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WEAK TYPE $L^1(\mathbb{R}^n)$-ESTIMATE FOR CERTAIN MAXIMAL OPERATORS

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ABSTRACT. Let $\{A_t\}_{t>0}$ be a dilation group given by $A_t = \exp(-P \log t)$, where P is a real $n \times n$ matrix whose eigenvalues has strictly positive real part. Let ν be the trace of P and P^* denote the adjoint of P. Suppose that $\mathcal{K}$ is a function defined on $\mathbb{R}^n$ such that $|\mathcal{K}(x)| \leq \mathfrak{K}(|x|_Q)$ for a bounded and decreasing function $\mathfrak{K}(t)$ on $\mathbb{R}_+$ satisfying $\mathfrak{K} \circ |\cdot|_Q \in \cup_{\varepsilon>0} L^1((1+|x|)^{\varepsilon} dx)$ where $Q = \int_0^\infty \exp(-tP^*)\exp(-tP)dt$ and the norm $|\cdot|_Q$ stands for $|x|_Q = \sqrt{\langle Qx, x\rangle}, x \in \mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$, define $\mathfrak{M}f(x) = \sup_{t>0} |\mathcal{K}_t * f(x)|$ where $\mathcal{K}_t(x) = t^{-\nu}\mathcal{K}(A^*_{1/t}x)$. Then we show that $\mathfrak{M}$ is a bounded operator of $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

1. Introduction

Let P be a real $n \times n$ matrix with eigenvalues λ_i, $Re(\lambda_i) > 0$, and let ν be the trace of P and P^* denote the adjoint of P. Then we consider the associated dilation group defined by $A_t = \exp(P \log t), t > 0$. Then $A^*_t = \exp(P^* \log t)$. We see that in general $|A^*_t x|$ is not strictly increasing in terms of t. However, one has the following fact (see [6]); there exists a positive definite and symmetric real matrix Q so that $|A^*_t x|_Q = \sqrt{\langle QA^*_t x, A^*_t x\rangle}$ is strictly increasing in t, where the norm $|\cdot|_Q$ is defined by $|x|_Q = \sqrt{\langle Qx, x\rangle}$ for $x \in \mathbb{R}^n$; in fact, it turns out that

$$Q = \int_0^\infty \exp(-tP^*)\exp(-tP)\, dt.$$

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Observe that, since Q is positive definite and symmetric, this norm satisfies the triangle inequality and we have that for $x \in \mathbb{R}^n$

$$(1.1) \qquad \mu_0 |x| \le |x|_Q \le \sqrt{\|Q\|}\, |x|$$

where $\|Q\|$ denotes the operator norm of Q and $\mu_0 > 0$ is the smallest eigenvalue of the orthogonal matrix which makes Q diagonalized.

For a notation, we write $f \in L^{1,\infty}(\mathbb{R}^n)$ (which is called weak-L^1 space) if

$$\|f\|_{L^{1,\infty}} \risingdotseq \sup_{s>0} s|\{x \in \mathbb{R}^n |\, |f(x)| > s\}| < \infty.$$

For simplicity, we denote the weak-L^1 space by $L^{1,\infty}$. We introduce weighted integrable functions with a weight $\omega_\varepsilon(x) = (1+|x|)^\varepsilon, \varepsilon > 0$. For $\varepsilon > 0$, denote by $L^1(\omega_\varepsilon)$ the space of all measurable functions f defined on $\mathbb{R}^n$ for which

$$\int_{\mathbb{R}^n} |f(x)|\, \omega_\varepsilon(x)\, dx < \infty.$$

Then we note that $L^1(\omega_\varepsilon)$ decreases in terms of inclusion as $\varepsilon > 0$ increases. Even though we take the union on $\varepsilon > 0$ of such spaces $L^1(\omega_\varepsilon)$, the space $\cup_{\varepsilon>0} L^1(\omega_\varepsilon)$ is still a proper subspace of $L^1(\mathbb{R}^n)$ because a function

$$f(x) = \frac{1}{[\log(1+|x|)]^{3/2}(1+|x|)^n} \chi_{\mathbb{R}^n \setminus \mathcal{B}(0;1)}(x)$$

where $\mathcal{B}(0;1)$ denotes the unit ball with center $0 \in \mathbb{R}^n$ belongs to $L^1(\mathbb{R}^n)$ but not to $\cup_{\varepsilon>0} L^1(\mathbb{R}^n)$. In what follows, we denote by $L^1(\omega_*) = \cup_{\varepsilon>0} L^1(\omega_\varepsilon)$, which we call the maximal weighted L^1-space. Our main result is to obtain the weak type $L^1(\mathbb{R}^n)$-estimate for certain maximal operator to be defined in the following theorem.

THEOREM 1.1. *Suppose that $\mathcal{K}$ be a function defined on $\mathbb{R}^n$ such that*

$$|\mathcal{K}(x)| \le \mathfrak{K}(|x|_Q)$$

where $\mathfrak{K}(t)$ is a bounded and decreasing function defined on $\mathbb{R}_+$ and $\mathfrak{K} \circ |\cdot|_Q \in L^1(\omega_*)$. For $f \in L^1(\mathbb{R}^n)$, define

$$\mathfrak{M}f(x) = \sup_{t>0} |\mathcal{K}_t * f(x)|,$$

where $\mathcal{K}_t(x) = t^{-\nu}\mathcal{K}(A_{1/t}\, x)$ for $t > 0$. Then $\mathfrak{M}$ is a bounded operator of $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$; that is, there is a constant $C = C(n)$ such that for any $f \in L^1(\mathbb{R}^n)$,

$$\|\mathfrak{M}f\|_{L^{1,\infty}} \leq C\, \|f\|_{L^1}.$$

REMARK. We see that the maximal operator associated with isotropic dilation on the kernel which is in $L^1(\mathbb{R}^n)$ is dominated by the Hardy-Littlewood maximal operator. However, this is no longer true for anisotropic cases. If $1 < p \leq \infty$ and the kernel has a quasiradial majorant that is bounded, decreasing, and integrable, then it is well-known (see [4]) that the maximal operator $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^n)$.

2. Preliminary estimates

Let $\mathcal{H}^\varepsilon(x) = \dfrac{1}{(1+|x|_Q)^{n+\varepsilon}}$ for $\varepsilon > 0$, and for $f \in L^1(\mathbb{R}^n)$ define

$$\mathcal{M}_\varepsilon f(x) = \sup_{t>0} |\mathcal{H}_t * f(x)|$$

where $\mathcal{H}_t(x) = t^{-\nu}\mathcal{H}^\varepsilon(A^*_{1/t}\, x)$ for $t > 0$. Then we need the following elementary lemma in order to prove Lemma 2.2 below.

LEMMA 2.1. *If $\mathfrak{K}\circ|\cdot|_Q \in L^1(\omega_\varepsilon)$ for $\varepsilon > 0$, then $\sum_{k=1}^{\infty} 2^{k(n+\varepsilon)}\mathfrak{K}(2^{k-1}) < \infty$.*

Proof. It easily follows from (1.1) and simple computation. □

LEMMA 2.2. *If $\mathfrak{K} \circ |\cdot|_Q \in L^1(\omega_\varepsilon)$ for $\varepsilon > 0$, then there exists a constant $C = C(\varepsilon) > 0$ such that $\mathfrak{M}f(x) \leq C\, \mathcal{M}_\varepsilon f(x)$ for any $x \in \mathbb{R}^n$.*

Proof. For $k \in \mathbb{Z}$, set $\mathcal{A}_k = \{y \in \mathbb{R}^n | \, 2^{k-1} \le |y|_Q \le 2^k\}$ and $\mathcal{U} = \{y \in \mathbb{R}^n | \, |y|_Q \le 1\}$. By simple computation, we have that

$$\begin{aligned} |\mathcal{K} * f(x)| &\le \sum_{k\in\mathbb{Z}} [(\mathfrak{K} \circ |\cdot|_Q)\chi_{\mathcal{A}_k}] * |f|(x) \\ &\le [(\mathfrak{K} \circ |\cdot|_Q)\chi_{\mathcal{U}}] * |f|(x) + \sum_{k=1}^{\infty} [(\mathfrak{K} \circ |\cdot|_Q)\chi_{\mathcal{A}_k} * |f|(x) \\ &\le C\,\chi_{\mathcal{U}} * |f|(x) + C \sum_{k=1}^{\infty} \mathfrak{K}(2^{k-1})\chi_{\mathcal{A}_k} * |f|(x) \\ &\le C\,\chi_{\mathcal{U}} * |f|(x) + C \sum_{k=1}^{\infty} 2^{k(n+\varepsilon)} \mathfrak{K}(2^{k-1}) \frac{1}{2^{k(n+\varepsilon)}} \chi_{\mathcal{A}_k} * |f|(x) \end{aligned}$$

where χ_E denotes the characteristic function on a set E. Note that there is a constant $C > 0$ such that for any $k = 1, 2, 3, \cdots$,

$$\frac{1}{2^{k(n+\varepsilon)}} \chi_{\mathcal{A}_k}(x) \le \frac{C}{(1 + |x|_Q)^{n+\epsilon}}$$

because $|x|_Q \sim 2^k$ for $x \in \mathcal{A}_k$. Thus it follows from Lemma 2.1 that for any $x \in \mathbb{R}^n$,

$$|\mathcal{K} * f(x)| \le C\,|\mathcal{H}^{\varepsilon} * f(x)|.$$

Hence we conclude that $\mathfrak{M}f(x) \le C\,\mathcal{M}_{\varepsilon} f(x)$ for any $x \in \mathbb{R}^n$. □

3. Weak type $L^1(\mathbb{R}^n)$-estimate

In this section, we introduce a Vitali family [5] and the result of Stein and Weiss [7] on adding weak type functions.

Let $\{\mathcal{U}_s | \, s > 0\}$ be a family of open subsets of $\mathbb{R}^n$ whose closure is compact. Then we say that $\{\mathcal{U}_s | \, s > 0\}$ is a Vitali family with constant $A > 0$, if the followings are satisfied; (i) $\mathcal{U}_s \subset \mathcal{U}_{s'}$ for $s \le s'$ and $\cap_{s>0} \mathcal{U}_s = \{0\}$, (ii) $|\mathcal{U}_s - \mathcal{U}_s| \le A\,|\mathcal{U}_s|$ for all $s > 0$, and (iii) $\lim_{k\to\infty} |\mathcal{U}_{s_k}| = |\mathcal{U}_s|$ when $\lim_{k\to\infty} s_k = s$.

LEMMA 3.1. *Suppose that $\{h_j\}$ is a sequence of nonnegative functions on a measure space for which*

$$\|h_j\|_{L^{1,\infty}} \le A$$

where $A > 0$ is a constant. Let $\{\alpha_j\}$ be a sequence of positive numbers with $\sum_j \alpha_j = 1$. Then we have that

$$\|\sum_j \alpha_j h_j\|_{L^{1,\infty}} \leq 2A(N+2)$$

where $N = \sum_j \alpha_j \log(1/\alpha_j)$.

LEMMA 3.2. *Let $\mathcal{G}$ be a function defined on $\mathbb{R}^n$ such that $(1+|\cdot|_Q)^\gamma \mathcal{G} \in L^\infty(\mathbb{R}^n)$ for $\gamma > n$. For $f \in L^1(\mathbb{R}^n)$, define*

$$\mathcal{M}f(x) = \sup_{t>0} |\mathcal{G}_t * f(x)|$$

where $\mathcal{G}_t(x) = t^{-\nu}\mathcal{G}(A^*_{1/t}x)$ for $t > 0$. Then there is a constant $C = C(n) > 0$ such that for any $f \in L^1(\mathbb{R}^n)$,

$$\|\mathcal{M}f\|_{L^{1,\infty}} \leq C\|f\|_{L^1}.$$

Proof. Without loss of generality, we may assume that

$$|\mathcal{G}(x)| \leq C\,\mathfrak{G}(|x|_Q)$$

where $\mathfrak{G}(r) = \dfrac{1}{(1+r)^\gamma}$ for $r > 0$ and $\gamma > n$. For $k \in \mathbb{Z}$, let $\mathcal{A}^k_t = \{y \in \mathbb{R}^n \,|\, 2^{k-1} \leq |A^*_{1/t}y|_Q \leq 2^k\}$, and for $k = 1, 2, 3, \cdots$, let $\mathcal{U}^k_t = \{y \in \mathbb{R}^n \,|\, |A^*_{1/t}y|_Q \leq 2^k\}$. From simple computation, we have that

$$\begin{aligned}
|\mathcal{G}_t * f(x)| &\leq \sum_{k\in\mathbb{Z}} (|\mathcal{G}_t|\chi_{\mathcal{A}^k_t}) * |f|(x) \\
&\leq \sum_{k=1}^{\infty} (|\mathcal{G}_t|\chi_{\mathcal{U}^k_t}) * |f|(x) \\
&\leq \sum_{k=1}^{\infty} \frac{1}{t^\nu}\mathfrak{G}(2^{k-1})\chi_{\mathcal{U}^k_t} * |f|(x) \\
&= \sum_{k=1}^{\infty} 2^{kn}\mathfrak{G}(2^{k-1})\frac{1}{2^{kn}t^\nu}\chi_{\mathcal{U}^k_t} * |f|(x) \\
&\leq \sum_{k=1}^{\infty} 2^{kn}\mathfrak{G}(2^{k-1})\mathcal{M}_k f(x),
\end{aligned}$$

where

$$\mathcal{M}_k f(x) = \sup_{t>0} \frac{1}{2^{kn}t^{\nu}} \chi_{\mathcal{U}_t^k} * |f|(x).$$

Then it is not hard to show that for each k, $\{\mathcal{U}_t^k | t > 0\}$ is a Vitali family with constant 2^n; for, $\lim_{t\to\infty} |A_{1/t}^* y|_Q = 0$ and $\lim_{t\to\infty} |A_{1/t}^* y|_Q = \infty$, $A_t^* y$ is continuous in t and y, $\mathcal{U}_t^k - \mathcal{U}_t^k \subset 2\mathcal{U}_t^k$, and $|\mathcal{U}_t^k| = \omega_n 2^{kn} t^{\nu}$, where $\omega_n = |\{x \in \mathbb{R}^n | |x|_Q \leq 1\}|$. Thus by the maximal theorem [5] on a Vitali family, we have that for each $k \geq 1$,

$$\|\mathcal{M}_k f\|_{L^{1,\infty}} \leq 2^n \|f\|_{L^1}.$$

Set $\beta_k = 2^{kn}\mathfrak{G}(2^{k-1})$ and set $\beta = \sum_{k=1}^{\infty} 2^{kn}\mathfrak{G}(2^{k-1}) < \infty$. If $\alpha_k = \beta_k/\beta$, then $\sum_{k=1}^{\infty} \alpha_k = 1$. Then by Lemma 3.1, we get that

$$\|\sum_{k=1}^{\infty} \alpha_k \mathcal{M}_k f\|_{L^{1,\infty}} \leq 2^{n+1}(K+2)\|f\|_{L^1},$$

where $K = \sum_{k=1}^{\infty} \alpha_k \log(1/\alpha_k) < \infty$. Also we have that

$$\sup_{t>0} |\mathcal{G}_t * f(x)| \leq \beta \sum_{k=1}^{\infty} \alpha_k \mathcal{M}_k f(x).$$

Therefore, this implies that $\mathcal{M}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.□

Proof of Theorem 1.1. Assume that $\mathfrak{K} \circ |\cdot|_Q \in L^1(\omega_*)$, and let $\varepsilon_0 > 0$ be the supremum of numbers $\varepsilon > 0$ so that $\mathfrak{K} \circ |\cdot|_Q \in L^1(\omega_\varepsilon)$. Applying Lemma 3.2 to the maximal operator $\mathcal{M}_{\varepsilon_0}$ that we defined in Section 2, it follows that $\mathcal{M}_{\varepsilon_0}$ is a bounded operator of $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. Therefore, by Lemma 2.2, we conclude that $\mathfrak{M}$ is a bounded operator of $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. □

4. Application

In this section, we give an application of Theorem 1.1 to the maximal Bochner-Riesz operator. In order to do that, we first of all introduce quasi-homogeneous distance functions defined on $\mathbb{R}^n$.

We say that a function $\varrho \in C^\infty(\mathbb{R}^n\backslash\{0\})$ is an A_t-homogeneous distance function defined on $\mathbb{R}^n$, if $\varrho(\xi) > 0$ for $\xi \in \mathbb{R}^n\backslash\{0\}$, and $\varrho(A_t\xi) = t\varrho(\xi)$ for $t > 0$ and $\xi \in \mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$, let $\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx$ denote the Fourier transform. We consider quasiradial Bochner-Riesz means of index $\delta > 0$ defined by

$$\mathcal{B}^\delta_{\varrho,t} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}(1-\varrho(\xi)/t)^\delta_+ \hat{f}(\xi)\, d\xi$$

and the associated maximal operator

$$\mathfrak{M}^\delta_\varrho f(x) = \sup_{t>0} |\mathcal{B}^\delta_{\varrho,t} f(x)|.$$

In what follows, denote by $\Sigma_\varrho = \{\xi \in \mathbb{R}^n | \, \varrho(\xi) = 1\}$ the unit sphere with respect to the distance function ϱ.

COROLLARY 4.1. *Suppose that the Gaussian curvature of the unit sphere Σ_ϱ does not vanish. If $\delta > (n-1)/2$, then there is a constant $C = C(n) > 0$ such that for any $f \in L^1(\mathbb{R}^n)$,*

$$\|\mathfrak{M}^\delta_\varrho f\|_{L^{1,\infty}} \le C\,\|f\|_{L^1}.$$

REMARK. For another proof, the reader can refer to the result of Dappa and Trebels [1]. If the sphere Σ_ϱ satisfies a finite type condition, then we also have sharp weak type $(1,1)$-estimate for the maximal Bochner-Riesz operator (see [2] and [3] for related results).

Proof. It easily follows from Theorem 1.1 and the decay estimate of the Bochner-Riesz kernel by using the stationary phase method. □

In order to apply the Marcinkiewicz interpolation theorem, we use the standard linearization technique of the maximal Bochner-Riesz operator. It thus suffices to consider the analytic family of linear operators given by

$$z \mapsto \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}(1-\varrho(\xi)/t(x))^z_+ \hat{f}(\xi)\, d\xi,$$

where $x \mapsto t(x)$ is an arbitrary measurable function defined on $\mathbb{R}^n$ with values in $\mathbb{R}_+$. Combining this with Corollary 4.1, we obtain the L^p-estimate for $\mathfrak{M}^\delta_\varrho$ near $p = 1$. By the complex interpolation method, we interpolate this estimate with the L^2-estimate for $\mathfrak{M}^\delta_\varrho$ on $\delta > 0$. Then we easily get the following corollary.

COROLLARY 4.2. *Suppose that the unit sphere* Σ_ϱ *has nonvanishing Gaussian curvature. If* $\delta > (n-1)\left(\frac{1}{p}-\frac{1}{2}\right)$ *and* $1 < p \le 2$, *then there is a constant* $C = C(n,p) > 0$ *such that for any* $f \in L^p(\mathbb{R}^n)$,

$$\|\mathfrak{M}_\varrho^\delta f\|_{L^p} \le C\,\|f\|_{L^p}.$$

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