H^α, S^α) and (H^β, S^β) which is ϵ-close in C^1-topology to the (possibly non-regular) homotopy

$$(\rho(\tau)H^\beta + (1-\rho(\tau))H^\alpha, (\rho(\tau)S^\beta + (1-\rho(\tau))S^\alpha)$$

(See [O1] for similar arguments). We compute $\mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) - \mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha)$ for a pair x^α, x^β connected by trajectory U satisfying (10). Since

$$\frac{d}{d\tau}\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U(\tau)) = d\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U)\frac{\partial U}{\partial \tau} - \int_0^1 \frac{\partial \overline{H}_\tau}{\partial \tau}(U(\tau))dt + \frac{\partial \overline{S}_\tau}{\partial \tau}(U(\tau,0))$$

and the last two terms are ϵ-close to

$$-\int_0^1 \rho'(\tau)(H^\beta - H^\alpha)dt + \rho'(\tau)(S^\beta - S^\alpha),$$

we have

$$\begin{aligned}
\mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) \;-\; \mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha) &= \int_{-\infty}^{-\infty} \frac{d}{d\tau}\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U(\tau))d\tau \\
&\le \int_{-\infty}^{-\infty} \Big\{ \int_0^1 \Big(d\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U)\frac{\partial U}{\partial \tau} - \rho'(\tau)(H^\beta - H^\alpha)(U)dt \Big) \\
&\quad + \rho'(\tau)(S^\beta - S^\alpha)(U(\tau,0)) \Big\} d\tau + \epsilon \\
&\le -\int_{-\infty}^{+\infty} \left|\frac{\partial U}{\partial \tau}\right|_J^2 \\
&\quad - \int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha) + \epsilon.
\end{aligned}$$

Here we used the first variation formula (2.16) [O1] and (10). Hence, we have the well defined homomorphism

$$h_{\alpha\beta} : HF_*^\lambda(J, H^\alpha : S^\alpha : N) \to HF_*^{\lambda+\epsilon^{\alpha\beta}_{(H,S)}-\epsilon}(J, H^\beta; S^\beta : N)$$

such that the diagram

$$\begin{array}{ccc}
HF_*^{\lambda+\epsilon^{\alpha\beta}_{(H,S)}+\epsilon}(J, H^\beta, S^\beta, N : E) & \xrightarrow{j_*^{\lambda+\epsilon^{\alpha\beta}_{(H,S)}-\epsilon}} & HF_*(J, H^\beta, S^\beta, N : E) \\
\uparrow h_{\alpha\beta} & & \uparrow h_{\alpha\beta} \\
HF_*^\lambda(J, H^\alpha, S^\alpha, N : E) & \xrightarrow{j_*^\lambda} & HF_*(J, H^\alpha, S^\alpha, N : E)
\end{array}$$

commutes. The conclusion follows by letting $\epsilon \to 0$. □

Now, let us examine two special cases, one for the case where $H \equiv 0$ and the other for the case where $S \equiv Q$. Without loss of any generality, by deforming Q, we may assume that $Q \equiv Q_0$ is constant over $q \in M$. This is because we assume that $E = M \times \mathbb{R}^m$ and the set of nondegenerate quadratic forms with a fixed signature is contractible.

When $S \equiv Q_0$, (8) becomes separated into

$$\begin{cases} \overline{\partial}_{J,H}(u) = 0 \\ \overline{\partial} v = 0 \\ u(\tau,1) \in \nu^* N, v(\tau,1) \in o_{\mathbb{R}^m} \\ u(\tau,0) \in o_M, v(\tau,0) \in \ \text{Graph} \ (dQ_0) \end{cases}$$

i.e., into the equation (2)

$$\begin{cases} \overline{\partial}_{J,H}(u) = 0 \\ u(\tau,1) \in \nu^* N, u(\tau,0) \in o_M \end{cases}$$

and the equation

$$\begin{cases} \overline{\partial} v = 0 \\ v(\tau,1) \in o_{\mathbb{R}^m} = \mathbb{R}^m \subset \mathbb{C}^m, v(\tau,0) \in \ \text{Graph} \ (dQ_0). \end{cases} \tag{12}$$

From the fact that Graph (dQ_0) is a (linear) Lagrangian subspace in $\mathbb{C}^m$ and Graph (dQ_0) $\mathbb{R}^m$, it follows that the only bounded solution of (12) is the constant solution

$$v \equiv 0 \in \mathbb{C}^m.$$

Therefore the moduli space $\mathcal{M}_{(J,H,Q)}(N : E)$ becomes equivalent to $\mathcal{M}_{(J,H)}(N : M)$ in Section 3. Furthermore this equivalence preserves the values of corresponding critical levels of $\mathcal{A}_{(H,Q_0)}$ and $\mathcal{A}_H$.

On the other hand when $H \equiv 0$, for given $J = \{J_t\}_{0 \le t \le 1}$, we choose

$$J_S = \left\{ (\phi^t_{\pi^* S})_* (J_t \oplus i) \right\}_{0 \le t \le 1}$$

with respect to which (8) becomes

$$\begin{cases} \overline{\partial}_{J_S} U = 0 \\ U(\tau,0) \in \ \text{Graph} \ (dS), \quad U(\tau,1) \in \nu^* N \times o_{\mathbb{R}^m}, \end{cases} \tag{13}$$

where $U(\tau,t) = (u(\tau,t), v(\tau,t))$. We denote by $\mathcal{U}$ a tubular neighborhood of N in M and denote by $\pi_N : \mathcal{U} \to N$ the projection. Motivated by the arguments in [F2, Pz], we deform S to another GFQI, S' such that

$$S' = \begin{cases} S & \text{outside } \mathcal{U} \text{ and outside a compact subset of } E \\ S_N \circ \pi_N & \text{on } E|_{\mathcal{U}}, \end{cases}$$

where $S_N = S|_{E_N}$ as before. By choosing $\mathcal{U}$ sufficiently small, we can make this deformation as small as we want in the C^0-norm (Note that it is not possible in general for this deformation to be made small in C^1-norm). By Theorem 7, the canonical isomorphism

$$HF_*(J,0,S,N:E) \to HF_*(J,0,S',N:E) \tag{14}$$

can be made the filtration level change as small as we want.

At this stage, *if we know that the covariant C^1-norm of dS' is "small" on the image of the solutions of* (13), then Floer-type argument in [F2, Pz] will imply that the moduli space of (13) with S replaced by S' will be diffeomorphic to the "bounded" Morse complex of S' on E. Such a smallness can be always obtained by changing the metric g to $\frac{1}{\epsilon^2}g$ (see Remark 3.4.10 [Pz]) and so this proves that $\mathcal{M}(J,0,S',N:E)$ is diffeomorphic to

$$\mathcal{M}_{\frac{1}{\epsilon^2}g}(S',N:E).$$

The analytic details of this will be given in [M]. Hence we have

$$HF^*(J,0,S';N:M) \cong H^*(S',N:E). \tag{15}$$

Both of the above equivalence and the deformation of metrics preserve the values of corresponding critical levels of $\mathcal{A}_{(0,S')}$ and S' and so induce the level preserving isomorphisms.

Since $S_N = S'_N$ by the way how we define S'_N, we have

$$H^\lambda(S',N:E) = H^\lambda(S,N:E). \tag{16}$$

Now composition of the isomorphisms (14), (15) and (16),

$$FH^*(J,0,S;N:M) \cong H^*(S,N:E). \tag{17}$$

can be made so that it changes the filtration level as small as we want. By making this change sufficiently small, compared to $\mathrm{dist}(\lambda, \mathrm{Spec}(0, S, N : E))$, (17) in fact preserves the level. This is because we have

$$\mathrm{Spec}(0, S, N : E) = \text{the set of critical values of } S_N.$$

5. Proof of main theorem

In this section, we finally prove the main theorem stated in the introduction. Denote by $S_t : E \to \mathbb{R}$ a generating function of $(\phi_t^H)^{-1}(L_S)$, such that $S_0 = S$, $S_1 = Q$. Let $H(t)$ denote a path of Hamiltonians such that $\phi_1^{H(t)} = \phi_t^H$. Note that

$$\begin{aligned} \phi_1^{H(t)}(L_{S_t}) &= \phi_1^{H(t)}(\phi_t^H)^{-1}(L_S) \\ &= \phi_t^H(\phi_t^H)^{-1}(L_S) \\ &= L_S \end{aligned}$$

and therefore, after a suitable normalization (see [O1], [MO]), the action spectrum

$$\mathrm{Spec}(H(t) \oplus 0, S_t, N : E)$$

is fixed as a set. Denote the common set by Spec $(L_S, N : E)$ We define the "gap"

$$\epsilon := \min_{\lambda,\mu}\{|\lambda - \mu| \mid \lambda \neq \mu, \in \mathrm{Spec}(L_S, N : E)\}.$$

Since $t \mapsto (H(t), S_t)$ is a smooth path, there exists some $\delta > 0$ such that

$$\|H(u) - H(s)\|_{C^0} + \|S_u - S_s\|_{C^0} < \frac{\epsilon}{3} \tag{18}$$

for all $u, s \in [0, 1]$ with $|u - s| < \delta$. Choose a partition

$$0 = t_0 < t_1 < \ldots < t_k = 1$$

such that

$$|t_j - t_{j-1}| < \delta \quad \text{for all } j.$$

By Theorem 7, the canonical chain map

$$h_{us} : CF_*(H(u) \oplus 0, S_u, N : E) \to CF_*(H(s) \oplus 0, S_s, N : E)$$

restricts to

$$h_{us} : CF_*^{\lambda}(H(u) \oplus 0, S_u, N : E) \to CF_*^{\lambda+\frac{\epsilon}{3}}(H(s) \oplus 0, S_s, N : E)$$

for any $u, s \in [0,1]$ with $|u - s| < \delta$. Similarly, we have

$$h_{su} : CF_*^\lambda(H_s \oplus 0, S_s, N : E) \to CF_*^{\lambda+\frac{\epsilon}{3}}(H(u) \oplus 0, S_u, N : E)$$

for any $\lambda \in \mathbb{R}$. Combining these two, we have the composition

$$h_{su} \circ h_{us} : CF_*^\lambda(H(u) \oplus 0, S_u, N : E) \to CF_*^{\lambda+\frac{2\epsilon}{3}}(H(u) \oplus 0, S_u, N : E).$$

To simplify notation, we will write $HF_*(J, H, S, N : E)$ instead of $HF_*(J \oplus i, H \oplus 0, S, N : E)$. By the condition (18) and the gap assumption, all of these three maps in fact preserve same levels and induce homomorphisms

$$h_{us}^\lambda : HF_*^\lambda(J, H(u), S_u, N : E) \to HF_*^\lambda(J, H(s), S_s, N : E)$$

$$h_{su}^\lambda : HF_*^\lambda(J, H(s), S_s, N : E) \to HF_*^\lambda(J, H(u), S_u, N : E)$$

and

$$h_{su}^\lambda \circ h_{us}^\lambda : HF_*^\lambda(J, H(u), S_u, N : E) \to HF_*^\lambda(J, H(u), S_u, N : E),$$

provided λ is chosen sufficiently close to Spec $(L_S, N : E)$. However, if we choose δ sufficiently small, it follows (see [M] for details) that

$$h_{su}^\lambda \circ h_{us}^\lambda = h_{uu}^\lambda (= \text{ id }) \text{ on } HF_*^\lambda(J, H(u), S_u, N : E)$$

which implies that h_{su}^λ is an isomorphism for all u, s with $|u - s| < \delta$. By repeating the above to $(u, s) = (t_j, t_{j+1})$ for $j = 0, \ldots, N-1$, we conclude that the composition

(19)
$$h_{t_jt_{j-1}} \circ \cdots \circ h_{t_1t_0} : CF_*(H(0) \oplus 0, S_0, N : E) \to CF_*(H(t_j) \oplus 0, S_{t_j}, N : E)$$

restricts to

$$h_{t_jt_{j-1}}^\lambda \circ \cdots \circ h_{t_1t_0}^\lambda : CF_*^\lambda(H(0) \oplus 0, S_0 : N : E) \to CF_*^\lambda(H(t_j) \oplus 0, S_{t_j}, N : E)$$

for all $1 \le j \le N$, and so induces the composition

$$h_{t_jt_{j-1}}^\lambda \circ \cdots \circ h_{t_1t_0}^\lambda : HF_*^\lambda(J, H(0), S_0; N, E) \to HF_*^\lambda(J, H(t_j), S_{t_j}, N : E)$$

which becomes an isomorphism. In particular, we have the isomorphism

$$(20) \quad h_{t_Nt_{N-1}}^\lambda \circ \cdots \circ h_{t_1t_0}^\lambda : HF_*^\lambda(J, 0, S; N, E) \to HF_*^\lambda(J, H, Q, N : E)$$

Since both (20) and the global homomorphism

$$(21) \quad h_{t_Nt_{N-1}} \circ \cdots \circ h_{t_1t_0} : HF_*(J, 0, S; N, E) \to HF_*(J, H, Q, N : E)$$

are induced by the chain map (19), the commutativity

$$\begin{array}{ccc} HF_*^\lambda(J,H,Q,N:E) & \stackrel{(20)}{\to} & HF_*^\lambda(J,0,S,N:E) \\ j_*^\lambda \downarrow & & \downarrow j_*^\lambda \\ HF_*(J,H,Q,N:E) & \stackrel{(21)}{\to} & HF_*(J,0,S,N:E) \end{array}$$

follows. However the global map (21) is the same as the canonical isomorphism

$$h_{01} : HF_*(J,0,S,N:E) \to HF_*(J,H,Q,N:E).$$

Since we have proven in Section 4

$$HF_*^\lambda(J,H,Q,N:E) \cong HF_*^\lambda(J,H,N:M)$$

and that the isomorphism (17)

$$HF_*(J,0,S,N:E) \cong H_*(S,N:E)$$

can be chosen so that its filtration level is preserved, this finally finishes the proof of the main theorem.

REMARK 8. We would like to emphasize that the isomorphism in the main theorem depends on the path $t \mapsto (H(t), S_t)$. It is not clear to us whether the canonical chain map

$$h_{01} : CF_*(0,S,N:E) \to CF_*(H \oplus 0,Q,N:E)$$

in terms of the isotopy $t \mapsto (H(t), S_t)$ induces level preserving isomorphism in homology, unless the path partitioned as above.

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Department of Mathematics
University of Wisconsin–Madison
Madison, WI 53706