J. Korean Math. Soc. **34** (1997), No. 4, pp. 1037–1048

CONFORMAL COMPACTIFICATION OF ASYMPTOTICALLY EUCLIDEAN SCALAR-FLAT KÄHLER SURFACES

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ABSTRACT. We show that an asymptotically Euclidean scalar-flat Kähler metric on a complex surface can be smoothly conformally compactified at the infinity point. We discuss some implication of this, characterizing such metrics on $\mathbb{C}^2$ blown up at some points.

1. Introduction

In this paper we study *asymptotically Euclidean Kähler surfaces of zero scalar curvature.* In [10], C. LeBrun observed that a complex surface which admits such a metric is biholomorphic to $\mathbb{C}^2$ blown up at a finite number of points (see also P. Li and S. T. Yau [14]) and constructed ones on $\mathbb{C}^2$ blown up at a finite number of *distinct* points. Though this marks a significant step toward the classification, a few important questions remain unanswered in the way.

The first question is whether one can construct such a metric on $\mathbb{C}^2$ blown up at a finite number of *any* points, not just *distinct* points.

The second question is on the uniqueness; Is such a metric on a blown-up $\mathbb{C}^2$ unique in a natural sense? The analogous question in compact Kähler manifolds of constant scalar curvature is an interesting nontrivial problem and settled only for Kähler Einstein metrics, see [5]. Affirmative answers to both questions will lead to the complete classification of asymptotically Euclidean Kähler surfaces of zero scalar curvature.

Received August 15, 1997.

1991 Mathematics Subject Classification: Primary 53C21; Secondary 32L25.

Key words and phrases: Conformal compactification, Scalar-flat Kähler surface, elliptic regularity.

The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1997.

The first question is rather successfully resolved by the author in [7]. To attack the second uniqueness problem, we need a sufficiently good geometric characterization of those metrics. The main point of this paper is to prove a characteristic property of such a metric, namely that an asymptotically Euclidean scalar-flat Kähler metric on a complex surface can be conformally compactified with the infinity point. This type of compactification may well have been studied through some standard theory but we have not found a general theorem with rigorous proof. So we choose an elliptic regularity approach to this as in M. Anderson [1]. We will discuss some implication of this compactification property in section 4, giving an answer to the second question in some special cases.

In section 2, we explain some definitions and exhibit examples of asymptotically Euclidean scalar-flat Kähler surfaces.

In section 3, we prove the conformal compactification. In fact, first we show that the Kähler metric when multiplied by a suitable conformal factor function can be extended upto the infinity point in such a way that it becomes a weak solution of a system of elliptic diferential equations on the one-point compactified manifold. Then by some ellitic regularity argument we show that the conformally compactified metric is a smooth solution of the system.

In section 4, we make remarks on applications of the compactification property.

2. Preliminaries

2.1 Blowing up

We begin with explaining the blowing up construction. It is a holomorphic surgery. Let M be a complex surface and p a point on it. We choose holomorphic coordinates near p so that we may assume $M = \mathbb{C}^2$ and $p = (0,0)$. We set $\tilde{p} = \mathbb{CP}^1$, the complex projective line.

Now we define a complex structure on $\tilde{M} = (M - \{p\}) \cup \tilde{p}$ as a complex submanifold of $\mathbb{C}^2 \times \mathbb{CP}^1$ as follows. Let (z_1, z_2) be the standard coordinates on $\mathbb{C}^2$ and $[\zeta_1, \zeta_2]$ the homogeneous coordinates on $\mathbb{CP}^1$ i.e. $[\zeta_1, \zeta_2] = (\zeta_1, \zeta_2)/\{(\zeta_1, \zeta_2) \sim \lambda(\zeta_1, \zeta_2), \lambda \in \mathbb{C} - \{0\}\}$. Then $\tilde{M}$ is the analytic subset of $\mathbb{C}^2 \times \mathbb{CP}^1$ defined by $\zeta_1 z_2 = \zeta_2 z_1$.

Let π be the restriction to $\tilde{M}$ of the projection $(z, [\zeta]) \to z$ of $\mathbb{C}^2 \times$

$\mathbb{CP}^1$ onto $\mathbb{C}^2$. Then π is a holomorphic map of $\tilde{M}$ onto $M = \mathbb{C}^2$, and $\pi^{-1}((0,0)) = \{(0,0)\} \times \mathbb{CP}^1$. Further π is a biholomorphic map of $\tilde{M} - \{(0,0)\} \times \mathbb{CP}^1$ to $M - \{p\}$.

Therefore we defined a complex structure on $\tilde{M} = (M - \{p\}) \cup \tilde{p}$. Topologically $\tilde{M}$ is obtained from M by replacing a point p by 2-dimensional sphere $\tilde{p} \cong S^2$, called the exceptional curve of the blowing up. Some argument shows that the normal complex line bundle to $\tilde{p} = S^2$ has Euler number -1. Therefore $\tilde{M}$ is diffeomorphic to $M \sharp \overline{\mathbb{CP}}^2$, the connected sum of M and a complex projective plane with reversed orientation.

The blowing up can be done at a point of an exceptional curve of previous blowing-ups. In this case we simply say that we blow up (more than once) at the same point. This is subtle but important as the first question in the introduction exactly points out the issue.

2.2 Aymptotically Locally Euclidean Metric

We define a Riemannian metric g on a noncompact manifold M to be *aymptotically locally Euclidean* if M has just one end and if there exists a compact subset K in M such that a finite covering $\tilde{V}$ of $M - K$ is diffeomorphic to the complement of the unit ball in $\mathbb{R}^4$; the pulled-back Riemannian metric g_{ij} on $\tilde{V}$ is required to approximate asymptotically the Euclidean metric δ_{ij} on $\mathbb{R}^4$, so that in the natural coordinates x_i, one has

$$g_{ij} = \delta_{ij} + a_{ij} \tag{2.1}$$

with $\partial^p a_{ij} = O(r^{-2-p})$, $p = 0, 1, 2$ where $r^2 = \sum x_i^2$ and ∂ denotes differentiation with respect to the coordinates x_i.

We call g *asymptotically Euclidean* if we can take $M - K$ as the covering $\tilde{V}$ in the above.

2.3 Examples

Now we describe some known examples of asymptotically Euclidean scalar-flat Kähler surfaces. Let $r \in (1, \infty)$ and $\sigma_1, \sigma_2, \sigma_3$ be a left invariant coframe for S^3. Then the following metric on $(1, \infty) \times S^3$

$$g = \frac{dr^2}{1 - \frac{1}{r^2}} + r^2(\sigma_1^2 + \sigma_2^2 + (1 - \frac{1}{r^2})\sigma_3^2). \tag{2.2}$$

can be compactified with S^2 corresponding to $r = 1$. The normal bundle of this S^2 has Euler number -1. So the compactified manifold is diffeomorphic to $\mathbb{C}^2$ blown up at one point as described in subsection 2.1. It is clear that g is aymptotically Euclidean.

Next, there are metrics found by LeBrun [11], which can be viewed as superpositions of (2.2) metrics, on a blow-up of $\mathbb{C}^2$ at a finite number of distinct points along a complex line in $\mathbb{C}^2$; it is of the form

$$g = f^2(Vh - V^{-1}\omega^2) \tag{2.3}$$

where h is a hyperbolic metric, $\Delta_h V = 0$ and $d\omega = \star_h dV$ with $\star_h$ being the Hodge-star operator with respect to h. In [10], LeBrun perturbed these metrics to ones on blow-ups of $\mathbb{C}^2$ at a finite number of any distinct points.
These so far are essentially all the known asymptotically Euclidean examples.

There are some more locally asymptotically Euclidean examples; There are ricci-flat multi-Eguchi Hanson metrics, see [6]. Related to these, Kronheimer [9] did a general construction and classification of locally asmptotically Euclidean ricci-flat Kähler surfaces.

There are other locally asmptotically Euclidean scalar-flat Kähler metrics [12] which have explicit expressions in terms of local coordinates. They are on the total spaces of some complex line bundles over S^2 and generalize (2.2) metric; For r and $\sigma_1, \sigma_2, \sigma_3$ be as above,

$$g = \frac{dr^2}{1 + \frac{A}{r^2} + \frac{B}{\mathbb{R}^4}} + r^2(\sigma_1^2 + \sigma_2^2 + (1 + \frac{A}{r^2} + \frac{B}{r^4})\sigma_3^2). \tag{2.4}$$

Here $A = n - 2$ and $B = 1 - n$, $n = 1, 2, 3, ..$ and the first chern class of the bundle corresponds to $-n$.

3. Compactification

Now suppose that g is an asymptotically Euclidean scalar flat Kähler metric on a noncompact complex surface M. We use the notation of subsection 2.2. Let $h = \frac{1}{\phi^2} g$ where ϕ is positive, smooth and equal to r^2 outside some compact subset of M. The following lemma is particularly well known in compact case as (3.1) is the Euler-Lagrange equation for the Riemannian functional $\mathcal{W}(g) = \int |W_g|^2 dvol_g$ where W_g is the Weyl curvature tensor and an anti-self-dual metric, meaning the positive part of the Weyl curvature tensor being zero, gives a minimum value of $\mathcal{W}(g)$. We denote the ricci curvature by ric and the scalar curvature by s.

LEMMA 3.1. *The metric h is anti-self-dual and satisfies the following system of partial differential equations, called the Bach equation.*

$$\delta^D d^D(ric - \frac{1}{6}sh) = 2\breve{W}ric, \quad on\ M. \tag{3.1}$$

Here d^D is the differential operator on symmetric 2-tensors ψ acting by $d^D\psi(x,y,z) = D_x\psi(y,z) - D_y\psi(x,z)$, and δ^D is the dual operator of d^D. For a curvature tensor A (such as the Riemannian curvature tensor R and the Weyl tensor W), $\breve{A}$ is the linear map on symmetric 2-tensors such that $(\breve{A}\psi)(x,y) = \sum_{i=1}^n \psi(A(x,e_i)y, e_i)$ for an orthonormal basis $e_1, e_2, .., e_n$.

REMARK 3.1. The Bach equation is the Euler-Lagrange equation of the functional $\mathcal{W}(g)$ on a compact 4-dimensional manifold. This equation is conformally invariant, i.e. if g satisfies the Bach equation then $f^2 \cdot g$ also does, for any positive function f.

REMARK 3.2. The anti-self-duality of a metric is also a conformally invariant condition. We call a Riemannian metric *self-dual* if $W^- = 0$. Under orientation change an anti-self-dual (self-dual resp.) metric becomes self-dual (anti-self-dual resp.) but the Weyl tensor $W = W^+ + W^-$ is invariant.

Proof of Lemma 3.1. A scalar-flat Kähler metric is anti-self-dual [13, p.274]. As h is conformal to a scalar-flat Kähler metric, h is anti-self-dual by Remark 3.2.

The Bach equation can be written in terms of *spinor* notation as follows [16, p127], [8]

$$B_{ab} = 2(\nabla^C_{A'}\nabla^D_{B'} + \Phi^{CD}_{A'B'})W^-_{ABCD}.$$

which can be translated to

$$B_{ab} = \nabla^c\nabla^d W_{acbd} - \frac{1}{2}(\breve{W}ric)_{ab} = -\frac{1}{2}(\delta^D D^* W)_{ab} - \frac{1}{2}(\breve{W}ric)_{ab}.$$

This is equivalent to (3.1) by Diffential Bianchi identity.

From above the Bach equation clearly holds for metrics with $W^- = 0$, i.e. self-dual metrics. But as (3.1) is independent of orientation, it holds for the anti-self-dual metric h too. □

We have a formula [4, (4.71)]

$$\delta^D d^D ric = 2D^* D ric - 2\breve{R} ric + 2 ric \circ ric + D ds, \tag{3.2}$$

and

$$\delta^D d^D (sh) = (\Delta s) h, \tag{3.3}$$

where $(ric \circ ric)_{ij} = \sum ric_i^k ric_{kj}$ in local coordinates. From (3.1)-(3.3) we get

$$D^* D ric = \breve{W} ric + \breve{R} ric - ric \circ ric - \frac{1}{2} D ds + \frac{1}{12} \Delta s h, \tag{3.4}$$

Now we have

$$\begin{aligned} DD^*(sh) &= -Dds \\ D^*D(sh) &= (\Delta s) h \\ DD^*(ric) &= -\frac{1}{2} Dds \end{aligned} \tag{3.5}$$

From these we have

$$(D^* D - \frac{6}{5} DD^*)(ric - \frac{1}{12} sh) = \breve{W} ric - ric \circ ric + \breve{R} ric. \tag{3.6}$$

Now substituting the variables by $y_i = x_i / r^2$, $i = 1, .., 4$, we can view the above equation as defined on an origin-deleted ball in $\mathbb{R}^4$.

LEMMA 3.2. *The metric h can be extended upto the origin as a $L^{2,\infty}$ weak solution of* (3.6).

Proof. As h is a smooth solution away from the origin, we only need to consider h near the origin. Then $h = \frac{1}{r^4}(dx_i \otimes dx_i + a)$ with $\partial^p a \in O(r^{-2-p})$, $p = 0, 1, 2$. Just compute $\frac{1}{r^4}(dx_i \otimes dx_i) = dy_i \otimes dy_i$.

Setting $\rho^2 = \sum y_i^2 = \frac{1}{r^2}$, we compute that $\frac{\partial}{\partial y_k}(\frac{a}{r^4})$ is $O(\rho^5)$ and $\frac{\partial}{\partial y^k}\frac{\partial}{\partial y^l}(\frac{a}{r^4})$ is $O(\rho^4)$. We then have in y_i coordinates

$$\begin{aligned} h_{ij} &= \delta_{ij} + O(\rho^6) \\ \frac{\partial}{\partial y^k} h_{ij} &= O(\rho^5) \\ \frac{\partial}{\partial y^k}\frac{\partial}{\partial y^l} h_{ij} &= O(\rho^4) \end{aligned} \tag{3.7}$$

Now h is at least twice differentiable near the origin of y_i coordinates from the first two equations and the second derivative of h is bounded (indeed differentiable) from the third equation. So h is in $L^{2,\infty}$.
Next, it is easy to show that (3.6) holds weakly with h; just apply integration on both sides after multiplying by functions $\eta \in C_c^\infty$ and see that both sides are equal with $h \in L^{2,\infty}$. Therefore h is a weak solution. □

Now the basic line of arguments follows that in [1, section 4]. We set $u = ric - \frac{1}{12}sh$. Now u is a weak L^∞ solution to the system (3.6) when it is viewed as a second order differential system in u.

Set $R_1 = \check{W}ric - ric \circ ric + \check{R}ric$, which is quadratic in curvature tensors. We then write (3.6) simply

$$L_1 u = R_1, \text{ with } R_1 \in L^\infty. \tag{3.8}$$

The system (3.8) is elliptic second order; we compute the symbol of L_1 acting on symmetric 2 tensors $S^2(M) \subset TM \otimes TM$. For $\xi \in TM$

$$\sigma_\xi(D^*D - \frac{6}{5}DD^*)(\eta \otimes v) = (< \xi, \xi > \eta - \frac{6}{5} < \xi, \eta > \xi) \otimes v.$$

This implies that L_1 is ellptic.
For a Riemannian metric with $L^{2,\infty}$ bound, there exists harmonic coordinates in which h_{ij} are bounded in $L^{2,\infty}$ [2]. So from now on we use

harmonic coordinates of h on a ball centered at the origin. For convenience we denote these harmonic coordintes again by y_i, $i = 1, 2, 3, 4$. Then the operator takes the form

$$
\begin{aligned}
(L_1 u)_{rs} &= h^{ij}\partial_i\partial_j u_{rs} + \frac{3}{4}\partial_r\partial_s(h^{ab}u_{ab}) + Q^1_{rs}(h, \partial h) \\
&= h^{ij}\partial_i\partial_j u + \frac{3}{4}(\partial_r\partial_s h^{ab})u_{ab} + \frac{3}{4}(\partial_r h^{ab})(\partial_s u_{ab}) \\
&\quad + \frac{3}{4}(\partial_s h^{ab})(\partial_r u_{ab}) + \frac{3}{4}h^{ab}\partial_r\partial_s u_{ab} + Q^1_{rs}(h, \partial h).
\end{aligned}
\tag{3.9}
$$

In the first equality we used (3.5) and $Q^1_{rs}(h, \partial h)$ is a term quadratic in h and ∂h and so belongs to L^∞.

As $(\partial_r\partial_s h^{ab})u_{ab}$ is in L^∞, setting

$$
L_2 u = h^{ij}\partial_i\partial_j u + \frac{3}{4}(\partial_r h^{ab})(\partial_s u_{ab}) + \frac{3}{4}(\partial_s h^{ab})(\partial_r u_{ab}) + \frac{3}{4}h^{ab}\partial_r\partial_s u_{ab},
\tag{3.10}
$$

we have

$$
L_2 u = f
\tag{3.11}
$$

with $f = R_1 - \frac{3}{4}(\partial_r\partial_s h^{ab})u_{ab} - Q^1_{rs}(h, \partial h) \in L^\infty$.

As the coefficients of L_2 are in C^α for any $\alpha < 1$, u is in $L^{2,p}$ for any $p < \infty$ by standard L^p elliptic regularity [15, section 6.2] . Taking trace of u, we see that ric is in $L^{2,p}$, for any $p < \infty$.

Now we use the ricci curvature equation in harmonic coordinates

$$
h^{ij}\frac{\partial^2 h_{rs}}{\partial x_i \partial x_j} + Q_{rs}(h, \partial h) = ric_{rs}
\tag{3.12}
$$

where Q is a term quadratic in h and its first derivatives, of the general form $h^{-2}(\partial h)^2$, which is in $L^{1,\infty}$. We see that h is in W^p_3, for any $p < \infty$. We return to the system (3.8)-(3.12) and repeat the argument. Then we can finally show that h is in W^p_k for any $k, p > 2$, and so h is in C^∞.

So we have:

THEOREM 3.3. *Let g be an asymptotically Euclidean scalar flat Kähler metric on a noncompact complex surface M. Then g can be conformally smoothly compactified, i.e. there exists a positive function f on M such that $f^2 \cdot g$ is a smooth Riemannian metric on $M \cup \{\infty\}$.*

REMARK 3.3. If we had assumed the asymptotic Euclidean condition as having the appropriate decay of all derivatives of the metric for $p = 0, 1, 2, 3, ..$ in (2.1), then we could get h with better regularity in Lemma 3.2. Then we proceed through (3.8)-(3.12) similarly as above.

4. Application and concluding remarks

In this section we make some remarks on an application of theorem 3.3.

To prove this application one needs to discuss some complex 3-dimensional geometry, so called twistor theory, which has much different flavour from that of section 3. We'd have to avoid some part of details, but we do a complete line of arguments giving necessary references.

The *twistor space* Z of an oriented 4-dimensional Riemannian manifold ($N^4, h,$ orientation) is the S^2-bundle of pointwise almost complex structures on N which are compatible with both h and the orientation; the fiber over each point p of N consists of the orientation preserving almost complex structures $J : T_pN \to T_pN$ i.e. $J^2 = -Id$ such that $h(JX, JY) = h(X, Y)$ for any $X, Y \in T_pN$. So this fiber can be identified with the quotient space $SO(4)/U(2) \cong S^2$. In [3] it is explained that Z itself admits a natural almost complex structure which makes it into a complex 3-manifold exactly when h is anti-self-dual. By definition this twistor space Z only depends on the conformal structure $[h]$ of the metric h.

Consider $t : Z \to M$ the bundle projection and then by definition a hermitian structure J on M is exactly a section of t whose image we denote by D, while another hermitian structure $-J$ is another section whose image is $\bar{D}$.

An important point of the twistor theory is that the twistor space Z of an anti-self-dual metric h completely determines the conformal class $[h]$ and that in the hermitian anti-self-dual case the pair (Z, D) or triple $(Z, D, \bar{D})$ completely determines the conformal class $[h]$ as well as the complex structure J. The proof of Theorem 4.1 below involves a detailed analysis of Z using this point.

So we return to the compactified manifold $\bar{M} = M \cup \{\infty\}$ of section 3. We denote the conformally compactified metric on $\bar{M}$ by h as before. The conformal class $[g]$ is equal to $[h]$ restricted to M. So the twistor space Z_M of (M, g) is an open submanifold of the twistor space $Z_{\bar{M}}$ of $(\bar{M}, h)$. Z_M and $Z_{\bar{M}}$ are complex 3-manifolds by Lemma 3.1.

By above description, the twistor space Z_M has two disjoint divisors D and $\bar{D}$ as (M, g) is assumed hermitian. Consider the closures of D and $\bar{D}$ in $Z_{\bar{M}}$, denoted by $cl(D)$ and $cl(\bar{D})$ respectively, which are complex codimension 1 hypersurfaces.

Now we show

THEOREM 4.1.

(1) *Any aymptotically Euclidean scalar-flat Kähler metric on $\mathbb{C}^2$ blown up at 1 point is the Burns metric* (2.2).
(2) *Any aymptotically Euclidean scalar-flat Kähler metric on $\mathbb{C}^2$ blown up at any 2 points is one of LeBrun's metrics* (2.3).

Sketch of Proof. The complex hypersurfaces $cl(D)$ and $cl(\bar{D})$ in $Z_{\bar{M}}$ are called elemenatry effective divisors. The existence of these implies [17, (3.3) and (4.3)] that the conformal class $[h]$ is of positive type; in the notation therein, $cl(D) + cl(\bar{D})$ is equivalent to $K_Z^{-\frac{1}{2}} \cong \mathcal{O}_Z(2)$ as a divisor.

The rest of argument follows by applying Poon's theorems; for (1), use Corollary 2.8 in [18] and for (2), use theorem 5.5 in the same paper.□

REMARK 4.1. Proving uniqueness of scalar-flat Kähler metrics on *compact* complex surfaces is easy in some cases if there exists a nontrivial holomorphic vector field; one argues by Matsushima-Lichnerowicz Theorem [5] that the vector field gives rise to a family of isometries which help to identify the scalar-flat metric. Here in noncompact case an obstacle is that even if the blowing up points are collinear, i.e. blowing-up points are along a line in $\mathbb{C}^2$, so that there is a nontrivial holomorphic vector field, we can not immediately say that it gives isometries.

When the points blown up in $\mathbb{C}^2$ are generically distributed so that the surface admits no nontrivial holomorphic vector field, the compactification property still gives a good deal of informations necessary to answer our uniqueness question.

Some more applications along this line and constructions alluded in section 1 will appear in [7].

ACKNOWLEDGMENT. The author thanks Michael Anderson for helpful communications related to section 3.

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