J. Korean Math. Soc. **34** (1997), No. 4, pp. 949–957

EXACTNESS THEOREM AND POOR M-COSEQUENCES

K. KHASHYARMANESH AND SH. SALARIAN

ABSTRACT. The purpose of this paper is to establish connection between certain complex of modules of generalized fractions and the concept of cosequence in commutative algebra. The main theorem of the paper leads to characterization, in terms of modules of generalized fractions, of regular (co) sequences.

1. Introduction

The construction, for a module M over a commutative ring R (with identity) and a multiplicatively closed subset S of R, of the module of fractions $S^{-1}M$ is, of course, one of the most basic ideas in commutative algebra. In [**8**], Sharp and Zakeri, for a triangular subset U of R^n, give a procedure for constracting so-called modules of generalized fractions $U^{-n}M$ which generalize the usual theory of localization of modules. In subsequent paper [**9**] they have shown, under Noetherian hypothesis on R, that there is a connection between modules of generalized fractions and the concept of regular sequences in commutative algebra. Although they first proved this result, a shorter proof, which applies also in the case in which the underlying commutative ring is not necessarily Noetherian, was later provided by O'Carroll [**7**].

In [**6**], Melkersson and Schenzel defined the co-localization $\mathrm{Hom}_R(S^{-1}R, M)$ of an R-module M with respect to a multiplicatively closed subset S of R. Hence, for a triangular subset U of R^n, the R-module $\mathrm{Hom}_R(U^{-n}R, M)$ is a natural extention of the difinition of co-localization. In this paper we establish connection between the R-modules $\mathrm{Hom}_R(U^{-n}R, M)$ $(n \geq 1)$ and the concept of cosequences which is a generalization

Received June 7, 1997.
1991 Mathematics Subject Classification: 13B30, 13B10.
Key words and phrases: poor M-sequences, poor M-cosequences, generalized fractions.

of 'only if' half of [**10**, 3.3]. Next we show that the main result of [**9, 7**] is then deduced quickly.

2. Preliminaries

Throughout this paper, R is a commutative ring with identity and M an R-module. We use ^{T} to denote matrix transpose and $D_n(R)$ $(n \geq 1)$ to denote the set of $n \times n$ lower triangular matrices over R. For $H \in D_n(R)$, $|H|$ denotes the determinant of H. Let $(a_1, \dots, a_i)R$ be the ideal of R generated by $\{a_1, \dots, a_i\}$ and $(a_1, \dots, a_i)M$ the submodule of M generated by $\{a_j m : j = 1, \dots, i \text{ and } m \in M\}$. We use $\mathbb{N}$ to denote the set of positive integers.

Let $x_1, \dots, x_n$ be a sequence of elements of R and M an R-module. Then $x_1, \dots, x_n$ is said to be a poor M-sequence if multiplication by x_i on $M/(x_1, \dots, x_{i-1})M$ is a monomorphism for all $i = 1, \dots, n$ (where $x_0 = 0$). If, in addition, $M/(x_1, \dots, x_n)M \neq 0$, we call $x_1, \dots, x_n$ an M-sequence.

If $\mathfrak{b}$ is an ideal of R, we set $\mathrm{Ann}_M \mathfrak{b} = \{m \in M : \mathfrak{b}m = 0\}$. We have a dual definition; $x_1, \dots, x_n$ is said to be a poor M-cosequence if multiplicative by x_i on $\mathrm{Ann}_M(x_1, \dots x_{i-1})R$ is an epimorphism for all $i = 1, \dots, n$ (where $x_0 = 0$). Similarly, if $\mathrm{Ann}_M(x_, \dots, x_n)R \neq 0$, $x_1, \dots, x_n$ is called an M-cosequence (see [**4**]).

Let E be an injective envelope of the direct sum of all of the simple R-modules, and define the functor * by $^* =$Hom(.,E), then * is a faithfully exact contravariant functor; that is, a sequence of R-modules is exact if and only if its * is exact. Recall that a module M is Matlis reflexive if $M \cong (M^*)^*$.

Now we gather together the well known properties of M-sequences and M-cosequences which are needed in this paper.

REMARK 2.1. ([**5**, 1, 5 and 6] and [**2**, 1.2]). (1) $x_1, \dots, x_n$ is a poor M-sequence if and only if $x_1, \dots, x_n$ is a poor M^*- cosequence.

(2) $x_1, \dots, x_n$ is a poor M-sequence if and only if $x_1, \dots, x_n$ is a poor M^*-cosequence.

(3) $x_1, \dots, x_n$ is a poor M-sequence if and only if $x_1^{\alpha_1}, \cdots, x_n^{\alpha_n}$ is a poor M-sequence for any positive integers $\alpha_1, \dots, \alpha_n$.

(4) $x_1, \dots, x_n$ is a poor M-cosequence if and only if $x_1^{\alpha_1}, \cdots, x_n^{\alpha_n}$ is a poor M-cosequence for any positive integers $\alpha_1, \dots, \alpha_n$.

As mentioned in the introduction, this paper is connected with the concept of modules of generalized fractions. The reader is referred to [**8**, **9**] for details of the following brief résumé of the theory of modules of generalized fractions.

A non-empty subset U of R^n is called triangular if
(i) Given $(u_1, \dots, u_n) \in U$, $(u_1^{\alpha_1}, \dots, u_n^{\alpha_n}) \in U$ for all $\alpha_i \in \mathbb{N}$, $1 \leq i \leq n$;
(ii) Given $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ in U, there exist $(w_1, \dots, w_n) \in U$ and
$H, K \in D_n(R)$ such that $H[u_1 \dots u_n]^T = [w_1 \dots w_n]^T = K[v_1 \dots v_n]^T$.

Whenever we can do so without ambiguity, we shall denote $(u_1, \dots, u_n) \in R^n$ by u, and $[u_1 \dots u_n]^T$ by u^T.

Given such a triangular subset U of R^n, we can form the module of generalized fractions $U^{-n}M = \{a/u : a \in M, u \in U\}$, where a/u denotes the equivalence class of the pair $(a, u) \in M \times U$ under the following equivalence relation $\sim$ on $M \times U$: $(c, x) \sim (d, y)$ precisely when there exist $z \in U$ and $P, Q \in D_n(R)$ such that $Px^T = z^T = Qy^T$, with $|P|c - |Q|d \in (z_1, \dots, z_{n-1})M$.

Now $U^{-n}M$ is an R-module under the operations

$$a/u + b/v = (|H|a + |K|b)/w,$$
$$r.(a/u) = (ra)/u$$

for $r \in R$, $a, b \in M$, $u, v \in U$, and any choice of $H, K \in D_n(R)$ and $w \in U$ such that $Hu^T = w^T = Kv^T$.

We shall need the following basic properties of generalized fractions.

PROPOSITION 2.2. [**8,9**]. *Let $m \in M$ and $u = (u_1, \dots, u_n) \in U$. Then*

(i) *$m/u = |H|m/v$ for any choice of $H \in D_n(R)$ and $v \in U$ such that $Hu^T = v^T$;*
(ii) *for $i = 1, \dots, n-1$, $u_i m/u = 0$; and*
(iii) *$U^{-n}R \otimes_R M \cong U^{-n}M$ under the natural map.*

A family $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ is called a chain of triangular subsets on R if the following conditions are satisfied:

(i) *U_n is a triangular subset of R^n for all $n \in \mathbb{N}$;*
(ii) *$(1) \in U_1$;*
(iii) *whenever $(u_1, \dots, u_n) \in U_n$ with $n \in \mathbb{N}$, then $(u_1, \dots, u_n, 1) \in U_{n+1}$; and*
(iv) *whenever $(u_1, \dots, u_n) \in U_n$ with $1 < n \in \mathbb{N}$, then $(u_1, \dots, u_{n-1}) \in U_{n-1}$.*

Each U_n leads to a module of generalized fractions $U_n^{-n}M$, and we can, in fact, arrange these modules into a complex

$$0 \xrightarrow{e^{-1}} M \xrightarrow{e^{0}} U_1^{-1}M \longrightarrow \dots \longrightarrow U_n^{-n}M \xrightarrow{e^{n}} U_{n+1}^{-n-1}M \longrightarrow \dots,$$

denoted by $C(\mathcal{U}, M)$, for which $e^0(m) = m/(1)$ for all $m \in M$ and

$$e^n(a/(u_1, \dots, u_n)) = a/(u_1, \dots, u_n, 1)$$

for all $n \in \mathbb{N}$, $a \in M$ and $(u_1, \dots, u_n) \in U_n$. Also we shall write the induced complex $Hom(C(\mathcal{U}, R), M)$ by

$$\dots \longrightarrow Hom_R(U_{n+1}^{-n-1}R, M) \xrightarrow{e_n} Hom_R(U_n^{-n}R, M) \longrightarrow \dots \longrightarrow Hom_R(U_1^{-1}R, M) \xrightarrow{e_0} M \xrightarrow{e_{-1}} 0,$$

where $e_n = Hom(e^n, M)$ for all $n \geq 0$.

3. Main results

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a chain of triangular subsets on R. Let $\mathcal{T}$ be the set of all sequences $x = \{x_i : i \in \mathbb{N}\}$ of elements of R such that (i) there exists $i_0 \geq 1$ such that $x_i = 1$ for all $i \geq i_0$, and (ii) $(x_1, \dots, x_n) \in U_n$ for all (sufficiently large) $n \geq 1$. (As in §1 we use obvious extensions of this notation, in particular denoting the infinite vector $[x_1\ x_2 \dots]^T$ by x^T.) Define a relation $\leq$ on $\mathcal{T}$ as follows: for x and y in $\mathcal{T}$, $x \leq y$ precisely when $Hx^T = y^T$ for some $H \in D_\infty(R)$ (throughout, we use $D_\infty(R)$ to denote the set of all infinite lower triangular matrices over R). Clearly $x \leq y$ if and only if $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ in U_n for all (sufficiently large) n. It is immediate therefore that $(\mathcal{T}, \leq)$ forms a directed set under the quasi-order $\leq$.

Suppose that $x = \{x_i : i \in \mathbb{N}\} \in \mathcal{T}$ and $\mathcal{B}(x, M)$ be the complex

$$\dots \longrightarrow \operatorname*{Ann}_M(x_1, \dots, x_n)R \xrightarrow{d_n^x} \operatorname*{Ann}_M(x_1, \dots, x_{n-1})R \longrightarrow \dots \longrightarrow \operatorname*{Ann}_M(x_1)R \xrightarrow{d_1^x} M \xrightarrow{d_0^x} M \xrightarrow{d_{-1}^x} 0$$

of R-modules and R-homomorphisms, where $d_0^x(m) = x_1 m$ for all $m \in M$ and, for $n \in \mathbb{N}$, $d_n^x(m) = x_{n+1}m$ for all $m \in M$. Let $x, y \in \mathcal{T}$ with $Hx^T = y^T$ for some $H \in D_\infty(R)$. For each $n \in \mathbb{N}$, let H_n be the $n \times n$

submatrix of H in the top left corner. Then, in view of [**8**, 2.2], the multiplication by $|H_n|$ provides an R-homomorphism

$$\delta_H^n : \operatorname*{Ann}_M(y_1, \ldots, y_{n-1})R \to \operatorname*{Ann}_M(x_1, \ldots, x_{n-1})R.$$

Hence there is induced a morphism of complexes

$$\delta_H : \mathcal{B}(y, M) \to \mathcal{B}(x, M),$$

which in n-th place restricts to the R-homomorphism δ_H^n. (For $n = 0$, δ_H^0 is the identity map.) Under these morphisms the complexes $\mathcal{B}(x, M)$, for $x \in \mathcal{T}$, form a directed set.

Let $x, y \in \mathcal{T}$ and suppose that $Hx^T = Kx^T = y^T$ for some $H, K \in D_\infty(R)$. Then, in view of [**10**, 3.1], there exist $z \in \mathcal{T}$ and $D \in D_\infty(R)$ such that $\delta_H^n \delta_D^n = \delta_K^n \delta_D^n$ for all $n \geq 0$. It follows that the above directed system has the usual properties of standard inverse limit systems where there is only one morphism between the comparable objects.

We have the following analogue of [**10**, 3.4].

PROPOSITION 3.1. $\varprojlim_{x \in \mathcal{T}} \mathcal{B}(x, M) \cong \mathit{Hom}(C(\mathcal{U}, R), M)$.

We now come to the main theorem in this paper.

THEOREM 3.2. *Let $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ be a chain of triangular subsets on R. Assume that $Hom(C(\mathcal{U}, R), M)$ is exact. Then, for each $i \in \mathbb{N}$, every element of U_i is a poor M-cosequence.*

Proof. Let $\mathcal{T}$ be as above. It follows from the hypothesis and 2.1 that the complex $\varprojlim_{x \in \mathcal{T}} \mathcal{B}(x, M)$ is exact. We use the notation

$$\ldots \longrightarrow M_{n+1} \xrightarrow{d_n} M_n \longrightarrow \ldots \longrightarrow M_2 \xrightarrow{d_1} M_1 \xrightarrow{d_0} M \xrightarrow{d_{-1}} 0$$

for the complex $\varprojlim_{x \in \mathcal{T}} \mathcal{B}(x, M)$.

Let $m \in M$. Then there exists $m_0 = \{m_{0,x}\}_{x \in \mathcal{T}} \in M_1$ such that $d_0(m_0) = m$. Hence $x_1 m_{0,x} = m$ for all $x = \{x_i : i \in \mathbb{N}\} \in \mathcal{T}$. Therefore $M = x_1 M$ for all $(x_1) \in U_1$ and so each element of U_1 is a poor M-cosequence.

Let $n \geq 2$, and suppose, inductively, that U_{n-1} consists of poor M-cosequences. Let $(x_1, \ldots, x_n) \in U_n$. We have to show that $x_1, \ldots, x_n$ is a poor M-cosequence. It suffices to show that the multiplication by x_n is surjective on $\operatorname*{Ann}_M(x_1, \ldots, x_{n-1})R$, since $(x_1, \ldots, x_{n-1}) \in U_{n-1}$ and so is a poor M-cosequence. Let $m \in \operatorname{Ann}_M(x_1, \ldots, x_{n-1})R$ and let $x = \{x_i : i \in \mathbb{N}\}$, where x_r is interpreted as 1 whenever $r > n$. It is easy to see that the inductive step will be completed if, for each $1 \leq p \leq n$, we show that, for all $i_1, \ldots, i_p \in \mathbb{N}$ with $1 \leq i_1 < \cdots < i_p < n$, there exists

$$m_{i_1,\ldots,i_p} = \{m_{i_1,\ldots,i_p,y}\}_{y \in \mathcal{T}} \in M_{p+1}$$

such that

(i) $d_p(m_{i_1,\ldots,i_p}) = \sum_{r=1}^{p}(-1)^{r-1}x_{i_r}m_{i_1,\ldots,\hat{i}_r,\ldots,i_p}$, where the character with $\hat{}$ means that it is deleted; and

(ii) $(-1)^{1+\sum_{r=1}^{p+1} r} x_{p+1}m_{1,\ldots,p,x} = m.$ □

To achieve this, we use induction on p. For the case in which $p = 1$, there exists $m_0 = \{m_{0,y}\}_{y\in\mathcal{T}} \in M_1$ such that $d_0(m_0) = m$. Since $m \in \operatorname{Ann}_M(x_1, \ldots, x_{n-1})R$, we have, for each $i = 1, \ldots, n-1$, that $x_i m_0 \in \ker d_0$. Hence, for each $i = 1, \ldots, n-1$, there exists $m_i = \{m_{i,y}\}_{y\in\mathcal{T}} \in M_2$ such that $d_1(m_i) = x_i m_0$. Therefore $x_2 m_{1,x} = d_1^x(m_{1,x}) = x_1 m_{0,x} = d_0^x(m_{0,x}) = m$. Hence points (i) and (ii) have been verified.

Now suppose, inductively, that $1 < p \leq n$ and the result has been proved for smaller values of p. Let $i_1, \ldots, i_p \in \mathbb{N}$ with $1 \leq i_1 < \cdots < i_p < n$. It is immediately follows from this inductive hypothesis that

$$d_{p-1}\Big(\sum_{r=1}^{p}(-1)^{r-1}x_{i_r}m_{i_1,\ldots,\hat{i}_r,\ldots,i_p}\Big) = \sum_{r=1}^{p}$$

$$(-1)^{r-1}x_{i_r}\Big(\sum_{l=1,l\neq r}^{p}(-1)^k x_{i_l}m_{i_1,\ldots,\hat{i}_r,\ldots,\hat{i}_l,\ldots,i_p}\Big),$$

where

$$k = \begin{cases} l, & if\ l > r, \\ l-1, & if\ l < r, \end{cases}$$

which is zero. Hence there exists $m_{i_1,\ldots,i_p} = \{m_{i_1,\ldots,i_p,y}\}_{y\in\mathcal{T}} \in M_{p+1}$ such that

$$d_p(m_{i_1,\ldots,i_p}) = \sum_{r=1}^{p}(-1)^{r-1}x_{i_r}m_{i_1,\ldots,\hat{i}_r,\ldots,i_p},$$

for all $i_1, \dots, i_p \in \mathbb{N}$ with $1 \le i_1 < \cdots < i_p < n$.
Also, since $m_{1,\dots,\hat{r},\dots,p,x} \in \operatorname{Ann}_M(x_1, \dots, x_{p-1})R$, for all $r = 1, \dots, p-1$, we have that

$$\begin{aligned}(-1)^{1+\sum_{r=1}^{p-1} r} x_{p+1} m_{1,\dots,p,x} &= (-1)^{1+\sum_{r=1}^{p+1} r} d_p^x(m_{1,\dots,p,x}) \\ &= (-1)^{1+\sum_{r=1}^{p+1} r} \quad (-1)^{p-1} x_p m_{1,\dots,p-1,x} \\ &= m.\end{aligned}$$

Hence points (i) and (ii) have been verified, and we are therefore able to complete the inductive step, and the proof.

Theorem 2.2 has some consequences which we record here.

CONSEQUENCES 3.3. Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a chain of triangular subsets on R. Then

(1) ([**9**, 3.3] and [**7**, 3.1]) $C(\mathcal{U}, M)$ is exact if and only if, for all $i \in \mathbb{N}$, each element of U_i is a poor M-sequence.

(2) ([**10**, 3.3]) Let M be an Artinian R-module. Then, $\operatorname{Hom}(C(\mathcal{U}, R), M)$ is exact if and only if, for all $n \in \mathbb{N}$, each element of U_n is a poor M-cosequence.

(3) Assume that, for all $n \in \mathbb{N}$, U_n is countable set (and so, in particular, if

$$U_n = \{(x_1^{\alpha_1}, \dots, x_n^{\alpha_n}) : \text{ there exists } j \text{ with } 0 \le j \le n \text{ such that } \alpha_1, \dots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \cdots = \alpha_n = 1\},$$

for some sequence $\{x_i : i \in \mathbb{N}\}$ of elements of A). Then $\operatorname{Hom}(C(\mathcal{U}, R), M)$ is exact if and only if, for all $n \in \mathbb{N}$, each element of U_n is a poor M-cosequence.

(4) If M is Matlis reflexive, then $\operatorname{Hom}(C(\mathcal{U}, R), M)$ is exact if and only if, for all $n \in \mathbb{N}$, each element of U_n is a poor M-cosequence.

Proof. (1) The 'if' part is clear. Hence we shall prove the 'only if' half. Since the functor * is exact as we have remarked earlier, the complex $C(\mathcal{U}, M)^*$ is also exact. Note that, for all $n \in \mathbb{N}$, $(U_n^{-n}R \otimes_R M)^* \cong \operatorname{Hom}_R(U_n^{-n}R, M^*)$ under the natural map. Hence, by 1.2(iii), the complex $\operatorname{Hom}(C(\mathcal{U}, R), M^*)$ is exact. Now the claim immediatley follows from 2.2 and 1.1(1).

(2) By 2.2, we only need to prove the 'if' part. We use the notation established before Proposition 2.1. Consider the corresponding set $\mathcal{T}$ of sequence $\{x_i, : i \in \mathbb{N}\}$. Since each element of U_n is a poor M-cosequence,

for all $n \in \mathbb{N}$, it is easy to see that the complex $B(x, M)$ is exact for all $x \in \mathcal{T}$. Also we can deduced from [**2**, 2.2], for $x, y \in \mathcal{T}$ with $x \leq y$, that the morphism $\delta_H : \mathcal{B}(y, M) \to \mathcal{B}(x, M)$ is surjective for all $H \in D_\infty(R)$ such that $Hx^T = y^T$. The claim now follows from [**3**, p.391] and 2.1.
(3) This holds, because, by [**1**, 10.2] insteed of Lemma 1 of [**3**, p.391], the same arguments in (2) still work for any R-module M.
(4) By 2.2, we only need to prove the 'if' part. By 1.1(2), each element of U_n is a poor M^*-sequence. Hence by (1) the complex $C(\mathcal{U}, M^*)$ is exact. Therefore the complex $C(\mathcal{U}, M^*)^*$ is exact. Note that, for all $n \in \mathbb{N}$, $(U_n^{-n}R \otimes_R M^*)^* \cong \mathrm{Hom}_R(U_n^{-n}R, M^{**})$ under the natural map. Thus the complex $\mathrm{Hom}(C(\mathcal{U}, R), M)$ is exact, since M is Matlis reflexive. □

ACKNOWLEDGMENT. We would like to thank Dr. H. Zakeri for helpful discussions, and also Institute for Studies in Theoretical Physics and Mathematics for the Financial support.

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative algebra* (Addison Wesley, 1969).

[2] S. C. Chung, *Epimorphisms of annihilators of poor M-cosequences*, Bull. Korean Math. Soc. **32** (1995), 359-365.

[3] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323-448.

[4] E. Matlis, *Modules with descending chain conditions*, Trans, AMS **97** (1960), 495-508.

[5] E. Matlis, *The higher properties of R-sequences*, J. Alg. **50** (1978), 77-112.

[6] L. Melkersson and P. Schenzel, *The Co-localization of an Artinian module*, Proc. Edinburgh Math. Soc. **38** (1995), 121-131.

[7] L. O'Carroll, *On the generalized fractions of Sharp and Zakeri*, J. London Math. Soc. **28** (1983), 417-427.

[8] R. Y. Sharp and H. Zakeri, *Modules of generalized fractions*, Mathematika **29** (1982), 32-41.

[9] R. Y. Sharp and H. Zakeri, *Modules of generalized fractions and balanced big Cohen-Macaulay modules*, (in commutative algebra: Durham 1981, London Mathematical Society Lecture Notes **72**), Cambridge University Press, 1982, 61-82.

[10] Z. Tang and H. Zakeri, *Co-Cohen-Macaulay modules and modules of generalized fractions*, Communications in Algebra **22** (1994), 2173-2204.

K. Khashyarmanesh
School of Sciences
Tarbiat Modarres University
P.O.Box 14155-4838 Tehran, Iran

Institute for Studies in Theoretical Physics and Mathematics
P.O.Box 14155-4838
Tehran, Iran
E-mail: khashyar@rose.ipm.ac.ir