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LIPSCHITZ REGULARITY OF M-HARMONIC FUNCTIONS

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ABSTRACT. In the paper we introduce Hausdorff measures which are suitable or the study of Lipschitz regularity of M-harmonic function in the unit ball B in $\mathbb{C}^n$. For an M-harmonic function h which satisfies certain integrabilty conditions, we show that there is an open set Ω, whose Hausdorff content is arbitrarily small, such that h is Lipschitz smooth on $B \setminus \Omega$.

1. Introduction and statement of the main results

Let B denote the open unit ball in $\mathbb{C}^n$ and let $\mathcal{M} = \operatorname{Aut}(B)$ be the group of all biholomorphic selfmaps of B. We denote by ν the Lebesgue measure on B normalized so that $\nu(B) = 1$. For any real number $s < 1$ consider the probability measure

$$d\nu_s(z) := \frac{\Gamma(n+1-s)}{n!\Gamma(1-s)} \frac{d\nu(z)}{(1-|z|^2)^s}.$$

By [1] the limit of $d\nu_s$ as $s \to 1$ is $d\nu_1(\zeta) = d\sigma(\zeta)$, where σ is the rotation invariant probability measure on $S = \partial B$, the boundary of B. It is also well-known that the measure $d\tau(z) := (1-|z|^2)^{n+1} d\nu(z)$ on B is invariant under the group $\mathcal{M}$.

For $z = |z|\eta \in B$, let φ_z denote the Möbius transformation exchanging z and the origin. This map is given [6] by

$$(1.1) \quad \varphi_z(w) = \frac{z - \langle w, \eta\rangle\eta - \sqrt{1-|z|^2}(w - \langle w, \eta\rangle\eta)}{1 - \langle w, z\rangle}, \quad \text{for } w \in B.$$

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It is a well-known fact [4] that the function

$$\varrho(z,w) := |\varphi_z(w)|, \quad z,w \in B$$

defines a distance function on B, the pseudohyperbolic distance. This is a Möbius invariant distance on B; that is,

$$\varrho(\varphi(z),\varphi(w)) = \varrho(z,w), \text{ for all } \varphi \in \mathrm{Aut}(B).$$

The corresponding balls are given by

$$E(z,r) = \{w \in B : \varrho(z,w) < r\} = \varphi_z(rB)$$

for $z \in B$ and $0 < r \leq 1$. These balls are called the pseudohyperbolic balls.

A function $f : B \to \mathbb{C}$ is called $\mathcal{M}$-harmonic or invariant harmonic on B, if f is continuous on B

$$(1.2)\quad (f\circ\varphi)(0) = \int_S (f\circ\varphi)(r\zeta)\,d\sigma(\zeta), \text{ for all } \varphi \in \mathcal{M} \text{ and } 0 \leq r < 1.$$

The $\mathcal{M}$-harmonic functions are precisely those C^∞-functions which are annihilated by the Laplacian $\widetilde{\Delta}$ of the Bergman metric. This is given by

$$(\widetilde{\Delta} f)(z) = (\Delta(f\circ\varphi_z))(0), \quad \text{for } f \in C^2(B), z \in B.$$

The symbol ω will stand once and for all for a gauge function. This is a function $\omega : [0,1) \to [0,\infty)$ which is nondecreasing and vanishes at $r = 0$. For $\Omega \subset B$ and $\varepsilon > 0$, set

$$\mathcal{H}^\varepsilon_{\omega,s}(\Omega) := \frac{\Gamma(n+1-s)}{n!\Gamma(1-s)} \inf\left\{\sum_{j=1}^\infty (1-|z_j|^2)^{n+1-s}\omega(r_j)\right\},$$

where the infimum is taken over all the pseudohyperbolic balls $\{E(z_j, r_j)\}_1^\infty$ such that $r_j \leq \varepsilon$ and $\Omega \subset \cup_{j=1}^\infty E(z_j,r_j)$. The corresponding Hausdorff measure of Ω is defined by

$$(1.3)\qquad \mathcal{H}_{\omega,s}(\Omega) := \lim_{\varepsilon\to 0} \mathcal{H}^\varepsilon_{\omega,s}(\Omega).$$

For $0 < \varepsilon_0 < \frac{1}{2}$, let $\widehat{\mathcal{H}}_{\omega,s}(\Omega) := \mathcal{H}^{\varepsilon_0}_{\omega,s}(\Omega)$.

REMARK 1.1. For $d > 0$ and $\omega(t) = t^d$, let

$$\mathcal{H}_d = \mathcal{H}_{\omega,0} \text{ and } \widehat{\mathcal{H}}_d = \widehat{\mathcal{H}}_{\omega,0} \text{ for } s = 0,$$

$$\mathcal{H}'_d = \mathcal{H}_{\omega,0} \text{ and } \widehat{\mathcal{H}}'_d = \widehat{\mathcal{H}}_{\omega,1} \text{ for } s = 1.$$

It should be noted that $\mathcal{H}_d$ is essentially the d-dimentional Hausdorff measure on B constructed with respect to pseudohyperbolic balls. This is so due to the fact that $\nu(E(z, r^{\frac{d}{2n}})) \approx (1-|z|^2)^{n+1} r^d$ for $z \in B$ and $0 \leq r \leq 1/2$.

For $0 < \kappa \leq 1$ we say that a function $f : \Omega \to \mathbb{C}$ is in the Möbius invariant Lipschitz class $\Gamma_\kappa(\Omega)$ if there exists a positive constant M such that

$$|f(z) - f(w)| \leq M|\varphi_z(w)|^\kappa, \text{ for all } z, w \in \Omega.$$

In order to define the Möbius invariant Lipschitz class in the case of higher order smoothness $\kappa > 1$, denote by $[\kappa]$ the greatest integer smaller than or equal to κ. We shall say that f is in the class $\Gamma_\kappa(\Omega)$ if there exist functions $\{f^{l\overline{m}} : l, m \in \mathbb{N}_0^n \text{ and } |l+m| \leq [\kappa]\}$ and a positive constant M such that $f^{0\overline{0}} = f$ and

$$\left| f^{u\overline{v}}(z) - \sum_{|l+m|<[\kappa]-|u+v|} \frac{f^{l\overline{m}}(w)}{(l+m)!} (\varphi_w(z))^l (\overline{\varphi_w(z)})^m \right| \leq M|\varphi_z(w)|^{\kappa-|u+v|},$$

for $z, w \in \Omega$ and $|u+v| \leq [\kappa]$. We have made use of the standard multi-index notations

$$|m| = m_1 + \cdots + m_n, \quad m! = m_1! \cdots ! m_n$$

$$z^m = z_1^{m_1} \cdots z_n^{m_n} \text{ and } \overline{z} = (\overline{z}_1, \cdots \overline{z}_n)$$

for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $m = (m_1, \cdots, m_n) \in \mathbb{N}_0^n$.

By looking at the differential definition of the classical (euclidian) Lipschitz spaces [7] one can think of the functions $f^{l\overline{m}}$ as partial derivatives which are induced by the action of the group $\mathcal{M}$. This is indeed our motivation for defining the Möbius invariant Lipschitz space in this manner.

Finally define the quantity

$$(\widehat{Q}_\kappa f)(z) := \sum_{j=1}^{\kappa} \sum_{|u+v|=j} \left| \left(\frac{\partial^u}{\partial z^u} \frac{\partial^v}{\partial \overline{z}^v}\right)(f \circ \varphi_z)(0) \right|, \quad z \in B.$$

Our main results are the following.

THEOREM A. *Let ω be an arbitrary gauge function, $\kappa \in \mathbb{N}$. Let $s \le 1$ and $p \ge 1$. If an M-harmonic function h on B satisfies*

$$|||h|||_{p,s} := \left\{ \int_B |(\widehat{Q}_\kappa h)(z)|^p \, d\nu_s(z) \right\}^{1/p} < \infty$$

then for each $\varepsilon > 0$ there exists an open subset $\Omega \subseteq B$ depending on all the parameters such that $\widehat{\mathcal{H}}_{\omega,s}(\Omega) < \varepsilon$ and $h \in \Gamma_\kappa(B \setminus \Omega)$.

THEOREM B. *Let ω be an arbitrary gauge function, $\kappa \in \mathbb{N}$ and $p \ge 1$. For $0 \le s \le 1$ and an M-harmonic function h on B let*

$$\|h\|_{p,s} := \begin{cases} \left\{ \int_B |h(z)|^p \log \dfrac{1}{1-|z|^2} \, d\nu(z) \right\}^{1/p} & s = 0, \\ \left\{ \int_B |h(z)|^p \, d\nu_s(z) \right\}^{1/p}, & 0 < s < 1 \\ \lim_{t \to 1} \|h\|_{p,t} & s = 1. \end{cases}$$

If $\|h\|_{p,s} < \infty$ then for each $\varepsilon > 0$ there exists an open subset $\Omega \subseteq B$ depending on all the parameters such that $\widehat{\mathcal{H}}_{\omega,s}(\Omega) < \varepsilon$ and $h \in \Gamma_\kappa(B \setminus \Omega)$.

COROLLARY C. *Let $d > 0$, $\kappa > 0$ and h be an M-harmonic function on B.*

(1) *If $p \ge 1$ and h satistfies the growth condition*

$$\|h\|_{H^p} := \sup_{0 \le r \le 1} \left\{ \int_S |h(r\zeta)|^p \, d\sigma(\zeta) \right\}^{1/p} < \infty,$$

then for each $\varepsilon > 0$ there exists an open subset $\Omega \subseteq B$ such that $\widehat{\mathcal{H}}'_d(\Omega) < \varepsilon$ and $h \in \Gamma_\kappa(B \setminus \Omega)$.

(2) *If $p > 1$ and h satistfies the growth condition*

$$\|h\|_p = \left\{ \int_B |h(w)|^p \, d\nu(w) \right\}^{1/p} < \infty,$$

then for each $\varepsilon > 0$ there exists an open subset $\Omega \subseteq B$ such that $\widehat{\mathcal{H}}_d(\Omega) < \varepsilon$ and $h \in \Gamma_\kappa(B \setminus \Omega)$.

2. Lipschitz regular points of M-harmonic functions

Let $\kappa \in \mathbb{N}$ and let f be a C^κ-function on B. For $z \in B$ and $\alpha, \beta \in \mathbb{N}_0^n$, set

$$\widetilde{\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}} f(z) = \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(f \circ \varphi_z)(0)$$

and for $\zeta \in \mathbb{C}^n$ let

(2.1)
$$D^\kappa f(z).\zeta = \sum_{|\alpha+\beta|=\kappa} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta} f(z)\zeta^\alpha\overline{\zeta}^\beta$$

(2.2)
$$\widetilde{D}^\kappa f(z).\zeta = \sum_{|\alpha+\beta|=\kappa} \widetilde{\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}} f(z)\zeta^\alpha\overline{\zeta}^\beta = D^\kappa(f \circ \varphi_z)(0).\zeta$$

Since $D^\kappa(f \circ U)(z).\zeta = (D^\kappa f)(Uz).U\zeta$ for all $U \in \mathcal{U}$, the group of all unitary transformations of $\mathbb{C}^n$. It follows that the quantities

$$|D^\kappa f(z)| := \sup_{|\zeta|=1} |D^\kappa f(z).\zeta| \quad\text{and}\quad \left|\widetilde{D}^\kappa f(z)\right| := \sup_{|\zeta|=1} \left|\widetilde{D}^\kappa f(z).\zeta\right|$$

are unitarily invariant. But for $a, b \in B$ and $c = \varphi_a(b)$ we have $\varphi_a \circ \varphi_b = \varphi_c \circ U$ for some $U \in \mathcal{U}$. Thus the quantity $|\widetilde{D}^\kappa f(z)|$ is indeed Möbius invariant; that is,

$$\left|\widetilde{D}^\kappa(f \circ \varphi)(z)\right| = \left|(\widetilde{D}^\kappa f)(\varphi(z))\right|, \quad \text{for all } \varphi \in \mathcal{M} \text{ and } z \in B. \tag{2.3}$$

For $\kappa = 1$, the function $\left|\widetilde{D}^1 f(z)\right|$ was shown to play an interesting role in the study of M-harmonic functions even in the more general context of bounded symmetric domains. See [3] and [4] on this matter.

It is clear that for some $C > 0$ independent of f and z we have

(2.4)
$$C \sum_{|\alpha+\beta|=\kappa} \left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta} f(z)\right| \le |D^\kappa f(z)| \le \frac{1}{C} \sum_{|\alpha+\beta|=\kappa} \left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta} f(z)\right|$$

(2.5)
$$C \sum_{|\alpha+\beta|=\kappa} \left|\widetilde{\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}} f(z)\right| \le \left|\widetilde{D}^\kappa f(z)\right| \le \frac{1}{C} \sum_{|\alpha+\beta|=\kappa} \left|\widetilde{\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}} f(z)\right|$$

Finally define the quantity

$$(Q_\kappa f)(z) := \sum_{j=1}^{\kappa} \left| \widetilde{D}^j f(z) \right|, \quad z \in B.$$

This $Q_\kappa f$ is a Möbius invariant quantity and by (1.7), (2.4) and (2.5) we have

$$C\widehat{Q}_\kappa f \leq Q_\kappa f \leq \frac{1}{C}\widehat{Q}_\kappa f. \tag{2.6}$$

LEMMA 2.1. *For $\kappa \in \mathbb{N}$ and $0 \leq s < 1$ there exist a positive constant C such that*

$$\int_B \frac{|\widetilde{D}^\kappa h(z)|^p}{(1-|z|^2)^s}\, d\nu(z) \leq C \begin{cases} \displaystyle\int_B \frac{|h(z)|^p}{(1-|z|^2)^s}\, d\nu(z), & s > 0 \\ \displaystyle\int_B |h(z)|^p \log\frac{1}{1-|z|^2}\, d\nu(z) & s = 0, \end{cases}$$

for all M-harmonic functions h on B.

Proof. For an M-harmonic function h on B, we have

$$h(z) = \int_B h(w) \frac{(1-|z|^2)^{n+1}}{|1-<z,w>|^{2(n+1)}}\, d\nu(w), \quad z \in B$$

from which it follows that

$$|\widetilde{D}^\kappa h(0)|^p \leq \int_B |h(w)|^p\, d\nu(w) \tag{2.7}$$

Replacing h by $h \circ \varphi_z$ and applying the change of variable formula, we obtain from (2.7) that

$$|\widetilde{D}^\kappa h(z)|^p \leq \int_B |h(w)|^p \frac{(1-|z|^2)^{n+1}}{|1-<z,w>|^{2(n+1)}}\, d\nu(w), \tag{2.8}$$

so that by Fubini's theorem we see that

$$\int_B \frac{|\widetilde{D}^\kappa h(z)|^p}{(1-|z|^2)^s} d\nu(z) \leq \int_B |h(w)|^p \int_B \frac{(1-|z|^2)^{-s+n+1}}{|1-<z,w>|^{2(n+1)}}\, d\nu(z)\, d\nu(w).$$

The lemma now follows from Proposition 1.4.10 in [6]. □

LEMMA 2.2. *For $\kappa \in \mathbb{N}$ and $0 < \delta < 1$ there exists $C = C(\kappa, \delta) > 0$ such that*

$$(Q_\kappa h)(z) \leq \frac{C}{\varepsilon^{2n}} \int_{E(z,\varepsilon)} (Q_\kappa h)(w)\, d\tau(w), \quad z \in B, \tag{2.9}$$

for all $0 < \varepsilon < \delta$, for all M-harmonic functions h on B.

Proof. Due to the invariant property of $Q_\kappa h$ it is enough to establish (2.9) at the origin $z = 0$. Using the mean value property, a little computing shows that for $j = 1, \cdots, \kappa$, we have

$$(D^j h)(0).\zeta = \frac{1}{c(\varepsilon)} \int_{\varepsilon B} (D^j h)(w).a(w,\zeta)\, d\tau(w) \tag{2.10}$$

where $a(w, \zeta) = \zeta - < w, \zeta > w$ for $w \in \delta B, \zeta \in \mathbb{C}^n$ and

$$c(\varepsilon) := \frac{1}{2n} \int_0^\varepsilon \frac{t^{2n-1}}{(1-t^2)^{n+1}}\, dt \geq \varepsilon^{2n}. \tag{2.11}$$

Also we have

$$(\widetilde{D}^j h)(w).\zeta = (D^j h)(w).\zeta + \sum_{|\alpha+\beta| \leq j-1} a_{\alpha,\beta}(w,\zeta) \widetilde{\frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta}} h(w) \zeta^\alpha \bar{\zeta}^\beta$$

for some functions $a_{\alpha,\beta}(w, \zeta)$ which are bounded for $(w, \zeta) \in \delta B \times S$ and such that $a_{\alpha,\beta}(w, .)$ is a polynomial in ζ and $\bar{\zeta}$ whose degree is at most $j - 1 - |\alpha + \beta|$. This fact combined with (2.4) and (2.5) implies that for some positive constant $C = C(j, \delta)$ independent of f and w we have

$$\Big| |(\widetilde{D}^j h)(w)| - |(D^j h)(w)| \Big| \leq C \sum_{l=1}^{j-1} |D^l h)(w)|, \quad \text{for all } w \in \delta B. \tag{2.12}$$

Now an induction process invoking (2.12) implies that some $C > 0$ independent of h we have

$$C(Q_j h)(w) \leq \sum_{l=j}^{j} |(D^l h)(w)| \leq \frac{1}{C}(Q_j h)(w), \quad \text{for all} w \in \delta B. \tag{2.13}$$

Putting together (2.10), (2.11) and (2.13), we obtain that

$$(2.14)\qquad |D^j h)(0)| \le C\frac{1}{\varepsilon^{2n}}\int_{\varepsilon B}(Q_j h)(w)\,d\tau(w),\quad \text{for all}\ \ 0<\varepsilon<\delta$$

so tha by the invariant property of $Q_j h$ and τ we obtain

$$(2.15)\qquad |\widetilde{D}^j h)(z)| \le C\frac{1}{\varepsilon^{2n}}\int_{E(z,\varepsilon)}(Q_j h)(w)\tau(w),\quad \text{for all}\ \ 0<\varepsilon<\delta.$$

The lemma is now follows from (2.15). □

For an M-harmonic function h let us call (h,ω,κ,p)-regular points those elements $z\in B$ for which there is a positive constant C such that

$$\sup_{\zeta\in E(z,r)} (r-|\varphi_\zeta(z)|)^{2n}(Q_\kappa h)^p(\zeta) \le C\omega(r),\quad \text{for all}\ 0<r<\delta.$$

Let $\mathcal{S} = \mathcal{S}(h,\omega,\kappa,p)$ be the set of all points in B which are not (h,ω,κ,p)-regular.

LEMMA 2.3. *For each $0<\delta<1/2$ there is a positive integer $N = N(\delta)$ such that for any $\Omega\subset B$ and any covering $\mathcal{B} = \{E(z,r(z))\}_{z\in\Omega}$ by pseudohyperbolic balls with radii $r(z)<\delta$, there exist N subfamilies $\mathcal{B}_1,\cdots,\mathcal{B}_N$ of $\mathcal{B}$ such that each $\mathcal{B}_j$ consists of pairwise disjoint pseudohyperbolic balls and Ω is covered by $\cup_{j=1}^N\mathcal{B}_j$.*

Proof. Follows from the fact that (B,ϱ) is directionnally limitted in the sense of Federer. See ([2], p.150) or ([5], p. 89). □

LEMMA 2.4. *Let $p\ge 1$ and h be M-harmonic function on B. Let (ω,K,δ) be a triple consisting of a gauge function ω and two positive numbers K,δ. Then there is an open set $\Omega = \Omega(h,\omega,s,K,p,\delta)\subset B$ with the following properties.*

1. *Ω contains $\mathcal{S}$.*
2. *There is a positive integer $N = N(\delta)$ which does not depend on K and there is a sequence of pseudohyperbolic balls $\{E(z_j,r_j)\}_1^\infty$ with $r_j<\delta$ with $z_j\in\Omega$ and points $\zeta_j\in E(z_j,r_j)$ such that*

$$(2.16)\qquad (r_j-|\varphi_{\zeta_j}(z_j)|)^{2n}(\widehat{Q}_\kappa h)^p(\zeta_j) > K\omega(r_j).$$

and each point of Ω is in at most N of these balls.

Proof. Let D be a dense sequence in $\mathcal{S} \times [0, \delta)$ and let O_K the set of all pairs (z, r) in $B \times [0, \delta)$ such that

$$\sup_{\zeta \in E(z,r)} (r - |\varphi_\zeta(z)|)^{2n} (\widehat{Q}_\kappa h)^p(\zeta) > K\omega(r), \quad \text{for some } 0 < r < \delta.$$

We set $\Omega := P(O_K)$ where $P_1(z,t) = z$ is the first coordinate projection from $\mathbb{C}^n \times \mathbb{R}$ onto $\mathbb{C}^n$. Then Ω is open and $\mathcal{S}$ is clearly contained in Ω. If $(z, r) \in O_K$ then there exists a pair $(w, t) \in D \cap O_K$ and $|\varphi_z(\zeta)| < t$. This shows that the sequence $\mathcal{B}^K$ consisting of those balls $E(w,t)$ for which $(w,t) \in D_K := D \cap O_K$ forms a covering of Ω. Appealing to Lemma 2.3, we can find subsequences $\mathcal{B}_1^K, \cdots, \mathcal{B}_N^K$ of $\mathcal{B}^K$ such that each $\mathcal{B}_j^K$ consists of pairwise disjoint pseudohyperbolic balls and Ω is covered by $\cup_{j=1}^N \mathcal{B}_j^K$. This completes the proof of the lemma. □

LEMMA 2.5. *Under the hypothesis of Lemma 2.4 if, in addition, h satisfies*

$$\int_B (Q_\kappa h)^p(z)\, d\nu_s(z) < \infty \quad \textit{for some } s \le 1,$$

then for each $\varepsilon > 0$ there exist an open subset $\Omega \subseteq B$ that contains $\mathcal{S}$ and a positive constant C such that $\widehat{\mathcal{H}}_{\omega,s}(\Omega) < \varepsilon$ and

$$\sup_{\zeta \in E(z,\rho)} (\rho - |\varphi_\zeta(z)|)^{2n} (Q_\kappa h)^p(\zeta) \le C\omega(\rho), \quad \textit{for all } 0 < \rho < \delta. \tag{2.17}$$

In particular, $\mathcal{S}$ is zero set for the Hausdorff measure $\mathcal{H}_{\omega,s}$.

Proof. Let $K, \delta > 0$ and let Ω and $E(z_j, r_j)$, be as in Lemma 2.4. Then the characteristic functions $\chi_{E(z_j,r_j)}$ satisfy

$$\sum_1^\infty \chi_{E(z_j,r_j)} \le N < \infty. \tag{2.18}$$

Since for $0 < r < 1$

$$\frac{1-r}{1+r}(1 - |z|^2) \le 1 - |w|^2 \le \frac{1+r}{1-r}(1 - |z|^2) \;\; \text{for } w \in E(z,r),$$

by virtue of Lemma 2.2 we see that

$$\begin{aligned}
&\sum_1^\infty (1-|z_j|^2)^{n+1-s}(r_j - |\varphi_{\zeta_j}(z_j)|)^{2n}(Q_\kappa h)^p(\zeta_j)\\
&\le C\sum_1^\infty (1-|z_j|^2)^{n+1-s}\int_{E(\zeta_j, r_j - |\varphi_{\zeta_j}(z_j)|)} (Q_\kappa h)^p(w)\, d\tau(w)\\
&\le C\sum_1^\infty \left(\frac{1+r_j}{1-r_j}\right)^{n+1-s}\int_{E(z_j,r_j)} \frac{(Q_\kappa h)^p(w)}{(1-|w|^2)^s}\, d\nu(w)\\
&\le CN\left(\frac{2+\delta}{2-\delta}\right)^{n+1-s}\int_B \frac{(Q_\kappa h)^p(w)}{(1-|w|^2)^s}\, d\nu(w).
\end{aligned}$$

This combined with (2.16) and (2.6) yields

$$\begin{aligned}
\widehat{\mathcal{H}}_{\omega,s}(\Omega) &\le \frac{\Gamma(n+1-s)}{n!\Gamma(1-s)}\sum_1^\infty (1-|z_j|^2)^{n+1-s}\omega(r_j)\\
&\le C\frac{N}{K}\left(\frac{2+\delta}{2-\delta}\right)^{n+1-s}\frac{\Gamma(n+1-s)}{n!\Gamma(1-s)}\int_B \frac{(Q_\kappa h)^p(w)}{(1-|w|^2)^s}\, d\nu(w)\\
&= C\frac{N}{K}\left(\frac{2+\delta}{2-\delta}\right)^{n+1-s}\int_B (Q_\kappa h)^p(w)\, d\nu_s(w) \to 0, \text{ as } K\to\infty.
\end{aligned}$$

□

3. Proof of the main result

For $z \in B$ define the functions

$$h^{u\overline{v}}(z) := \widetilde{\frac{\partial^{|u+v|}}{\partial z^u \partial \bar{z}^v}} h(z), \quad u, v \in \mathbb{N}_0^n. \tag{3.1}$$

LEMMA 3.1. *For $\kappa \in \mathbb{N}$ and $0 < \delta < 1$ there exist a positive constant $C = C(\kappa, \delta)$ such that for each $\zeta \in B$ we have*

$$\begin{aligned}
&\left| h^{u\overline{v}}(z) - \sum_{|l+m|<[\kappa]-|u+v|} \frac{h^{l\overline{m}}(\zeta)}{(l+m)!}(\varphi_\zeta(z))^l(\overline{\varphi_\zeta(z)})^m \right|\\
&\qquad \le C\frac{|\varphi_\zeta(z)|^{\kappa-|u+v|}}{(r-|\varphi_\zeta(z)|)^{2n}}\int_{E(z,r)} (Q_\kappa h)(w)\, d\tau(w)
\end{aligned}$$

for all $z \in E(\zeta, r)$ with $0 < r < \delta$ and M-harmonic functions h on B.

Proof. First we assume that $\zeta = 0$. This is no loss of generality due to invariant property of Q_κ and τ. Now Taylor's formula we see that for some positive constant C

$$\left| h^{u\overline{v}}(z) - \sum_{|l+m|<[\kappa]-|u+v|} \frac{h^{l\overline{m}}(0)}{(l+m)!} z^l \overline{z}^m \right| \leq C|z|^{\kappa-|u+v|} \int_0^1 (1-t)^{\kappa-1} (Q_\kappa h)(tz)\, dt$$

for $|z| \leq \delta$. This fact, combined with Lemma 2.2 with $\varepsilon = r - |z|$ yields

$$\left| h^{u\overline{v}}(z) - \sum_{|l+m|<[\kappa]-|u+v|} \frac{h^{l\overline{m}}(0)}{(l+m)!} z^l \overline{z}^m \right| \leq C \frac{|z|^{\kappa-|u+v|}}{(r-|z|)^{2n}} \int_0^1 \int_{E(tz,r-|z|)} (Q_\kappa h)(w)\, d\tau(w)\, dt, \quad z \in B.$$

But for $w \in E(tz, r-|z|)$ and $|z| < r$ we have

$$|\varphi_z(w)| \leq |\varphi_{tz}(z)| + |\varphi_{tz}(w)| < \frac{(1-t)|z|}{1-t|z|^2} + r - |z| \leq r.$$

Thus $E(tz, r-|z|) \subseteq E(z,r)$. From this the lemma now follows. □

Proof of Theorem A. Let the parameters h, ω, s, κ, p be as the hypothesis of Theorem A. For $\varepsilon > 0$ let Ω be the open set constructed in Lemma 2.5. Fix $0 < r < \rho < \delta < 1/2$. If $z \in B \setminus \Omega$, then by virtue of 2.17 we have

$$(Q_\kappa h)^p(w) \leq C \frac{\omega(\rho)}{(\rho - r)^{2n}}, \quad \text{for } w \in E(z,r),$$

from which it follows that

$$\int_{E(z,r)} (Q_\kappa h)(w)\, d\tau(w) \leq \tau(rB) \left\{ C \frac{\omega(\rho)}{(\rho-r)^{2n}} \right\}^{\frac{1}{p}}. \tag{3.2}$$

For $\zeta \in B$ with $|\varphi_z(\zeta)| < \frac{r}{2}$ then applying (3.2) and Lemma 3.1 shows that for some constant $C > 0$ depending on r, ρ and δ we have

$$\begin{aligned}\left| h^{u\overline{v}}(z) - \sum_{|l+m|<[\kappa]-|u+v|} \frac{h^{l\overline{m}}(\zeta)}{(l+m)!}(\varphi_\zeta(z))^l(\overline{\varphi_\zeta(z)})^m \right| \\ \leq C\frac{|\varphi_\zeta(z)|^{\kappa-|u+v|}}{(r-|\varphi_\zeta(z)|)^{2n}} \\ \leq C\frac{2^{2n}}{r^{2n}}|\varphi_\zeta(z)|^{\kappa-|u+v|}.\end{aligned}$$

□

Proof of Theorem B. By Lemma 2.1, (1.7) and (2.5) we see that the hypothesis of Theorem B implies $|||f|||_{p,s} < \infty$. Theorem B now follows from Theorem A. □

Proof of Corollary C. Part (1) is a consequence of Theorem B, Remark 1.1 and the fact that $\|h\|_{H_p} = \|h\|_{p,1}$. Part (2) follows from Theorem A, Remark 1.1 and the fact that if $p > 1$, then $\|h\|_{\frac{p}{s},0} \leq C_s\|h\|_p$, for $1 < s < p$, where

$$C_s = \left\{ \int_B \left(\log \frac{1}{1-|z|^2} \right)^{\frac{s}{s-1}} d\nu(z) \right\}^{\frac{s-1}{p}} < \infty.$$

□

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