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SINGULAR SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS IN SEVERAL SPACE DIMENSIONS

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ABSTRACT. We study the existence and uniqueness of nonnegative singular solutions $u(\mathbf{x}, t)$ of the semilinear parabolic equation

$$u_t = \Delta u - \mathbf{a} \cdot \nabla(u^q) - u^p,$$

defined in the whole space $\mathbf{R}^N$ for $t > 0$, with initial data $M\delta(\mathbf{x})$, a Dirac mass, with $M > 0$. The exponents p, q are larger than 1 and the direction vector $\mathbf{a}$ is assumed to be constant.

We here show that a unique singular solution exists for every $M > 0$ if and only if $1 < q < (N+1)/(N-1)$ and $1 < p < 1 + (2q^*)/(N+1)$, where $q^* = \max\{q, (N+1)/N\}$. This result agrees with the earlier one for $N = 1$. In the proof of this result, we also show that a unique singular solution of a diffusion-convection equation without absorption,

$$u_t = \Delta u - \mathbf{a} \cdot \nabla(u^q),$$

exists if and only if $1 < q < (N - 1)/(N - 1)$.

1. Introduction

This paper is concerned with the Cauchy problem for the nonlinear diffusion-convection equation

$$\text{(E)} \qquad u_t = \Delta u - \mathbf{a} \cdot \nabla(u^q), \quad q > 1$$

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and for the nonlinear diffusion-convection equation with absorption

$$(\mathrm{F}) \qquad u_t = \Delta u - \mathbf{a} \cdot \nabla(u^q) - u^p, \quad p, q > 1$$

in the domain $Q = \{(\mathbf{x}, t) : \mathbf{x} \in \mathbf{R}^N, t > 0\}$. We assume for convenience that the convection direction $\mathbf{a}$ is a constant vector in $\mathbf{R}^N$.

Following Escobedo and Zuazua (1991) (see [5]), it is easy to see that for given nonnegative initial data $u_0(\mathbf{x}) \in L^1(\mathbf{R}^N)$ there exists a unique nonnegative solution $u(\mathbf{x}, t)$ of (E) and (F) such that

$$u \in C((0,\infty); W^{2,l}(\mathbf{R}^N)) \cap C^1((0,\infty); L^l(\mathbf{R}^N))$$

for every $l \in (1, \infty)$. Moreover, $u(\mathbf{x}, t)$ is positive unless $u \equiv 0$ in Q by the Maximum Principle and becomes C^∞ smooth in Q by the standard regularity theory.

A singular solution is the one which develops a singularity at $(\mathbf{x}, t) = (0, 0)$ as $t \to 0$. More precisely, a singular solution (it is also called a fundamental solution or a source-type solution) is the one corresponding to the initial data $M\delta(\mathbf{x})$ (a Dirac mass) with $M > 0$, namely, a classical solution $u(\mathbf{x}, t)$ of (E) or (F) such that $u(\mathbf{x}, t) \to M\delta(\mathbf{x})$ as $t \to 0$ in the sense of measures, that is,

$$\lim_{t \to 0} \int_{\mathbf{R}^N} u(\mathbf{x}, t)\phi(\mathbf{x})d\mathbf{x} = M\phi(0),$$

for every bounded continuous function ϕ on $\mathbf{R}^N$.

By rotating the coordinate axes and rescaling, we may assume that the convection direction $\mathbf{a}$ satisfies $\mathbf{a} = qe_N$, $e_N = (0, \cdots, 0, 1)$. With the convenient notation $\mathbf{x} = (x, y) \in \mathbf{R}^{N-1} \times \mathbf{R}$, (E) and (F) become

$$(1.1) \qquad u_t = \Delta u - u^{q-1} u_y$$

and

$$(1.2) \qquad u_t = \Delta u - u^{q-1} u_y - u^p,$$

respectively.

Escobedo, Vazquez and Zuazua (see [4]) have studied the asymptotic behaviour of solutions of (1.1) with nonnegative initial data $u_0(\mathbf{x}) \in$

$\mathbf{L}^1(R^N)$ in the range $1 < q < (N+1)/N$. They have shown that the asymptotic behaviour of generic solutions of this problem is given in terms of the fundamental solution of the reduced equation

$$\text{(R)} \qquad u_t = \Delta_x u - u^{q-1}u_y,$$

where Δ_x denotes the Laplace operator acting only on the variable x which spans the hyperplane perpendicular to $\mathbf{a}$ in $\mathbf{R}^N$. Thus the diffusion in the direction $\mathbf{a}$ disappears in the limit as $t \to \infty$. The question of existence of nonnegative singular solutions of (E) and (R) has been treated but not completed.

The main purpose of this paper is to complete the question of existence of nonnegative singular solutions of (E), (R) and (F) for all the range of p and q. In fact, we prove that

THEOREM A. *There exists a unique singular solution of* (E) (*and* R)) *for every total mass* $M > 0$ *if and only if* $1 < q < (N+1)/(N-1)$.

THEOREM B. *There exists a unique singular solution of* (F) *for every* $M > 0$ *if and only if* $1 < q < (N+1)/(N-1)$ *and* $1 < p < 1 + (2q^*)/(N+1)$. *Here* $q^* = \max\{q, (N+1)/N\}$

We recall that the heat equation with absorption $u_t = \Delta u - u^p$ admits a unique singular solution only for $1 < p < 1 + 2/N$, see [2]. The case $N = 1$ has been considered in [1] and the existence range becomes $1 < p < 1 + q^*$ for any $q > 1$.

For the proof, we first obtain the so-called $L^1 - L^\infty$ regularizing effect for (E):

$$0 \le u(\mathbf{x}, t) \le C(q, N) M^{\frac{1}{q}} t^{-\frac{N+1}{2q}},$$

for any $q > 1$, which generalizes the main estimate of [4]. Here and in the sequel, $C(q, N)$ will denote a constant depending only on q and N. In fact, the estimate is essential in proving both existence and nonexistence of singular solutions. Together with this estimate we use a comparison argument for the proof of existence and uniqueness, see section 3 and 4.

For nonexistence, we treat three cases separately. For $1 < q < (N+1)/(N-1)$ and $1 < p < 1 + (2q^*)/(N+1)$, following [1], we take a function of the form $\eta(k(|x|^2 + |y|^\alpha + t))$ with $\alpha = \frac{2q^*}{N+1-(N-1)q^*}$ as a test function to lead to a contradiction. For $q \geq (N+1)/(N-1)$ and $p \geq 1 + (2q^*)/(N+1)$, using a similarity transformation

$$x \to \lambda^{1/2}x, \quad y \to \lambda^{\frac{N+1-(N-1)q}{2q}}, \quad t \to \lambda t,$$

we will show that a family $\{u_\lambda\}$ defined by

$$u_\lambda(x,y,t) = \lambda^{\frac{N+1}{2q}} u(\lambda^{\frac{1}{2}}x, \lambda^{\frac{(N+1)-(N-1)q}{2q}}y, \lambda t)$$

has a limit as $\lambda \to 0$ and its limit is a singular solution of the reduced equation (R), which contradicts the nonexistence result of singular solution for (R). For the completion of proof of limiting process, we bring a compensated compactness argument from Evans [6] and Tartar [9]. In case $q \geq (N+1)/(N-1)$ and $1 < p < 1 + (2q^*)/(N+1)$, we use the same similarity transformation and consider the limit of

$$\int_0^1 \int_{|x|<1} \int_{|y|<1} u_\lambda(x,y,t)\,dxdydt$$

as $\lambda \to 0$ to lead to another contradiction. See section 5 for details. This nonexistence argument is also applicable for (E) in the range $q \geq (N+1)/(N-1)$.

The asymptotic behaviour of solutions of (F) as $t \to \infty$ (also as $t \to 0$) will be discussed in other space.

2. Preliminary Estimates

For the diffusion-convection equation (E), it is shown in [5] (see estimate (2.32) in that paper) that

$$0 \leq u(x,t) \leq C(M)(t^{-N/2} + t^{(1-Nq)/2}), \qquad \forall t > 0, \tag{2.1}$$

where $C(M)$ is a constant depending only on $M := \int_{\mathbf{R}^N} u_0(\mathbf{x})d\mathbf{x}$. This estimate is not sharp and for $1 < q \leq 2$, Escobedo, Vazquez and Zuazua obtained a better (for large t) bound for u:

LEMMA 2.1 ([4], LEMMA 2.2). *For $1 < q \leq 2$ we have*

$$0 \leq u(\mathbf{x},t) \leq C(q,N)M^{1/q}t^{-(N+1)/(2q)}, \quad t > 0. \tag{2.2}$$

We here extend the above estimate (2.2) to the whole range of q.

LEMMA 2.2. *For $q > 2$ we have*

$$0 \le u(\mathbf{x}, t) \le C(q, N) M^{1/q} t^{-(N+1)/(2q)}, \quad t > 0. \tag{2.3}$$

Proof. We define the new variable $z = u^{q-1}$ as explained in [4]. In terms of z, equation (E) reads

$$z_t - \Delta z + z z_y - \beta \frac{|\nabla z|^2}{z} = 0, \tag{2.4}$$

where $\beta = (2 - q)/(q - 1)$.

Let $v(x, t)$ be a solution of the Burgers type equation

$$v_t - \Delta v + v v_y = 0 \tag{2.5}$$

with initial data $v(\mathbf{x}, 0) = u^{q-1}(\mathbf{x}, s)$, $s > 0$. For $q > 2$, $\beta < 0$ and $v(\mathbf{x}, t)$ is a supersolution of (2.5) and Nagumo's Lemma implies that

$$u^{q-1}(\mathbf{x}, t + s) \le v(\mathbf{x}, t), \quad t > 0.$$

Moreover from Lemma 2.1 we obtain

$$0 \le v(\mathbf{x}, t) \le C(2, N) \left(\int v(\mathbf{x}, 0) d\mathbf{x} \right)^{1/2} t^{-(N+1)/4}$$

and from the conservation of mass of solutions of (2.5) we get

$$0 \le u^{q-1}(\mathbf{x}, t + s) \le C(2, N) \left(\int u^{q-1}(\mathbf{x}, s) d\mathbf{x} \right)^{1/2} t^{-(N+1)/4}, \quad t > 0. \tag{2.6}$$

We also have

$$\int u^{q-1}(\mathbf{x}, s) d\mathbf{x} \le |u^{q-2}(\mathbf{x}, s)|_{L^\infty(\mathbf{R}^N)} \int u(\mathbf{x}, s) d\mathbf{x} = M |u(\mathbf{x}, s)|^{q-2}_{L^\infty(\mathbf{R}^N)}.$$

We now let $s = t + \epsilon$, with $\epsilon > 0$ and define

$$w(t) = \sup_{0 < \tau \le t} \tau^{(N+1)/(2q)} |u(\mathbf{x}, \tau + \epsilon)|_{L^\infty(\mathbf{R}^N)},$$

then, for $t > 0$, $0 \le w(t) < \infty$ and

$$0 \le w(t)^{q-1} \le w(2t)^{q-1} \le C(q, N) M^{1/2} w(t)^{(q-2)/2}.$$

Therefore we have

$$w(t) \le C(q, N) M^{1/q}, \qquad \forall t > 0. \tag{2.7}$$

This implies that $0 \le t^{(N+1)/(2q)} u(\mathbf{x}, t + \epsilon) \le C(q, N) M^{1/q}$ and (2.3) in the limit as $\epsilon \to 0$. □

3. Existence

We discuss in this section the existence of fundamental solutions for equations (E) and (F).

THEOREM 3.1. *For* $1 < q < (N+1)/(N-1)$ *there exists a fundamental solution of* (E) *for every* $M > 0$.

Proof. The proof of existence for the reduced equation (R) given in [4] can be directly applied to this case. The only different point is when checking the initial data where they use Lemma 2.1 and the restriction $(N+1)/(N-1) \leq 2$ appears. Due to Lemma 2.2, no such a restriction is needed. (See [4], Theorem 8.1 for details.) □

We now turn to the equation (F). We denote by $E_M(\mathbf{x}, t)$ the singular solution for (E) corresponding to the initial data, a Dirac mass, $M\delta(\mathbf{x})$. We note that $\int E_M(\mathbf{x}, t) d\mathbf{x} = M$ for all $t > 0$. We also denote $q^* = \max\{q, (N+1)/N\}$ and we prove

THEOREM 3.2. *For* $1 < q < (N+1)/(N-1)$ *and* $1 < p < 1 + (2q^*)/(N+1)$, *there exists a fundamental solution of* (F) *for every* $M > 0$.

Proof. Let us define $\tilde{u}_n(\mathbf{x}, t)$ as the solution of (F) for $t > 1/n$ with initial data $\tilde{u}_n(\mathbf{x}, 1/n) = E_M(\mathbf{x}, 1/n)$ at $t = 1/n$. By the Maximum Principle (see [8] for example) we know that

$$\tilde{u}_n(\mathbf{x}, t) \leq E_M(\mathbf{x}, t) \qquad \forall t \geq 1/n \quad \text{and} \quad \mathbf{x} \in \mathbf{R}^N.$$

Since $\{\tilde{u}_n(\mathbf{x}, t)\}$ is monotone decreasing as $n \to \infty$, its limit

$$u(\mathbf{x}, t) = \lim_{n \to \infty} \tilde{u}_n(\mathbf{x}, t)$$

exists and $u(\mathbf{x}, t)$ is a weak solution of (F) in Q. By standard regularity results, we may conclude that $u(\mathbf{x}, t)$ is a classical solution in Q.

It is clear that $u(\mathbf{x}, t) \leq E_M(\mathbf{x}, t)$ in Q and $u(\mathbf{x}, 0) = 0$ for every

$\mathbf{x} \neq 0$. Moreover, for $1/n < t \leq 1$

$$
\begin{aligned}
I_n(t) = &\left| \int_{\mathbf{R}^N} (\tilde{u}_n(\mathbf{x}, t) - \tilde{u}_n(\mathbf{x}, 1/n))\, d\mathbf{x} \right| \\
= &\int_{1/n}^{t} \int_{\mathbf{R}^N} \tilde{u}_n^p(\mathbf{x}, s) d\mathbf{x} ds \\
\leq &\int_{1/n}^{t} \int_{\mathbf{R}^N} E_M^p(\mathbf{x}, s) d\mathbf{x} ds \\
\leq &\int_{1/n}^{t} M \sup_{\mathbf{R}^N} E_M^{p-1}(\cdot, s) ds \\
\leq &C(q, M, N) \int_{1/n}^{t} s^{-(N+1)(p-1)/(2q^*)} ds \quad \text{(from (2.1) and (2.3))} \\
\leq &C(q, M, N) \frac{2q^*}{2q^* - (N+1)(p-1)} t^{\frac{2q^* - (N+1)(p-1)}{2q^*}}.
\end{aligned}
$$

Note that $2q^* - (p-1)(N+1) > 0$ if $p < 1 + (2q^*)/(N+1)$.

Since $\int \tilde{u}_n(\mathbf{x}, 1/n) d\mathbf{x} = M$ by the definition of $\tilde{u}_n$, first taking $n \to \infty$ and then taking $t \to 0$ we may conclude that $\lim_{t \to 0} \int u(\mathbf{x}, t) d\mathbf{x} = M$. For every continuous bounded function ϕ,

$$
0 \leq \int u(\mathbf{x}, t)[\phi(\mathbf{x}) - \phi(0)]_+ d\mathbf{x} \leq \int E_M(\mathbf{x}, t)[\phi(\mathbf{x}) - \phi(0)]_+ d\mathbf{x}
$$

and since the second integral tends to 0 as $t \to 0$, the same will be true for the first integral. Similarly

$$
\lim_{t \to 0} \int u(\mathbf{x}, t)[\phi(x) - \phi(0)]_- d\mathbf{x} = 0
$$

and writing

$$
\phi(\mathbf{x}) = \phi(0) + [\phi(\mathbf{x}) - \phi(0)]_+ - [\phi(\mathbf{x}) - \phi(0)]_-,
$$

we finally obtain

$$
\lim_{t \to 0} \int u(\mathbf{x}, t) \phi(\mathbf{x}) d\mathbf{x} = M\phi(0).
$$

This completes the proof. □

4. Uniqueness

The uniqueness of singular solutions of (E) and (R) is proved in [4], section 3. For equation (F), we prepare the following lemmas.

LEMMA 4.1. *Let $u(\mathbf{x},t)$ be a fundamental solution of* (F), *then*

$$\int u(\mathbf{x},t)d\mathbf{x} - M = -\int_0^t \int u^p(\mathbf{x},s)d\mathbf{x}ds. \tag{4.1}$$

In particular $\int_0^\infty \int u^p d\mathbf{x}ds \le M$ and $\int u(\mathbf{x},t)d\mathbf{x} \le M$ for every $t > 0$.

Proof. An integration gives

$$\int u(\mathbf{x},t)d\mathbf{x} - \int u(\mathbf{x},\tau)d\mathbf{x} = -\int_\tau^t \int u^p(\mathbf{x},s)d\mathbf{x}ds$$

for $t > \tau > 0$. By definition $\lim_{\tau\to 0}\int u(\mathbf{x},\tau)dx = M$ and (4.1) holds. The last inequality also holds obviously. □

LEMMA 4.2. *Let $u(\mathbf{x},t)$ be a fundamental solution of* (F), *then $u(\mathbf{x},t) \le E_M(\mathbf{x},t)$ in Q.*

Proof. Let $w_n(\mathbf{x},t)$ be a solution of

$$w_t = \Delta w - (w^q/q)_y, \quad w(\mathbf{x},1/n) = u(\mathbf{x},1/n)$$

for $t > 1/n$. Similarly to the proof of Theorem 3.2, one can see that w_n is increasing and $u(\mathbf{x},t) \le w_n(\mathbf{x},t)$ for $t > 1/n$. Let $w = \lim_{n\to\infty} w_n$, then $u(\mathbf{x},t) \le w(\mathbf{x},t)$ for $t > 0$ and $w(\mathbf{x},t)$ is a weak solution of (E) and also becomes a classical solution in Q.

We now note that

$$\int w_n(\mathbf{x},t)d\mathbf{x} = \int w_n(\mathbf{x},1/n)d\mathbf{x} = \int u(\mathbf{x},1/n)d\mathbf{x} \le M$$

for $t > 1/n$. From the Monotone Convergence Theorem, we obtain

$$\int w(\mathbf{x},t)d\mathbf{x} \le M.$$

Since $\int u(\mathbf{x},t)d\mathbf{x}$ goes to M as t tends to 0, the same will be true for $\int w(\mathbf{x},t)dx$. We now apply the same argument as the proof of Theorem 3.2 to conclude that

$$\lim_{t\to 0}\int w(\mathbf{x},t)\phi(\mathbf{x})d\mathbf{x} = M\phi(0) \tag{4.2}$$

for every continuous bounded function ϕ. Thus $w(\mathbf{x},t)$ is a fundamental solution of (E) and the completion of the proof of Lemma follows from the uniqueness of the fundamental solution of (E). But here we take $\psi(\mathbf{x}) = |\phi|_{L^\infty} - [\phi(\mathbf{x}) - \phi(0)]_+$ as a test function. Then

$$0 \le \int u(\mathbf{x},t)\psi(\mathbf{x})d\mathbf{x} \le \int w(\mathbf{x},t)\psi(\mathbf{x})d\mathbf{x} \le M|\phi|_{L^\infty}.$$

Since $\lim_{t\to 0}\int u(\mathbf{x},t)\psi(\mathbf{x})d\mathbf{x} = M|\phi|_{L^\infty}$, we obtain

$$\lim_{t\to 0}\int w(\mathbf{x},t)[\phi(\mathbf{x}) - \phi(0)]_+ d\mathbf{x} = 0. \tag{4.3}$$

Similarly we have

$$\lim_{t\to 0}\int w(\mathbf{x},t)[\phi(\mathbf{x}) - \phi(0)]_- d\mathbf{x} = 0$$

and (4.2). □

We now prove the uniqueness.

THEOREM 4.3. *There exists at most one fundamental solution of* (F) *for each* $M > 0$.

Proof. Let u, v be fundamental solutions of (F) with initial data $M\delta(\mathbf{x})$. Then from the contraction principle and Lemma 4.1 we have

$$\begin{aligned}\int |u(\mathbf{x},t) - v(\mathbf{x},t)|d\mathbf{x} &\le \int |u(\mathbf{x},1/n) - v(\mathbf{x},1/n)|d\mathbf{x} \\ &\le \int (E_M(\mathbf{x},1/n) - u(\mathbf{x},1/n))d\mathbf{x} \\ &\quad + \int (E_M(\mathbf{x},1/n) - v(\mathbf{x},1/n))d\mathbf{x} \\ &= 2M - \int u(\mathbf{x},1/n)d\mathbf{x} - \int v(\mathbf{x},1/n)d\mathbf{x},\end{aligned}$$

which tends to 0 as $n \to \infty$ from lemma 4.1. Hence $u \equiv v$ in Q. □

5. Nonexistence

We first consider the case $1 < q < (N+1)/(N-1)$ and $p \geq 1 + (2q^*)/(N+1)$. Recall that $q^* = \max\{q, (N+1)/N\}$. Let $u(\mathbf{x}, t)$ be a fundamental solution of (F) with initial data $M\delta$, then from Lemma 4.1 we see that $u(\mathbf{x}, t)$ is L^p-integrable over Q.

Following [1], we prove that

THEOREM 5.1. *For* $1 < q < (N+1)/(N-1)$ *and* $p \geq 1+(2q^*)/(N+1)$ *there is no fundamental solution of* (F).

Proof. Let $\eta(s)$ be any smooth non-decreasing function on $\mathbf{R}$ such that

$$\eta(s) = \begin{cases} 1 & \text{for} \quad s \geq 1 \\ 0 & \text{for} \quad s \leq 0, \end{cases}$$

and set $\eta_k(s) = \eta(ks)$.

If we define $\phi_k(x, y, t) = \eta_k(|x|^2 + |y|^\alpha + t)$, where $\alpha \geq 2$ is a constant to be determined, then we know that

$$\int_\epsilon^T \int_{\mathbf{R}^N} u_t \phi_k - \int_\epsilon^T \int_{\mathbf{R}^N} u \Delta \phi_k - \int_\epsilon^T \int_{\mathbf{R}^N} \frac{u^q}{q} \phi_{k,y} + \int_\epsilon^T \int_{\mathbf{R}^N} u^p \phi_k = 0$$

for $0 < \epsilon < T$. We first have

$$\int_\epsilon^T \int_{\mathbf{R}^N} u_t \phi_k d\mathbf{x} dt = \int_{\mathbf{R}^N} u(\mathbf{x}, T) \phi_k(\mathbf{x}, T) d\mathbf{x} - \int_{\mathbf{R}^N} u(\mathbf{x}, \epsilon) \phi_k(\mathbf{x}, \epsilon) d\mathbf{x} - \int_\epsilon^T \int_{\mathbf{R}^N} u \phi_{k,t}$$

and in the limit as $\epsilon \to 0$ we get

$$\int_0^T \int u^p \phi_k \leq \int_0^T \int u \phi_{k,t} + \int_0^T \int u \Delta \phi_k + \int_0^T \int \frac{u^q}{q} \phi_{k,y}. \tag{5.1}$$

We claim that all the integral in the right tends to 0 as $k \to \infty$. It then follows that $\int_0^T \int u^p d\mathrm{x}dt = 0$ and $u \equiv 0$ in Q. We have

$$(5.2)\qquad \begin{aligned} \left|\int_0^T \int u\phi_{k,t}\right| &\le Ck \iint_{D_k} u, \\ \left|\int_0^T \int u\Delta_x\phi_k\right| &\le Ck \iint_{D_k} u, \\ \left|\int_0^T \int u\phi_{k,yy}\right| &\le Ck^{2/\alpha} \iint_{D_k} u, \\ \left|\int_0^T \int \frac{u^q}{q}\phi_{k,y}\right| &\le Ck^{1/\alpha} \iint_{D_k} u^q, \end{aligned}$$

where $D_k = \{(x, y, t); t > 0, \quad 0 < |x|^2 + |y|^\alpha + t \le 1/k\}$. By Hölder inequality we get:

$$(5.3)\qquad \begin{aligned} \iint_{D_k} u &\le \left(\iint_{D_k} u^p\right)^{1/p} |D_k|^{1-1/p} \\ \iint_{D_k} u^q &\le \left(\iint_{D_k} u^p\right)^{q/p} |D_k|^{1-q/p}. \end{aligned}$$

Here we note that

$$p - q \ge 1 + \frac{2q^*}{N+1} - q \ge 1 - \frac{N-1}{N+1}q > 0.$$

We also have

$$(5.4)\qquad |D_k| \le 2^N k^{-((N+1)/2+1/\alpha)}$$

and our claim will follow from the fact that $\iint_{D_k} u^p \to 0$ as $k \to \infty$ if the next inequalities hold:

$$(5.5)\qquad 1 - \left(\frac{1}{\alpha} + \frac{N+1}{2}\right)\left(1 - \frac{1}{p}\right) \le 0,$$

$$(5.6)\qquad \frac{1}{\alpha} - \left(\frac{1}{\alpha} + \frac{N+1}{2}\right)\left(1 - \frac{q}{p}\right) \le 0.$$

As an example we take $\alpha = \frac{2q^*}{N+1-(N-1)q^*}$, then

$$\alpha - 2 = 2\frac{Nq^* - (N+1)}{N+1-(N-1)q^*} \geq 0$$

and the left sides of (5.5) and (5.6) become

$$-\frac{N+1}{2pq^*}\left(p - 1 - \frac{2q^*}{N+1}\right)$$

and

$$-\frac{N+1}{2pq^*}\left(p(q^* - q) + q(p - 1 - \frac{2q^*}{N+1})\right)$$

respectively. Both of them are nonpositive from assumption. Hence the proof is completed. □

We now turn to the case $q \geq (N+1)/(N-1)$. We introduce a similarity transformation

$$x \to \lambda^{\frac{1}{2}}x, \quad y \to \lambda^{\gamma}y, \quad t \to \lambda t$$

and define

$$u_\lambda(x, y, t) = \lambda^{\beta}u(\lambda^{\frac{1}{2}}x, \lambda^{\gamma}y, \lambda t).$$

Suppose that $u(x, y, t)$ be a fundamental solution of (F) with initial data $M\delta(x)$ and if we assume that

$$\beta - \frac{N-1}{2} - \gamma = 0, \tag{5.7}$$

then since

$$\int u_\lambda(x, y, t)dxdy = \lambda^{\beta-(N-1)/2-\gamma}\int u(x, y, \lambda t)dxdy,$$

we obtain that

$$\left|\int u_\lambda(x, y, t)dxdy\right| \leq M \tag{5.8}$$

and

$$\lim_{t\to 0}\int u_\lambda(x,y,t)dxdy = M \tag{5.9}$$

uniformly in $0<\lambda\le 1$. It is easy to see that u_λ satisfies

(5.10)
$$\begin{aligned} u_{\lambda,t} = \Delta_x u_\lambda + \lambda^{1-2\gamma} u_{\lambda,yy} - \lambda^{1-\gamma-(q-1)\beta} u_\lambda^{q-1} u_{\lambda,y} \\ - \lambda^{\beta+1-\beta p} u_\lambda^p. \end{aligned}$$

We also assume that $1-\gamma-(q-1)\beta=0$, then together with (5.7), we have

$$\beta = \frac{N+1}{2q}, \quad \gamma = \frac{(N+1)-(N-1)q}{2q}.$$

Now observe that

$$1-2\gamma = \frac{N}{q}\left(q-\frac{N+1}{N}\right) > 0, \tag{5.11}$$

$$\beta+1-\beta p = \frac{N+1}{2q}\left(1+\frac{2q}{N+1}-p\right). \tag{5.12}$$

We split into two cases:

Case I: $1<p<1+(2q)/(N+1)$.

In this case (5.12) becomes positive. Hence we may consider equation (5.10) as a small perturbation of the reduced equation (R) with convection in the y-direction and diffusion in the orthogonal directions:

$$u_t = \Delta_x u - u^{q-1}u_y. \tag{R}$$

The nonexistence result will consist precisely in showing that in the limit $\lambda\to 0$, we will obtain a fundamental entropy solution of (R) with mass $M>0$. This will lead to a contradiction.

It is easy to check that the uniform L^∞-estimates of u_λ is still valid. For every $\lambda>0$ and every $t>0$, we have

$$0\le u_\lambda(x,y,t)\le Ct^{-\frac{N+1}{2q}} \quad \text{for every} \quad (x,y)\in \mathbf{R}^N. \tag{5.13}$$

Hence the family $\{u_\lambda\}$ is uniformly bounded in $L^\infty(\mathbf{R}^N \times (\tau, T))$ and we may extract a subsequence $\{u_{\lambda_j}\}_{j=1}^\infty$ which converges in the weak star topology of L^∞. We now apply Theorem 26, page 202, of Tartar [9] (see also Theorem 6, page 57, of Evans [6]) with a minor modification (when we apply the Div-Curl Lemma, we mainly consider the vector fields

$$v_k = (0, F(u^{\epsilon_k}), u^{\epsilon_k}) \quad \text{and} \quad w_k = (0, \Phi(u^{\epsilon_k}), -\Psi(u^{\epsilon_k}))$$

in the variables $(x, y, t) \in \mathbf{R}^{N-1} \times \mathbf{R} \times (0, \infty)$, see page 58, [6].) to conclude that along such a sequence

$$\begin{aligned} u_{\lambda_j} &\overset{*}{\rightharpoonup} U \quad \text{in} \quad L^\infty_{loc}(Q), \\ u^q_{\lambda_j} &\overset{*}{\rightharpoonup} U^q \quad \text{in} \quad L^\infty_{loc}(Q). \end{aligned} \tag{5.14}$$

Thus U is an entropy solution of (R), see [4] for the precise definition.

We now want to check the initial condition. We see from (5.9) that the total mass of U is M. Moreover

$$v_\lambda(x,t) = \int u_\lambda(x,y,t)dy = \lambda^{(N-1)/2} \int u(\lambda^{1/2}x, y, \lambda t)dy$$

solves the equation

$$v_{\lambda,t} - \Delta_x v_\lambda = -\lambda^{\beta+1-\beta p} \int u^p_\lambda dy \leq 0, \tag{5.15}$$

with initial data a Dirac mass, $M\delta(x)$ in $\mathbf{R}^{N-1}$. Therefore v_λ is bounded from above by the fundamental solution $G(x,t)$ of the linear heat equation in $\mathbf{R}^{N-1}$ with mass M. Now for any $r > 0$,

$$\int_{|x|>r} \int_{\mathbf{R}} u_\lambda(x,y,t)dydx = \int_{|x|>r} v_\lambda(x,t)dx \leq \int_{|x|>r} G(x,t)dx,$$

which tends to 0 as $t \to 0$ uniformly in $0 < \lambda \leq 1$. One also sees that

$$\int_{\mathbf{R}^{N-1}} \int_{|y|>r} u_\lambda(x,y,t)dydx = \int_{\mathbf{R}^{N-1}} \int_{|\eta|\geq\lambda^\gamma r} u(\xi, \eta, \lambda t)d\eta d\xi,$$

which also tends to 0 as $t \to 0$ uniformly in $0 < \lambda \le 1$. Notice that $\gamma \le 0$ and $\lambda^\gamma r \ge r$. Hence we have from (5.9) that

$$\lim_{t\to 0}\int_{|x|<r}\int_{|y|<r} u_\lambda(x,y,t)dydx = M \tag{5.16}$$

uniformly in $0 < \lambda \le 1$ and $\lim_{t\to 0} U(x,y,t)$ is a Dirac mass with mass M. But this contradicts obviously to the nonexistence result of fundamental entropy solution of (R), see Corollary 5.5, [4].

The above argument does not apply to the second case.

Case II. $p \ge 1 + (2q)/(N+1)$.

Nevertheless (5.9), (5.15) and the arguments leading to (5.16) still holds and in particular (5.16) implies

$$\int_0^1\int_{|x|<1}\int_{|y|<1} u_\lambda(x,y,t)dydxdt \ge \epsilon \tag{5.17}$$

for all $0 < \lambda \le 1$ and for some $\epsilon > 0$. On the other hand

$$\begin{aligned}
&\int_0^1\int_{|x|<1}\int_{|y|<1} u_\lambda(x,y,t)dydxdt \\
&\quad = \int_0^1\int_{|x|<1}\int_{|y|<1} \lambda^\beta u(\lambda^{1/2}x,\lambda^\gamma y,\lambda t)dydxdt \\
&\quad = \int_0^\lambda\int_{|\xi|<\lambda^{1/2}}\int_{|\eta|<\lambda^\gamma} \lambda^{-1}u(\xi,\eta,s)d\xi d\eta ds,
\end{aligned}$$

which is bounded from the Hölder inequality by

$$\lambda^{\frac{N+1}{2pq}(p-1-\frac{2q}{N+1})}\left(\int_0^\lambda\int_{|\xi|<\lambda^{1/2}}\int_{|\eta|<\lambda^\gamma} u^p(\xi,\eta,s)d\xi d\eta ds\right)^{1/p}. \tag{5.18}$$

Lemma 4.1 implies the $L^p(Q)$-integrability of u and this in turn implies that (5.18) tends to 0 as λ tends to 0, which contradicts to (5.17). Therefore we have proved

THEOREM 5.2. *For $q \geq (N+1)/(N-1)$, there is no singular solution of* (F).

The above compensated compactness argument used in Case I also works for the equation (E) and we may conclude that

THEOREM 5.3. *For $q \geq (N+1)/(N-1)$, there is no singular solution of* (E).

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