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MINIMAL WALLMAN COVERS OF TYCHONOFF SPACES

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ABSTRACT. Observing that for any β_c-Wallman functor $\mathcal{A}$ and any Tychonoff space X, there is a cover $(C_1(\mathcal{A}(X), X), c_1)$ of X such that X is $\mathcal{A}$-disconnected if and only if $c_1 : C_1(\mathcal{A}(X), X) \longrightarrow X$ is a homeomorphism, we show that every Tychonoff space has the minimal $\mathcal{A}$-disconnected cover. We also show that if X is a weakly Lindelöf or locally compact zero-dimensional space, then the minimal G-disconnected (equivalently, cloz)-cover is given by the space $C_1(\mathcal{A}(X), X)$ which is a dense subspace of $E_{cc}(\beta X)$.

1. Introduction

All spaces in this paper are Tychonoff spaces and βX denotes the Stone-Čech compactification of a space X.

It is known that minimal covers of some spaces are given by certain filter spaces. Indeed, the minimal extremally disconnected cover EX of a space X is given by the filter space $\{\alpha : \alpha \text{ is a fixed } R(X)\text{-ultrafilter}\}$ ([5]) and for any locally weakly Lindelöf space X, the minimal basically disconnected (quasi-F, resp.) cover of X is characterized by the filter space $\Lambda X = \{\alpha : \alpha \text{ is a fixed } \sigma Z(X)^{\#}\text{-ultrafilter}\}$ ($QF(X) = \{\alpha : \alpha$ is a fixed $Z(X)^{\#}$-ultrafilter$\}$, resp.) ([6]). In [4], it is shown that every comapct space X has a minimal cloz-cover $(E_{cc}(X), \Psi_{G(X)})$ and in [8] ([3], resp.), a theory of basically disconnected (quasi-F, resp.) covers of Tychonoff spaces is developed and the relation between ΛX and $\Lambda\beta X$ (QF(X) and QF(βX), resp.) is explored. Henriksen, Vermeer, and Woods ([4]) introduced the notion of Wallman sublattices and showed that for any compact space X and Wallman sublattice $\mathcal{A}(X)$

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of R(X), X has a Wallman cover $\mathcal{L}(\mathcal{A}(X), X)$ of $\mathcal{A}(X)$. In fact, for any compact space X, $\Lambda X = \mathcal{L}(\sigma Z(X)^{\#})$, $QF(X) = \mathcal{L}(Z(X)^{\#})$, and $E_{cc}(X) = \mathcal{L}(G(X), X)$ (see [8], [3], and [4]).

Let $\mathbf{Tych}_c$ be the category of Tychonoff spaces and covering maps and **Lat** the category of lattices and lattice homomorphisms. In this paper, we first introduce the notion of β_c-Wallman functor $\mathcal{A} : \mathbf{Tych}_c \longrightarrow \mathbf{Lat}$, that is, $\mathcal{A}$ is a contravariant functor such that for any X and $f : Y \longrightarrow X$ in $\mathbf{Tych}_c$, $\mathcal{A}(X)$ is a Wallman sublattice of R(X), $\mathcal{A}(\beta X)_X = \{A \cap X : A \in \mathcal{A}(\beta X)\} = \mathcal{A}(X)$, $\mathcal{A}(f)(A) = cl_Y(int_Y(f^{-1}(A)))$ ($A \in \mathcal{A}(X)$), and a space X is $\mathcal{A}$-disconnected (that is, for any $A, B \in \mathcal{A}(X)$ with $A \wedge B = \emptyset$, $A \cap B = \emptyset$) if and only if βX is $\mathcal{A}$-disconnected. We show that for any β_c-Wallman functor $\mathcal{A}$, every space has the minimal $\mathcal{A}$-disconnected cover and if X is a weakly Lindelöf or locally compact zero-dimensional space, then the minimal G-disconnected (equivalently, cloz) cover is given by the space $C_1(G(X), X)$ which is a dense subspace of $E_{cc}(\beta X)$. For the terminology, we refer to [1] and [2].

2. β-Wallman sublattices

Recall that the collection R(X) of all regular closed sets in a space X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows: if $A \in R(X)$ and $\{A_i : i \in I\} \subseteq R(X)$, then

$$\bigvee\{A_i : i \in I\} = \mathrm{cl}_X(\cup\{A_i : i \in I\}),$$

$$\bigwedge\{A_i : i \in I\} = \mathrm{cl}_X(int_X(\cap\{A_i : i \in I\})), \text{ and}$$

$$A' = \mathrm{cl}_X(X - A)$$

and a sublattice of R(X) is a subset of R(X) that contains $\emptyset$, X and is closed under finite joins and meets.

DEFINITION 2.1. ([4]) Let X be a space and $\mathcal{A}(X)$ a sublattice of R(X). Then

(a) $\mathcal{A}(X)$ is said to be T_1 with respect to X if for any $A \in \mathcal{A}(X)$ and $x \notin A$, there is $B \in \mathcal{A}(X)$ such that $x \in int_X(B)$ and $A \wedge B = \emptyset$,

(b) $\mathcal{A}(X)$ is said to be *normal* if for any $A, B \in \mathcal{A}(X)$ with $A \wedge B = \emptyset$, there are C, D in $\mathcal{A}(X)$ such that $A \wedge C = B \wedge D = \emptyset$ and $C \cup D = X$, and

(c) if $\mathcal{A}$(X) satisfies (a) and (b) , then $\mathcal{A}$(X) is said to be a *Wallman sublattice* of R(X).

NOTATION 2.2. For any space X, let

(a) $Z(X)^{\#} = \{cl_X(int_X(A)) : A$ is a zero-set in X}, and

(b) $\sigma Z(X)^{\#}$ denote the smallest σ-complete Boolean subalgebra of R(X) containing $Z(X)^{\#}$, and

(c) $G(X) = \{cl_X(C)$: C is a cozero-set set in X and there is a cozero-set D in X such that $C \cap D = \emptyset$ and $C \cup D$ is dense in X}.

It is well-known that for any space X, R(X), $Z(X)^{\#}$, $\sigma Z(X)^{\#}$, and G(X) are Wallman sublattices of R(X) ([4]).

DEFINITION 2.3. Let (L, $\leq$) be a lattice with the top element 1 and the bottom element 0 and $F \subseteq L$. Then F is said to be

(a) an *L-filter* if

(1) $\emptyset \neq F$.

(2) for any $a, b \in F$, there is $c \in F$ with $0 \neq c \leq a \wedge b$.

(3) if $a \in F$ and $a \leq b \in L$, then $b \in F$.

(b) an *L-ultrafilter* if F is a maximal L-filter.

Let X be a compact space, $\mathcal{A}$(X) a Wallman sublattice of R(X), and $\mathcal{L}(\mathcal{A}(X)) = \{\alpha : \alpha$ is an $\mathcal{A}$(X)-ultrafilter on X}. For any $A \in \mathcal{A}(X)$, let $A^* = \{\alpha \in \mathcal{L}(\mathcal{A}(X)) : A \in \alpha\}$. Then $\{A^* : A \in \mathcal{A}(X)\}$ is a base for closed sets of some compact topology on $\mathcal{L}(\mathcal{A}(X))$. And let $\mathcal{L}(\mathcal{A}(X), X)$ be the subspace $\{(\alpha, x) \in \mathcal{L}(\mathcal{A}(X)) \times X : x \in \cap\alpha\}$ of the product space $\mathcal{L}(\mathcal{A}(X)) \times X$. Then the map $\Psi_{\mathcal{A}(X)} : \mathcal{L}(\mathcal{A}(X), X) \longrightarrow X$ ($\Psi_{\mathcal{A}(X)}(\alpha, x) = x$) is a covering map, that is, a continuous perfect irreducible map, and $\mathcal{A}$(X) is a base for closed sets in X if and only if the map $k : \mathcal{L}(\mathcal{A}(X), X) \longrightarrow \mathcal{L}(\mathcal{A}(X))$ defined by $k((\alpha, x)) = \alpha$ is a homeomorphism ([4]).

Recall that for any dense subspace X of a space Y, the map ϕ : $R(Y) \longrightarrow R(X)$ defined by $\phi(A) = A \cap X$ ($A \in R(Y)$) is a Boolean isomorphism.

For any space X and $\mathcal{A}(X) \subseteq R(X)$, let $\mathcal{A}(X)_\beta$ denote the set $\{A \in R(\beta X) : A \cap X \in \mathcal{A}(X)\}$.

DEFINITION 2.4. Let X be a space and $\mathcal{A}$(X) a Wallman sublattice of R(X). Then $\mathcal{A}$(X) is said to be β-Wallman if $\mathcal{A}(X)_\beta$ is a Wallman sublattice of R(βX).

Let X be a space, $\mathcal{A}(X)$ a β-Wallman sublattice of R(X), and $C_1(\mathcal{A}(X)) = \{\alpha : \alpha$ is a fixed $\mathcal{A}(X)$-ultrafilter on X$\}$. For any $A \in \mathcal{A}(X)$, let $A^+ = \{\alpha \in C_1(\mathcal{A}(X)) : A \in \alpha\}$. Then $\{A^+ : A \in \mathcal{A}(X)\}$ is a base for closed sets of some topology on $C_1(\mathcal{A}(X))$. And let $C_1(\mathcal{A}(X), X)$ be the subspace $\{(\alpha, x) \in C_1(\mathcal{A}(X)) \times X : x \in \cap\alpha\}$ of the product space $C_1(\mathcal{A}(X)) \times X$ and $\Psi_{\mathcal{A}(X)_{\beta X}} : \Psi^{-1}_{\mathcal{A}(X)_\beta}(X) \longrightarrow X$ the restriction and corestriction of the map $\Psi_{\mathcal{A}(X)_\beta} : \mathcal{L}(\mathcal{A}(X)_\beta, \beta X) \longrightarrow \beta X$ with respect to $\Psi^{-1}_{\mathcal{A}(X)_\beta}(X)$ and X, respectively. Then $\Psi_{\mathcal{A}(X)_{\beta X}}$ is a covering map.

PROPOSITION 2.5. *Let X be a space and $\mathcal{A}(X)$ a β-Wallman sublattice of R(X). Then there is a homeomorphism $h : C_1(\mathcal{A}(X), X) \longrightarrow \Psi^{-1}_{\mathcal{A}(X)_\beta}(X)$ and hence the map $c_1 = \Psi_{\mathcal{A}(X)_{\beta X}} \circ h$ from $C_1(\mathcal{A}(X), X)$ to X is a covering map.*

Proof. Since $\mathcal{A}(X)$ and $\mathcal{A}(X)_\beta$ are lattice isomorhpic, for any $\mathcal{A}(X)$-ultrafilter α, there is a unique $\mathcal{A}(X)_\beta$-ultrafilter $\alpha_\beta = \{A \in \mathcal{A}(X)_\beta : A \cap X \in \alpha\}$ and vice-versa. Let $T = \Psi^{-1}_{\mathcal{A}(X)_\beta}(X)$. Define a map $h : C_1(\mathcal{A}(X), X) \longrightarrow T$ by $h((\alpha, x)) = (\alpha_\beta, x)$. Then h is one-to-one and onto.

Let $A \in \mathcal{A}(X)$ and U be an open set in βX. Then there is a unique $A_\beta \in \mathcal{A}(X)_\beta$ with $A = A_\beta \cap X$. Let $G = [(C_1(\mathcal{A}(X)) - A^+) \times (U \cap X)] \cap C_1(\mathcal{A}(X), X)$ and $H = (\mathcal{L}(\mathcal{A}(X)_\beta) - A^*_\beta) \times U) \cap T$. Take any $(\gamma, y) \in G$. Then $\gamma \notin A^+$; $\gamma_\beta \notin A^*_\beta$. So $h((\gamma, y)) = (\gamma_\beta, y) \in (\mathcal{L}(\mathcal{A}(X)_\beta) - A^*_\beta) \times U$. Since $\Psi_{\mathcal{A}(X)_\beta}((\gamma_\beta, y)) = y \in X$, $h((\gamma, y)) = (\gamma_\beta, y) \in T$. Hence $h(G) \subseteq H$.

Let $(\delta_\beta, z) \in H$. Since $z \in X$ and $z \in \cap\delta$, $h^{-1}((\delta_\beta, z)) = (\delta, z) \in G$. Hence $h(G) = H$. Thus h is a homeomorphism and $c_1 = \Psi_{\mathcal{A}(X)_{\beta X}} \circ h$ is a covering map. □

DEFINITION 2.6. Let $\mathbf{Tych}_c$ be the category of Tychonoff spaces and covering maps and **Lat** the category of lattices and lattice homomorphisms. Then a contravariant functor $\mathcal{A} : \mathbf{Tych}_c \longrightarrow \mathbf{Lat}$ is said to be β-Wallman if

(a) for any $X \in \mathbf{Tych}_c$, $\mathcal{A}(X)$ is a β-Wallman sublattice of R(X) and $\mathcal{A}(X)_\beta = \mathcal{A}(\beta X)$,

(b) for any $f : Y \longrightarrow X$ in $\mathbf{Tych}_c$, $\mathcal{A}(f)(A) = \mathrm{cl}_Y(\mathrm{int}_Y(f^{-1}(A)))$ $(A \in \mathcal{A}(X))$.

Let R ($Z^{\#}$, $\sigma Z^{\#}$, G, resp.) : $\mathbf{Tych}_c \longrightarrow \mathbf{Lat}$ be a contravariant functor taking each Tychonoff space X to the lattice R(X) ($Z(X)^{\#}$, $\sigma Z(X)^{\#}$, G(X), resp.) and each covering map $f : Y \longrightarrow X$ to the lattice homomorphism $\mathcal{A}(f) : \mathcal{A}(X) \longrightarrow \mathcal{A}(Y)$ for which $\mathcal{A}(f)(A) = cl_Y(int_Y(f^{-1}(A)))$ ($A \in \mathcal{A}(X)$), where $\mathcal{A}$ = R, $Z^{\#}$, $\sigma Z^{\#}$, or G, resp. Then R, $Z^{\#}$, $\sigma Z^{\#}$, and G are β-Wallman functors.

DEFINITION 2.7. Let $\mathcal{A}$ be a β-Wallman functor and X a space. Then X is called $\mathcal{A}$-disconnected if for any A, B $\in \mathcal{A}(X)$ with $A \wedge B = \emptyset$, $A \cap B = \emptyset$.

Recall that a space X is called *extremally disconnected (basically disconnected, cloz*, resp.) if every element in R(X) ($\sigma Z(X)^{\#}$, G(X), resp.) is clopen in X and that a space X is called *quasi-F* if for any A, B $\in Z(X)^{\#}$, $A \wedge B = A \cap B$. Hence a space X is extremally disconnected (basically disconnected, quasi-F, cloz, resp.) space if and only if it is R ($\sigma Z^{\#}$, $Z^{\#}$, G, resp.)-disconnected.

THEOREM 2.8. *Let $\mathcal{A}$ be a β-Wallman functor and X a space. Then X is $\mathcal{A}$-disconnected if and only if $c_1 : C_1(\mathcal{A}(X), X) \longrightarrow X$ is a homeomorphism.*

Proof. ($\Rightarrow$) It is enough to show that c_1 is one-to-one. Take any $(\alpha, x) \neq (\gamma, y)$ in $C_1(\mathcal{A}(X), X)$. If $x \neq y$, then $c_1((\alpha, x)) = x \neq y = c_1((\gamma, y))$. Suppose that $\alpha \neq \gamma$, then there are A, B $\in \mathcal{A}(X)$ such that $A \in \alpha$, $B \in \gamma$ and $A \wedge B = \emptyset$. Since X is $\mathcal{A}$-disconnected, $A \cap B = \emptyset$ and hence $c_1((\alpha, x)) = x \neq y = c_1((\gamma, y))$. So c_1 is one-to-one.

($\Leftarrow$) Suppose that there are A, B $\in \mathcal{A}(X)$ such that $A \wedge B = \emptyset$ and $A \cap B \neq \emptyset$. Pick $x \in A \cap B$. Let $\alpha_0 = \{E \in \mathcal{A}(X) : x \in int_X(E)\} \cup \{A\}$. Then there is an $\mathcal{A}(X)$-ultrafilter α with $\alpha_0 \subseteq \alpha$. Suppose that $x \notin \cap\alpha$, then $x \notin E$ for some $E \in \alpha$. Since $\mathcal{A}(X)$ is T_1 with respect to X, there is $F \in \mathcal{A}(X)$ with $x \in int_X(F)$ and $E \wedge F = \emptyset$. Since $F \in \alpha$, we have a contradiction. Hence $x \in \cap\alpha$. Similarly there is a $\gamma \in C_1(\mathcal{A}(X))$ with $x \in \cap\gamma$ and $B \in \gamma$. Hence $(\alpha, x) \neq (\gamma, x)$ and so c_1 is not one-to-one. Thus we have a contradiction. □

3. Minimal $\mathcal{A}$-disconnected covers

Note that a space X is R ($\sigma Z^{\#}$, $Z^{\#}$, G, resp.)-disconnected if and only if βX is also R ($\sigma Z^{\#}$, $Z^{\#}$, G, resp.)-disconnected

DEFINITION 3.1. A β-Wallman functor $\mathcal{A}$ is said to be β_c-Wallman if for any $\mathcal{A}$-disconnected space X, βX is $\mathcal{A}$-disconnected.

Recall that a pair (Y, f) is called a *cover* of a space X if f : Y $\longrightarrow$ X is a covering map.

DEFINITION 3.2. Let $\mathcal{A}$ be a β-Wallman functor and X a space. Then

(a) a cover (Y, f) of X is called an *$\mathcal{A}$-disconnected cover* of X if Y is $\mathcal{A}$-disconnected,

(b) an $\mathcal{A}$-disconnected cover (Y, f) of X is called a *minimal $\mathcal{A}$-disconnected cover* of X if for any $\mathcal{A}$-disconnected cover (K, g) of X, there is a covering map h : K $\to$ Y with $f \circ h = g$.

LEMMA 3.3. ([4]) *Let X be a compact space and $\mathcal{A}(X)$ a Wallman sublattice of $R(X)$. Then $(\mathcal{L}(\mathcal{A}(X), X), \Psi_{\mathcal{A}(X)})$ is a cover of X such that*

(a) for any $A, B \in \mathcal{A}(X)$ with $A \wedge B = \emptyset$, $cl_{\mathcal{L}(\mathcal{A}(X),X)}(\Psi^{-1}_{\mathcal{A}(X)}(int_X(A))) \cap cl_{\mathcal{L}(\mathcal{A}(X),X)}(\Psi^{-1}_{\mathcal{A}(X)}(int_X(B))) = \emptyset$, and

(b) if $f : Y \longrightarrow X$ is a covering map such that for any $A, B \in \mathcal{A}(X)$ with $A \wedge B = \emptyset$, $cl_{\mathcal{L}(\mathcal{A}(X),X)}(f^{-1}(int_X(A))) \cap cl_{\mathcal{L}(\mathcal{A}(X),X)}(f^{-1}(int_X(B))) = \emptyset$, then there is a covering map $g : Y \longrightarrow \mathcal{L}(\mathcal{A}(X), X)$ with $f = \Psi_{\mathcal{A}(X)} \circ g$.

For any β_c-Wallman functor $\mathcal{A}$ and space X, the following diagram

$$\begin{array}{ccc} C_1(\mathcal{A}(X), X) & \xrightarrow{c_1} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathcal{L}(\mathcal{A}(\beta X), \beta X) & \xrightarrow{\Psi_{\mathcal{A}(\beta X)}} & \beta X. \end{array}$$

is a pullback in **Top**, where j is a dense embedding and **Top** is the category of topological spaces and continuous maps. Using this and the above lemma, we have the following:

THEOREM 3.4. *Let $\mathcal{A}$ be a β_c-Wallman functor and X a space. If (Y, f) is an $\mathcal{A}$-disconnected cover of X, then there is a covering map g : $Y \longrightarrow C_1(\mathcal{A}(X), X)$ with $c_1 \circ g = f$.*

COROLLARY 3.5. *Let $\mathcal{A}$ be a β_c-Wallman functor and X a space. If $C_1(\mathcal{A}(X), X)$ is $\mathcal{A}$-disconnected, then $(C_1(\mathcal{A}(X), X), c_1)$ is the minimal $\mathcal{A}$-disconnected cover of X.*

In the following, for any space X, let (EX, π_X) denote the minimal R-disconnected cover (absolute, or minimal extremally disconnected cover) of X. Indeed, for any space X, (C_1(R(X), X), c_1) is the minimal R-disconnected cover of X ([5]).

Note that for any β-Wallman functor $\mathcal{A}$ and X $\in$ **Tych**$_c$, (EX, π_X) is an $\mathcal{A}$-disconnected cover of X.

THEOREM 3.6. *Let $\mathcal{A}$ be a β_c-Wallman functor. Then for any space X, there is the mininmal $\mathcal{A}$-disconnected cover $(C(\mathcal{A}(X), X), C_{\mathcal{A}(X)})$ of X.*

Proof. Let $C_0(\mathcal{A}(X), X) = X$, $c_0^0 = 1_X$ and α be an ordinal. Suppose that for any ordinal β with $\beta < \alpha$,

(i) for any ordinal γ with $\gamma \leq \beta$, there is a cover $(C_\gamma(\mathcal{A}(X), X), c_0^\gamma)$ of X, and

(ii) for any ordinals δ, γ with $\delta < \gamma \leq \beta$, there is a covering map $c_\delta^\gamma : C_\gamma(\mathcal{A}(X), X) \longrightarrow C_\delta(\mathcal{A}(X), X)$ with $c_0^\gamma = c_0^\delta \circ c_\delta^\gamma$.

For any ordinals γ_1, γ_2, γ_3 with $\gamma_1 < \gamma_2 < \gamma_3 \leq \beta$, since $c_0^{\gamma_3} = c_0^{\gamma_1} \circ c_{\gamma_1}^{\gamma_3} = c_0^{\gamma_1} \circ (c_{\gamma_2}^{\gamma_1} \circ c_{\gamma_2}^{\gamma_3})$ and $c_0^{\gamma_1}$ is a covering map, $c_{\gamma_1}^{\gamma_3} = c_{\gamma_1}^{\gamma_2} \circ c_{\gamma_2}^{\gamma_3}$.

Let α be a non-limit ordinal, then there is an ordinal β with $\alpha = \beta + 1$. Let $C_\alpha(\mathcal{A}(X), X) = C_1(\mathcal{A}(C_\beta(\mathcal{A}(X), X)), C_\beta(\mathcal{A}(X), X))$ and $c_\beta^\alpha = c_1 : C_\alpha(\mathcal{A}(X), X) \longrightarrow C_\beta(\mathcal{A}(X), X)$. Then (i) and (ii) hold for α.

Let α be a limit ordinal. Let I = $\{\beta : \beta$ is an ordinal with $\beta < \alpha\}$. Define an inverse limit system D : I $\longrightarrow$ **Top** as follows: for any β and $\gamma < \beta$ in I, let $D(\beta) = C_\beta(\mathcal{A}(X), X)$ and $D(\gamma \leq \beta) = c_\gamma^\beta : C_\beta(\mathcal{A}(X), X) \longrightarrow C_\gamma(\mathcal{A}(X), X)$. Let $(C_\alpha(\mathcal{A}(X), X), c_\beta^\alpha)_{\beta<\alpha}$ be the inverse limit of D. Then (i) and (ii) hold for α.

By Theorem 3.4 and the transfinite induction, for any ordinal α, there is a covering map $f_\alpha : EX \longrightarrow C_\alpha(\mathcal{A}(X), X))$ with $c_0^\alpha \circ f_\alpha = \pi_X$. Hence all spaces $C_\alpha(\mathcal{A}(X), X)$ lie between X and EX and therefore there is the smallest ordinal α_0 such that $c_{\alpha_0}^{\alpha_0+1} : C_{\alpha_0+1}(\mathcal{A}(X), X) \longrightarrow C_{\alpha_0}(\mathcal{A}(X), X)$ is a homeomorphism. By Theorem 2.8, $C_{\alpha_0}(\mathcal{A}(X), X)$ is an $\mathcal{A}$-disconnected space. Let $C(\mathcal{A}(X), X) = C_{\alpha_0}(\mathcal{A}(X), X)$ and $C_{\mathcal{A}(X)} = c_0^{\alpha_0} : C(\mathcal{A}(X), X) \longrightarrow X$. By Theorem 3.4 and the transfinite induction, $(C(\mathcal{A}(X), X), C_{\mathcal{A}(X)})$ is the mininmal $\mathcal{A}$-disconnected cover of X. □

Let $\mathcal{A}$ and $\mathcal{B}$ be β_c-functors such that for any space X, $\mathcal{B}(X) \subseteq \mathcal{A}(X)$. If a space X is $\mathcal{A}$-disconnected, then X is also $\mathcal{B}$-disconnected. Using this and the fact that if f, g, and h are covering maps with $f \circ g = f \circ h$, then $g = h$, we have the following:

PROPOSITION 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be β_c-functors such that for any space X, $\mathcal{B}(X) \subseteq \mathcal{A}(X)$. Then for any space X, there is a homeomorphism $h : C(\mathcal{A}(Y), Y) \longrightarrow C(\mathcal{A}(X), X)$ with $C_{\mathcal{B}(X)} \circ C_{\mathcal{A}(Y)} = C_{\mathcal{A}(X)} \circ h$, where $Y = C(\mathcal{B}(X), X)$.*

Recall that a space X is *weakly Lindelöf* if every open cover $\mathcal{U}$ of X has a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ with $\mathrm{cl}_X(\cup\mathcal{V}) = X$ and that a space X is called *locally weakly Lindelöf* if every element of X has a weakly Lindelöf neighborhood in X.

In [6], it is shown that for any locally weakly Lindelöf space X, $(C_1(Z(X)^{\#}, X), c_1)$ $((C_1(\sigma Z(X)^{\#}, X), c_1)$, resp.) is the minimal $Z^{\#}$ ($\sigma Z^{\#}$, resp.)-disconnected cover of X. Now we will try to find suitable conditions for a space X for which $(C_1(G(X), X), c_1)$ is the minimal G-disconnected cover of X.

LEMMA 3.8. *Let $\mathcal{A}$ be a β-Wallman functor such that for any open subspace S of a space Y, $\mathcal{A}(Y)_S = \{A \cap S : A \in \mathcal{A}(Y)\} \subseteq \mathcal{A}(S)$. Then a space X is $\mathcal{A}$-disconnected if and only if every element of X has an $\mathcal{A}$-disconnected open neighborhood in X.*

Proof. Suppose that every element of X has an $\mathcal{A}$-disconnected open neighborhood in X. Let A, B $\in \mathcal{A}(X)$ with $A \wedge B = \emptyset$. Suppose that there is an $x \in A \cap B$. Let S be an $\mathcal{A}$-disconnected open neighborhood of x in X. Note that $A \cap S, B \cap S \in \mathcal{A}(S)$ and $(A \cap S) \wedge (B \cap S) = \emptyset$. Since S is $\mathcal{A}$-disconnected, $A \cap B \cap S = \emptyset$. This is a contradiction. Hence X is an $\mathcal{A}$-disconnected space. The converse is trivial. □

Recall that a space X is a G-disconnected (equivalently, cloz-) space if and only if every dense cozero-set C in X is D-extendable in X, that is, for any continuous map $f : C \longrightarrow D$, there is a continuous map $g : X \longrightarrow D$ with $g|_C = f$, where D denotes the two point discrete space ([4]) and that for any dense weakly Lindelöf subspace X of a space Y and $A \in Z(X)^{\#}$, there is $B \in Z(Y)^{\#}$ with $A = B \cap X$.

LEMMA 3.9. *If X is a clopen or dense weakly Lindelöf subspace of a G-disconnected space Y, then X is also G-disconnected*

Proof. Let X be a subspace of a G-disconnected space Y. Suppose that X is clopen in Y and take any dense cozero-set C in X. Then C is a cozero-set in Y and hence C ∪ (Y - X) is a dense cozero-set in Y. Since C and Y - X are disjoint clopen sets in C ∪ (Y - X), C is $\underline{D}$-extendable in C ∪ (Y - X). Sinec Y is G-disconnected, C ∪ (Y - X) is $\underline{D}$-extendable in Y and hence C is $\underline{D}$-extendable in Y. Thus X is G-disconnected.

Suppose that X is a dense weakly Lindelöf subspace of Y. Let $D_1 \in G(X)$, then there is a cozero-set D_2 in X such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2$ is dense in X. Since X is a dense weakly Lindelöf subspace of Y, there are cozero-sets E_1, E_2 in Y such that $D_i \subseteq \mathrm{cl}_Y(E_i)$ $(i = 1, 2)$, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2$ is dense in Y. Hence $\mathrm{cl}_Y(E_1) \cap \mathrm{cl}_Y(E_2) = \emptyset$ and thus $\mathrm{cl}_Y(D_1) \cap \mathrm{cl}_Y(D_2) = \emptyset$. Therefore X is G-disconnected. □

In [4], the minimal G-disconnected cover of a compact space X is characterized by $(\mathcal{L}(G(X), X), \Psi_{G(X)})$. In the literature, for a compact space X, $\mathcal{L}(G(X), X)$ is denoted by $E_{cc}(X)$.

It is known that if $f : Y \to X$ is a covering map and X is a weakly Lindelöf space, then Y is weakly Lindelöf ([3]).

THEOREM 3.10. *If X is a locally compact zero-dimensional or weakly Lindelöf space, then $(C_1(G(X), X), c_1)$ is the minimal G-disconnected cover of X.*

Proof. Let $l = \Psi_{G(\beta X)} : E_{cc}(\beta X) \longrightarrow \beta X$ and $W = l^{-1}(X)$.

Suppose that X is a locally compact zero-dimensional space. Take any $y \in W$, then there is a clopen compact neighborhood B of $l(y)$ in X. Since B is clopen in βX, $l^{-1}(B)$ is clopen in $E_{cc}(\beta X)$ and $l^{-1}(B) \subseteq W$. By Lemma 3.9, $l^{-1}(B)$ is a G-disconnected open neighborhood of y in W. Thus , by Lemma 3.8, W is G-disconnected and therefore, by Corollary 3.5, $(C_1(G(X), X), c_1)$ is the minimal G-disconnected cover of X. If X is a weakly Lindelöf space, then W is a weakly Lindelöf subspace of $E_{cc}(\beta X)$ and hence by Corollary 3.5 and Lemma 3.9, $(C_1(G(X), X)$, $c_1)$ is the minimal G-disconnected cover of X. □

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