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ON STOCHASTIC EVOLUTION EQUATIONS WITH STATE-DEPENDENT DIFFUSION TERMS

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ABSTRACT. The integral solution for a deterministic evolution equation was introduced by Benilan. Similarly, in this paper, we define the integral solution for a stochastic evolution equation with a state-dependent diffusion term and prove that there exists a unique integral solution of the stochastic evolution equation under some conditions for the coefficients. Moreover we prove that this solution is a unique strong solution.

1. Introduction

In this paper, we consider the following stochastic evolution equation

$$(1.1)\qquad \begin{cases} dX_t = AX_t\,dt + \Psi(t, X_t)\,dB_t, \ t \geq 0 \\ X_0 = x_0 \end{cases}$$

where A is an unbounded m-dissipative operator for a real separable Hilbert space K to itself, i.e. for all $x, y \in D(A)$, $(Ax - Ay, x - y)_K \leq 0$ and for any $\lambda > 0$, $I - \lambda A$ is surjective, $\Psi \in L^2([0, \infty) \times K \to \sigma_2(H, K))$, where $\sigma_2(H, K)$ is the set of all Hilbert-Schmidt operator from another Hilbert space H to K and B_t is a cylindrical Brownian motion on H. The second term of the right hand side of (1.1) is Itô's integral of $\Psi(t, X_t)$ with respect to B_t.

Stochastic evolution equations have been studied by many mathematicians. Curtain and Pritchard [3], Dawson [4] and Miyahara [6], by the semigroup approach, defined the strong solution, mild solution and weak solution and gave the relation between these solutions. By the

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Lion's approach, Pardoux [7] studied the uniqueness and existence of strong solution when A is coercive. Kim [5] introduced a new kind of approach. More precisely, he first defined the integral solution of (1.1) with $\Psi(t, X_t) \equiv \Phi(t)$, and studied the uniqueness and existence of integral solution and strong solution. The integral solution of deterministic evolution equations was first introduced by Benilan[2].

In this paper, we define the integral solution of (1.1) with state dependent noise and prove that a strong solution of (1.1) is an integral solution (Theorem 3.3). And in the case that A is linear, we prove that there is unique integral solution of (1.1) (Theorem 3.5) and that this integral solution is the unique strong solution(Theorem 3.7). These results are extensions of the ones in [5].

The organization of this paper is as follows. Section 2 establishes the basic notations and results in [5]. In Section 3, we give our main results.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with reference family $\{\mathcal{F}_t\}_{t\geq 0}$ and K be a real separable Hilbert space with inner product $(\cdot,\cdot)_K$. An K-valued function $X_t(\omega)$ defined on $[0,\infty) \times \Omega$ is called an K-valued process if for any $y \in K$, $(y, X_t)_K$ is a real valued process.

DEFINITION 2.1. A mapping $B_t(h,\omega) : [0,\infty) \times H \times \Omega \longrightarrow R^1$ is called a cylindrical Brownian motion on separable Hilbert space H if it satisfies the following conditions :

(1) $B_0(h,\cdot) = 0$ and $B_t(h,\cdot)$ is $\mathcal{F}_t$-adapted.
(2) For any $h \in H$, $h \neq 0$, $B_t(h,\cdot)/\| h \|_H$ is R^1-valued Brownian motion.
(3) For any $t \in [0,\infty)$ and $\alpha, \beta \in R^1$ and $h, k \in H$, the following formula holds

$$B_t(\alpha h + \beta k) = \alpha B_t(h) + \beta B_t(k) \quad P-\text{a.s.}$$

DEFINITION 2.2. Let $\phi(t)$ be an $\mathcal{F}_t$-adapted H-valued process such that for any $t \geq 0$,

$$E\,[\int_0^t \| \phi(s) \|_H \, ds] < \infty,$$

where $E[Y]$ is the expectation of a random variable Y. The stochastic integral $\int_0^t < \phi(s), dB_s >$ of $\phi(t)$ is the real valued process given by

$$(2.1) \qquad \int_0^t < \phi(s), dB_s >= \sum_{n=1}^{\infty} \int_0^t (\phi(s), e_n)_H \, dB_s(e_n),$$

where $\{e_n \mid n = 1, 2, \cdots\}$ is a complete orthonormal system of H and where the integral on the right hand side of (2.1) is usual Itô's integral with respect to real valued Brownian motions $B_t(e_n)$.

Let $\sigma_2(H, K)$ be the set of all Hilbert-Schmidt opertors from H to K. It is well known that $\sigma_2(H, K)$ is a Hilbert space when the Hilbert-Schmidt norm $\| \cdot \|_2$ and inner product $(\cdot, \cdot)_2$ are introduced on it.

DEFINITION 2.3. Let $\Phi(t)$ be an $\mathcal{F}_t$-adapted $\sigma_2(H, K)$-valued process such that for any $t \geq 0$

$$(2.2) \qquad E\left[\int_0^t \| \Phi(s) \|_2^2 \, ds\right] < \infty$$

The stochastic integral $\int_0^t \Phi(s) \, dB_s$ of $\Phi(t)$ is the K-valued continuous $\mathcal{F}_t$-adapted process determined by

$$(2.3) \qquad (k, \int_0^t \Phi(s) \, dB_s)_K = \int_0^t < \Phi^*(s) \, k, dB_s > \quad P - \text{a.s.}$$

for any $k \in K$. where $\Phi^*(s)$ is the dual operator of $\Phi(s)$ and the right hand side of (2.3) is the stochastic integral in the sense of Definition 2.2.

Consider the following stochastic evolution equation

$$(2.4) \qquad \begin{cases} dZ_t = AZ_t \, dt + \Phi(t) \, dB_t, \ t \geq 0 \\ Z_0 = z_0 \end{cases}$$

In [5], Kim defined the integral solution of (2.4) (cf.Definition 3.2 with $\Phi(t, X_t) = \Phi(t)$) and gave the following results which is necessary to prove our main results.

THEOREM 2.4. *Let $z_0 \in \overline{\mathcal{D}(A)}$ and $\Phi(t) \in \sigma_2(H, \mathcal{D}(A))$ with*

$$E\left[\int_0^t \| \Phi(s) \|^2_{\sigma_2(H,\mathcal{D}(A))} \, ds\right] < \infty \qquad \text{for each } t \in [0,T].$$

If A is linear m-dissipative, then equation (2.4) has a unique integral solution Z_t such that $Z_t \in \overline{\mathcal{D}(A)}$ for any $t \in [0,T]$. Moreover, when $\mathcal{D}(A) = K$, this integral solution is strong solution.

3. Main results

DEFINITION 3.1. An $\mathcal{F}_t$-adapted $\mathcal{D}(A)$-valued L^2-process X_t is called a strong solution of (1.1) if it satisfies the following conditions:

(1) For any $t \geq 0$,

$$E\left[\int_0^t \| \Psi(s, X_s) \|_2^2 \, ds\right] < \infty \tag{3.1}$$

(2) For any $t \geq 0$,

$$X_t = x_0 + \int_0^t A X_s \, ds + \int_0^t \Psi(s, X_s) \, dB_s \qquad P-\text{a.s.} \tag{3.2}$$

Using usual Picard approximation method, it is easy to prove that if A and Ψ are Lipschitz continuous then the equation (1.1) has a unique strong solution.

DEFINITION 3.2. An $\mathcal{F}_t$-adapted K-valued L^2-process X_t is called an integral solution of (1.1) if it satisfies the following conditions: (1) $\Psi(\cdot, X_\cdot) \in L^2([0,\infty) \times \Omega \to \sigma_2(H, K))$ i.e. for any $t \geq 0$,

$$E\left[\int_0^t \| \Psi(s, X_s) \|_2^2 \, ds\right] < \infty$$

(2) For any $t \geq 0$,

$$\begin{aligned} \frac{1}{2} \| X_t - x \|_K^2 \leq \frac{1}{2} \| X_s - x \|_K^2 &+ \int_s^t (Ax, X_\tau - x)_K \, d\tau \\ &+ \int_s^t < \Psi^*(\tau, X_\tau)(X_\tau - x), dB_\tau > \\ &+ \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau) \|_2^2 \, d\tau \qquad P-\text{a.s.} \end{aligned} \tag{3.3}$$

THEOREM 3.3. *If X_t is a strong solution of* (1.1), *then X_t is an integral solution.*

Proof. Since X_t is a strong solution of (1.1), the following equality holds. For $t \in [0,T]$

$$X_t = x_0 + \int_0^t AX_s\, ds + \int_0^t \Psi(s,\, X_s)\, dB_s \quad P-\text{a.s.}$$

For fixed $x \in \mathcal{D}(A)$, define $F : K \longrightarrow R^1$ by $F(z) = \frac{1}{2} \parallel z - x \parallel_K^2$. Then $F_z = z - x$ and $F_{zz} = \frac{\delta^2 F}{\delta z \delta z} = I$. Applying Itô's formula to F,

$$\begin{aligned}
&\frac{1}{2} \parallel X_t - x \parallel_K^2 \\
&\quad = \frac{1}{2} \parallel X_s - x \parallel_K^2 + \int_s^t (X_\tau - x\ ,\ AX_\tau)_K d\tau \\
&\qquad + \int_s^t < \Psi^*(\tau,\, X_\tau)(X_\tau - x)\ ,\ dB_\tau > + \frac{1}{2}\int_s^t \parallel \Psi(\tau,\, X_\tau) \parallel_2^2 d\tau \\
&\quad = \frac{1}{2} \parallel X_s - x \parallel_K^2 + \int_s^t (X_\tau - x\ ,\ AX_\tau - Ax)_K d\tau \\
&\qquad + \int_s^t (X_\tau - x\ ,\ Ax)_K d\tau + \int_s^t < \Psi^*(\tau,\, X_\tau)(X_\tau - x)\ ,\ dB_\tau > \\
&\qquad + \frac{1}{2}\int_s^t \parallel \Psi(\tau,\, X_\tau) \parallel_2^2 d\tau \\
&\quad \le \frac{1}{2} \parallel X_s - x \parallel_K^2 + \int_s^t (X_\tau - x\ ,\ Ax)_K d\tau \\
&\qquad + \int_s^t < \Psi^*(\tau,\, X_\tau)(X_\tau - x)\ ,\ dB_\tau > \\
&\qquad + \frac{1}{2}\int_s^t \parallel \Psi(\tau,\, X_\tau) \parallel_2^2 d\tau \quad P-\text{a.s.}
\end{aligned}$$

since A is dissipative. Hence X_t is an integral solution of (1.1). The proof is complete. □

THEOREM 3.4. *Let X_t, Y_t be strong solutions of $dX_t = AX_t dt + \Psi(t, X_t)\, dB_t$, $dY_t = AY_t dt + \Phi(t, Y_t)\, dB_t$, respectively, with the same initial value $X_0 = Y_0 = x_0$, and let A be dissipative. Then the following inequality holds :*

(3.4)

$$\frac{1}{2} \| X_t - Y_t \|_K^2 \leq \frac{1}{2} \| X_s - Y_s \|_K^2 + \int_s^t < (\Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau))^* (X_\tau - Y_\tau) \,,\, dB_\tau > + \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau) \|_2^2 \, d\tau \quad P-\text{a.s.}$$

Proof. Since X_t and Y_t are two strong solution of the above equations,

$$X_t - Y_t = \int_0^t (AX_s - AY_s) ds + \int_0^t (\Psi(s, X_s) - \Phi(s, Y_s)) dB_s \qquad P-\text{a.s.}$$

By Itô's formula,

$$\begin{aligned} \frac{1}{2} \| X_t - Y_t \|_K^2 &= \frac{1}{2} \| X_s - Y_s \|_K^2 + \int_s^t (X_\tau - Y_\tau \,,\, AX_\tau - AY_\tau)_K d\tau \\ &+ \int_s^t < (\Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau))^* (X_\tau - Y_\tau) \,,\, dB_\tau > \\ &+ \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau) \|_2^2 \, d\tau \\ &\leq \frac{1}{2} \| X_s - Y_s \|_K^2 \\ &+ \int_s^t < (\Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau))^* (X_\tau - Y_\tau)) \,,\, dB_\tau > \\ &+ \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau) \|_2^2 \, d\tau \qquad P-\text{a.s.} \end{aligned}$$

since A is dissipative. The proof is complete. □

REMARK. If $\Psi(t, X_t) = \Psi(t)$ and $\Phi(t, Y_t) = \Phi(t)$ in Theorem 3.4, then (3.4) with $\Psi(t, X_t) = \Psi(t)$ and $\Phi(t, Y_t) = \Phi(t)$ holds. We will use this inequality to prove Theorem 3.5.

THEOREM 3.5. *Let A be a linear m-dissipative operator with $\mathcal{D}(A) = K$ and for any $\mathcal{F}_t$ - adapted K-valued L^2-process $\{Z_t\}_{t\geq 0}$, $\Psi(\cdot\,, Z_\cdot) \in L^2([0,\infty) \to \sigma_2(H, K))$ and Ψ satisfies $\| \Psi(t, x) - \Psi(t, y) \|_2^2 \leq C \| x - y \|_K^2$ for all $t \in [0, \infty)$ and $x, y \in K$. Then the equation (1.1) has unique integral solution X_t.*

Proof. Consider for each $n = 0, 1, 2, \cdots$,

$$dX_t^n = AX_t^n dt + \Psi(t, X_t^{n-1})dB_t\,, X_0^n = x_0 \tag{3.5}$$

where $\psi(t, X_t^{-1}) = 0$. From Theorem 2.4, the equation (3.5) has the unique strong solution X_t^n, and so it holds

$$X_t^n = x_0 + \int_0^t AX_s^n ds + \int_0^t \Psi(s, X_s^{n-1})\, dB_s \qquad P - \text{a.s.}$$

Now we prove that $\{X_t^n\}_{n\geq 0}$ is a Cauchy sequence. Let $T > 0$ be given and fixed. From Remark of Theorem 3.4, it holds for any $t \in [0, T]$.

$$\begin{aligned}\frac{1}{2} \| X_t^n - X_t^{n-1} \|_K^2 \\ &\leq \int_0^t < (\Psi(\tau, X_\tau^{n-1}) - \Psi(\tau, X_\tau^{n-2}))^* (X_\tau^n - X_\tau^{n-1}), dB_\tau > \\ &\quad + \frac{1}{2} \int_0^t \| \Psi(\tau, X_\tau^{n-1}) - \Psi(\tau, X_\tau^{n-2}) \|_2^2\, d\tau \qquad P - \text{a.s.}\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{2} E[\| X_t^n - X_t^{n-1} \|_K^2] \\ &\leq E[\int_0^t < (\Psi(\tau, X_\tau^{n-1}) - \Psi(\tau, X_\tau^{n-2}))^* (X_\tau^n - X_\tau^{n-1}), dB_\tau >] \\ &\quad + \frac{1}{2} E[\int_0^t \| \Psi(\tau, X_\tau^{n-1}) - \Psi(\tau, X_\tau^{n-2}) \|_2^2\, d\tau]\end{aligned} \tag{3.6}$$

$$\leq \frac{1}{2} C \int_0^t E[\| X_\tau^{n-1} - X_\tau^{n-2} \|_K^2] d\tau$$

$$\leq \frac{1}{2} \int_0^t C^{n-1} \frac{\tau^{n-1}}{(n-1)!} E[\| X_\tau^1 - X_\tau^0 \|_K^2] d\tau$$

Since X_t^0 and X_t^1 are in K for any $t \in [0, T]$, there is a constant $D > 0$ such that $E[\| X_t^1 - X_t^0 \|_K^2] \leq D$. Hence

$$\frac{1}{2} E[\| X_t^n - X_t^{n-1} \|_K^2] \leq \frac{1}{2} D C^{n-1} \frac{t^n}{n!} \qquad t \in [0, T]$$

By usual argument, it is easy to see that $\{X_t^n\}_{n \geq 0}$ is a L^2-Cauchy sequence in K, and hence there exists an $\mathcal{F}_t$ - adapted K - valued process X_t such that X_t^n converges to X_t in mean square. Thus $\{X_t^n\}$ has a subsequence $\{X_t^{n_k}\}$ which converges uniformly in $t \in [0, T]$ with probability one. For simplicity, we use the same notation $\{X_t^n\}$ for this subsequence. Since X_t^n is an integral solution, we have for each $x \in K$

$$\begin{aligned} \frac{1}{2} \| X_t^n - x \|_K^2 \leq \frac{1}{2} \| X_s^n - x \|_K^2 &+ \int_s^t (Ax\,,\, X_\tau^n - x)_K d\tau \\ &+ \int_s^t < \Psi^*(\tau, X_\tau^{n-1})(X_\tau^n - x)\,,\, dB_\tau > \\ &+ \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau^{n-1}) \|_2^2 \, d\tau \qquad P - \text{a.s.} \end{aligned} \tag{3.7}$$

Since $X_t^n \longrightarrow X_t$ a.s. and Ψ is (Lipschitz) continuous,

$$\| \Psi(t, X_t^n) - \Psi(t, X_t) \|_2 \longrightarrow 0 \quad \text{a.s.} \tag{3.8}$$

And so it holds

$$\| \Psi^*(t, X_t^n) - \Psi^*(t, X_t) \|_2 \longrightarrow 0 \quad \text{a.s.} \tag{3.9}$$

We have

$$\begin{aligned} &\| \Psi^*(t, X_t^{n-1}) X_t^n - \Psi^*(t, X_t) X_t \|_H \\ &\leq \| \Psi^*(t, X_t^{n-1}) \|_2 \| X_t^n - X_t \|_K \\ &\quad + \| \Psi^*(t, X_t^{n-1}) - \Psi^*(t, X_t) \|_2 \| X_t \|_K \; . \end{aligned} \tag{3.10}$$

Since $\| \Psi^*(t, X_t^{n-1}) \|_2 < \infty$, the right hand side of (3.10) converges to 0 a.s. Hence $\Psi^*(t, X_t^{n-1}) X_t^n \longrightarrow \Psi^*(t, X_t) X_t$ a.s. Letting $n \to \infty$ in (3.7) , we obtain (3.3). The uniqueness of integral solution is clear by the following Theorem 3.6. The proof is complete. □

THEOREM 3.6. *Let the assumptions of Theorem 3.5 hold. If X_t and Y_t are integral solutions of $dX_t = AX_t dt + \Psi(t, X_t)\, dB_t$ and $dY_t = AY_t dt + \Phi(t, Y_t)\, dB_t$, respectively, with the same initial value $X_0 = Y_0 = x_0$, then*

$$\begin{aligned} \frac{1}{2} \| X_t - Y_t \|_K^2 \leq & \frac{1}{2} \| X_s - Y_s \|_K^2 \\ (3.11) \qquad & + \int_s^t < (\Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau))^* (X_\tau - Y_\tau) \,,\, dB_\tau > \\ & + \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau) - \Phi(\tau, Y_\tau) \|_2^2 \, d\tau \qquad P - \text{a.s.} \end{aligned}$$

for $0 \leq s \leq t \leq T$.

Proof. Let X_t^n and Y_t^n be the strong solution of the approximation problems as in the proof of Theorem 3.5. According to the Theorem 3.4 we can see that

$$\begin{aligned} \frac{1}{2} \| X_t^n - Y_t^n \|_K^2 \leq & \frac{1}{2} \| X_s^n - Y_s^n \|_K^2 \\ (3.12) \qquad & + \int_s^t < (\Psi(\tau, X_\tau^{n-1}) - \Phi(\tau, Y_\tau^{n-1}))^* (X_\tau^n - Y_\tau^n) \,,\, dB_\tau > \\ & + \frac{1}{2} \int_s^t \| \Psi(\tau, X_\tau^{n-1}) - \Phi(\tau, Y_\tau^{n-1}) \|_2^2 \, d\tau \qquad P - \text{a.s.} \end{aligned}$$

Since $X_t^n \to X_t$ and $Y_t^n \to Y_t$, from (3.12), we have (3.11). The proof is complete. □

THEOREM 3.7. *Let the assumtions of Theorem 3.5 hold. Then the equation* (1.1) *has unique strong solution.*

Proof. According to Theorem 3.5 , (1.1) has the unique integral solution $X_t \in \mathcal{D}(A) = K$ as the limit of the strong solution X_t^n of the equation (3.5). Hence for any $y \in \mathcal{D}(A^*) = K$, we have

(3.13)

$$\begin{aligned} & (y, X_t^n)_K \\ & = (y, x_0)_K + (y, \int_0^t AX_s^n \, ds)_K + (y, \int_0^t \Psi(s, X_s^{n-1}) \, dB_s)_K \\ & = (y, x_0)_K + \int_0^t (A^* y, X_s^n)_K \, ds + \int_0^t < \Psi^*(s, X_s^{n-1}) y, dB_s > \end{aligned}$$

Since $X_t^n \longrightarrow X_t$ a.s.,

$$(A^*y, X_s^n) \longrightarrow (A^*y, X_t) = (y, AX_t) \text{ uniformly in } t \in [0,T].$$

By (3.9), we have

$$\|\Psi^*(t, X_t^n)y - \Psi^*(t, X_t)y\|_H \leq \|\Psi^*(t, X_t^n) - \Psi^*(t, X_t)\|_2 \|y\|_K \longrightarrow 0.$$

Hence $\Psi^*(t, X_t^n)y$ converges to $\Psi^*(t, X_t)y$ uniformly in $t \in [0,T]$ and

$$\int_0^t < \Psi^*(s, X_s^{n-1})y, dB_s > \longrightarrow \int_0^t < \Psi^*(s, X_s)y, dB_s > .$$

Letting $n \to \infty$ in (3.13), we have for any $y \in K$

$$(y, X_t)_K = (y, x_0 + \int_0^t AX_s\, ds + \int_0^t \Psi(s, X_s)dB_s)_K$$

and hence

$$X_t = x_0 + \int_0^t AX_s\, ds + \int_0^t \Psi(s, X_s)dB_s \qquad P-\text{a.s.}$$

Thus X_t is a strong solution of (1.1). The proof is complete. □

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