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GAUSS SUMS FOR $\mathbf{U(2n+1, q^2)}$

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ABSTRACT. For a lifted nontrivial additive character λ' and a multiplicative character χ of the finite field with q^2 elements, the 'Gauss' sums $\sum \lambda'(\operatorname{tr} w)$ over $w \in SU(2n+1, q^2)$ and $\sum \chi(\det w)\lambda'(\operatorname{tr} w)$ over $w \in U(2n+1, q^2)$ are considered. We show that the first sum is a polynomial in q with coefficients involving certain new exponential sums and that the second one is a polynomial in q with coefficients involving powers of the usual twisted Kloosterman sums and the average (over all multiplicative characters of order dividing $q-1$) of the usual Gauss sums. As a consequence we can determine certain 'generalized Kloosterman sum over nonsingular Hermitian matrices' which were previously determined by J. H. Hodges only in the case that one of the two arguments is zero.

1. Introduction

Let λ' be the lifting of a nontrivial additive character λ of $\mathbb{F}_q$ to $\mathbb{F}_{q^2}$ (cf. (2.4)), and let χ be a multiplicative character of $\mathbb{F}_{q^2}$. Then we consider the exponential sum

$$\sum_{w \in SU(2n+1, q^2)} \lambda'(\operatorname{tr} w), \tag{1.1}$$

where $SU(2n+1, q^2)$ is a special unitary group over $\mathbb{F}_{q^2}$ (cf. (2.9)) and tr w is the trace of w. Also, we consider

$$\sum_{w \in U(2n+1, q^2)} \chi(\det w)\lambda'(\operatorname{tr} w), \tag{1.2}$$

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where $U(2n+1, q^2)$ is a unitary group over $\mathbb{F}_{q^2}$ (cf. (2.5) and (2.7)) and det w is the determinant of w.

The main purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). We will show that (1.1) is a polynomial in q with coefficients involving certain exponential sums (cf. (5.1) and (5.2)), which seem to be new. On the other hand, (1.2) is a polynomial in q with coefficients involving powers of the usual twisted Kloosterman sums and the average (over all multiplicative characters of order dividing $q-1$) of the usual Gauss sums.

In [2], Hodges expressed certain exponential sums in terms of what we call the 'generalized Kloosterman sum over nonsingular Hermitian matrices' $K_{Herm,t}(A, B)$, where A, B are $t \times t$ Hermitian matrices over $\mathbb{F}_{q^2}$ (cf. (8.1)). Some of its general properties were investigated in [2], and, for A or B zero, it was evaluated in [1]. However, they have never been explicitly computed for both A and B nonzero. From a corollary to the main theorem in [2] and Theorem 7.2, we will be able to find an explicit expression for $K_{Herm,2n+1}(a^2C^{-1}, C)$, where C is a nonsingular Hermitian matrix over $\mathbb{F}_{q^2}$ of size $2n+1$ and $a \in \mathbb{F}_q^\times$. On the other hand, $K_{Herm,2n}(a^2C^{-1}, C)$ was obtained in [6], where C is a nonsingular Hermitian matrix over $\mathbb{F}_{q^2}$ of size $2n$ and a is as before.

Similar sums for other classical groups over a finite field have been considered and the results for these sums will appear in various places ([3] - [7]).

Finally, we would like to state the main results of this paper. For some notation, one is referred to the next section.

THEOREM A. *The sum* $\sum_{w \in SU(2n+1,q^2)} \lambda'(tr\, w)$ *in* (1.1) *equals*

$$q^{2n^2+n-2}\sum_{r=0}^{n}(-1)^r q^{\frac{1}{2}r(r+3)}\begin{bmatrix} n \\ r \end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j)$$
$$\times \sum_{l=1}^{[\frac{n-r+2}{2}]} q^{2l}F_{n-r-2l+2}(\lambda'; 1, 1, (-1)^r)\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1),$$

where $F_r(\lambda'; a, b, c)$ *is the exponential sum defined by* (5.1) *and* (5.2), *and the innermost sum is over all integers* $j_1, \cdots, j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \cdots \le j_1 \le n-r+1$.

THEOREM B. *The sum* $\sum_{w \in U(2n+1,q^2)} \chi(\det w)\lambda'(tr\, w)$ *in* (1.2) *equals*

$$q^{2n^2+n-2}\left(\frac{1}{q-1}\sum_{j=0}^{q-2} G(\chi\psi^j, \lambda')\right)$$
$$\times \sum_{r=0}^{n}(-\chi(-1))^r q^{\frac{1}{2}r(r+3)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r}(q^j + (-1)^j)$$
$$\times \sum_{l=1}^{[\frac{n-r+2}{2}]} q^{2l} K(\lambda', \chi^{q-1}; 1, 1 : q^2)^{n-r-2l+2} \sum \prod_{\nu=1}^{l-1}(q^{2j_\nu - 4\nu} - 1),$$

where ψ *is a multiplicative character of* $\mathbb{F}_{q^2}$ *of order* $q-1$, $G(\chi\psi^j, \lambda')$ *is the usual Gauss sum as in* (2.10), $K(\lambda', \chi^{q-1}; 1, 1 : q^2)$ *is the twisted Kloosterman sum defined in* (2.11), *and the innermost sum is over all integers* $j_1, \cdots, j_{l-1}$ *satisfying the same inequalities as in the above Theorem A.*

THEOREM C. *Let* $a \in \mathbb{F}_q^\times$, *and let* C *be any nonsingular Hermitian matrix over* $\mathbb{F}_{q^2}$ *of size* $2n+1$. *Then the following Kloosterman sum over nonsingular Hermitian matrices (cf.* (8.1)) *is the same for any such a* C, *and*

$$K_{Herm,2n+1}(a^2C^{-1}, C) = -\sum_{w \in U(2n+1,q^2)} \lambda'_a(tr\, w),$$

so that it equals (-1) times the expression in Theorem B above with χ *trivial,* $\lambda' = \lambda'_a$ *(cf.* (2.3) *and* (2.4)).

The above Theorems A, B and C are respectively stated as Theorem 6.2, Theorem 7.2 and Theorem 8.1.

2. Preliminaries

In this section, we will fix some notations that will be used throughout this paper, describe some basic groups and mention the q-binomial theorem.

Let $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ denote respectively the finite field with q elements, $q = p^d$ (p any prime, d a positive integer), and the quadratic extension of $\mathbb{F}_q$.

The Frobenius automorphism τ of $\mathbb{F}_{q^2}$ is defined by

$$\alpha^{\tau} = \alpha^{q}. \tag{2.1}$$

Then, for $\alpha \in \mathbb{F}_{q^2}$,

$$\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\alpha = \alpha + \alpha^{\tau}, \qquad N_{\mathbb{F}_{q^2}/\mathbb{F}_q}\alpha = \alpha\alpha^{\tau}.$$

Also, for $\alpha \in \mathbb{F}_{q^2}^{\times}$, α^{τ}/α and α/α^{τ} are respectively denoted by

$$\alpha^{\tau}/\alpha = \alpha^{\tau-1}, \qquad \alpha/\alpha^{\tau} = \alpha^{1-\tau}. \tag{2.2}$$

Let λ be an additive character of $\mathbb{F}_q$. Then $\lambda = \lambda_a$ for a unique $a \in \mathbb{F}_q$, where, for $\alpha \in \mathbb{F}_q$,

$$\lambda_a(\alpha) = \exp\left\{\frac{2\pi i}{p}\left(a\alpha + (a\alpha)^p + \cdots + (a\alpha)^{p^{d-1}}\right)\right\}. \tag{2.3}$$

It is nontrivial if $a \neq 0$. For such a λ, λ' denotes the additive character λ lifted to $\mathbb{F}_{q^2}$. Thus

$$\lambda' = \lambda \circ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}. \tag{2.4}$$

Note that λ' is nontrivial if λ is. Likewise, for a multiplicative character ψ of $\mathbb{F}_q$, the lifting of that to $\mathbb{F}_{q^2}$ is denoted by ψ'. So $\psi' = \psi \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$.

Here tr A and det A denote respectively the trace and determinant of A for a square matrix A, and ${}^*B = {}^t(\beta_{ij}^{\tau})$ for any matrix $B = (\beta_{ij})$ over $\mathbb{F}_{q^2}$ (cf. (2.1)), where the 't' indicates the transpose. We will say that B is Hermitian if ${}^*B = B$.

$GL(n, q)$ denotes the group of all nonsingular $n \times n$ matrices with entries in $\mathbb{F}_q$. Then

$$U(2n+1, q^2) = \left\{w \in GL(2n+1, q^2) \mid {}^*wJw = J\right\}, \tag{2.5}$$

where

$$J = \begin{bmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6}$$

Write $w \in U(2n+1,q^2)$ as

$$w = \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix},$$

where A, B, C, D are of $n \times n$, e, f are of $n \times 1$, g, h are of $1 \times n$, and i is of 1×1. Then $U(2n+1,q^2)$ in (2.5) is also given by

(2.7)
$$\begin{aligned} &U(2n+1,q^2) \\ &= \left\{ \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \in GL(2n+1,q^2) \middle| \begin{array}{l} {}^*AC - {}^*CA + {}^*gg = 0, \\ {}^*BD + {}^*DB + {}^*hh = 0, \end{array} \right. \\ &\qquad \left. \begin{array}{l} {}^*AD + {}^*CB + {}^*gh = 1_n,\ {}^*ef + {}^*fe - i^T i = 1, \\ {}^*Af + {}^*Ce + {}^*gi = 0,\ {}^*Bf + {}^*De + {}^*hi = 0 \end{array} \right\} \\ &= \left\{ \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \in GL(2n+1,q^2) \middle| \begin{array}{l} A\,{}^*B + B\,{}^*A + e\,{}^*e = 0, \\ C\,{}^*D + D\,{}^*C + f\,{}^*f = 0, \end{array} \right. \\ &\qquad \left. \begin{array}{l} A\,{}^*D + B\,{}^*C + e\,{}^*f = 1_n,\ g\,{}^*h + h\,{}^*g + ii^T = 1, \\ A\,{}^*h + B\,{}^*g + ei^T = 0,\ C\,{}^*h + D\,{}^*g + fi^T = 0 \end{array} \right\}. \end{aligned}$$

$P(2n+1,q^2)$ is the maximal parabolic subgroup of $U(2n+1,q^2)$ defined by

(2.8)
$$\begin{aligned} &P(2n+1,q^2) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^*A^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^*h \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{l} A \in GL(n,q^2),\ ii^T = 1, \\ B + {}^*B + {}^*hh = 0 \end{array} \right\}. \end{aligned}$$

(2.9) $$SU(2n+1,q^2) = \{w \in U(2n+1,q^2) \mid \det w = 1\},$$

which is a subgroup of index $q+1$ in $U(2n+1,q^2)$.

For a multiplicative character ψ of $\mathbb{F}_q$ and an additive character λ of $\mathbb{F}_q$, the Gauss sum $G(\psi,\lambda:q)=G(\psi,\lambda)$ is defined by

$$G(\psi,\lambda:q)=G(\psi,\lambda)=\sum_{\alpha\in\mathbb{F}_q^\times}\psi(\alpha)\lambda(\alpha). \tag{2.10}$$

With ψ, λ just as above and for a, $b\in\mathbb{F}_q$, the 'twisted' Kloosterman sum
$K(\lambda,\psi;a,b:q)=K(\lambda,\psi;a,b)$ is defined as

$$K(\lambda,\psi;a,b:q)=K(\lambda,\psi;a,b)=\sum_{\alpha\in\mathbb{F}_q^\times}\psi(\alpha)\lambda(a\alpha+b\alpha^{-1}). \tag{2.11}$$

Further, if λ is nontrivial, then usual Kloosterman sum $K(\lambda;a,b:q)=K(\lambda;a,b)$ is given by

$$K(\lambda;a,b:q)=K(\lambda;a,b)=\sum_{\alpha\in\mathbb{F}_q^\times}\lambda(a\alpha+b\alpha^{-1}). \tag{2.12}$$

For integers n, r with $0\le r\le n$, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q=\prod_{j=0}^{r-1}(q^{n-j}-1)/(q^{r-j}-1). \tag{2.13}$$

The order of the group $GL(n,q)$ is given by

$$g_n(q)=\prod_{j=0}^{n-1}(q^n-q^j)=q^{\binom{n}{2}}\prod_{j=1}^{n}(q^j-1). \tag{2.14}$$

Then we have :

$$\frac{g_n(q)}{g_{n-r}(q)g_r(q)}=q^{r(n-r)}\begin{bmatrix} n \\ r \end{bmatrix}_q \tag{2.15}$$

for integers n, r with $0\le r\le n$.

For x an indeterminate, n a nonnegative integer

$$(x;q)_n=(1-x)(1-xq)\cdots(1-xq^{n-1}). \tag{2.16}$$

Then the q-binomial theorem says

$$\sum_{r=0}^{n}\begin{bmatrix} n \\ r \end{bmatrix}_q(-1)^r q^{\binom{r}{2}}x^r=(x;q)_n. \tag{2.17}$$

$[y]$ denotes the greatest integer $\le y$, for a real number y.

3. Bruhat decomposition

In this section, we will discuss the Bruhat decomposition of $U(2n+1,q^2)$ with respect to the maximal parabolic subgroup $P(2n+1,q^2)$ of $U(2n+1,q^2)$ (cf. (2.8)).

This decomposition (in fact, its slight variants (3.11) and (3.12)) will play a key role in deriving Theorem 6.2 and Theorem 7.2. As the next theorem about the decomposition can be proved by slightly modifying the corresponding proof in [4], a proof for that will not be provided. Instead, we demonstrate that this decomposition combined with the q-binomial theorem yields a well-known formula for the order of the group $U(2n+1,q^2)$.

THEOREM 3.1. (a) *There is a one-to-one correspondence*

$$P(2n+1,q^2)\backslash U(2n+1,q^2) \to P'(n+1,q^2)\backslash\Lambda$$

given by

$$P(2n+1,q^2)\begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \longmapsto P'(n+1,q^2)\begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix},$$

where

$$P'(n+1,q^2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(n+1,q^2) \,\middle|\, \begin{matrix} a \in GL(n,q^2),\ b=0 \\ N_{\mathbb{F}_{q^2}/\mathbb{F}_q} d = 1 \end{matrix} \right\},$$

$$\Lambda = \left\{ \begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix} \,\middle|\, \begin{matrix} C,\ D,\ f,\ g,\ h,\ i \text{ are respectively of} \\ n\times n,\ n\times n,\ n\times 1,\ 1\times n,\ 1\times n,\ 1\times 1 \\ \text{matrices over } \mathbb{F}_{q^2} \text{ subject to the conditions (3.1)} \\ \text{in below, and the matrix is of full rank } n+1 \end{matrix} \right\}.$$

$$\text{(3.1)}\qquad \begin{cases} C^*D + D^*C + f^*f = 0, \\ g^*h + h^*g + ii^T = 1, \\ C^*h + D^*g + fi^T = 0. \end{cases}$$

(b) *For given* $\begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix} \in \Lambda$, *there exists a unique* $r (0 \leq r \leq n)$, $p' \in P'(n+1, q^2)$, $p \in P(2n+1, q^2)$ *such that*

$$p' \begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix} p = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c)

$$U(2n+1, q^2) = \coprod_{r=0}^{n} P\sigma_r P, \tag{3.2}$$

where $P = P(2n+1, q^2)$ *and*

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_{n-r} & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in U(2n+1, q^2). \tag{3.3}$$

Put

$$\begin{aligned} Q = & Q(2n+1, q^2) = \{ w \in P(2n+1, q^2) \mid \det w = 1 \} \\ = & \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^*A^{-1} & 0 \\ 0 & 0 & (\det A)^{\tau - 1} \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^*h \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{l} A \in GL(n, q^2), \\ B + {}^*B + {}^*hh = 0 \end{array} \right\}, \end{aligned} \tag{3.4}$$

(3.5)

$$\begin{aligned} Q^- = & Q^-(2n+1, q^2) = \{ w \in P(2n+1, q^2) \mid \det w = -1 \} \\ = & \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^*A^{-1} & 0 \\ 0 & 0 & -(\det A)^{\tau - 1} \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^*h \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{l} A \in GL(n, q^2), \\ B + {}^*B + {}^*hh = 0 \end{array} \right\} \end{aligned}$$

(cf. (2.2)).
Then $Q(2n+1, q^2)$ is a subgroup of index $q+1$ in $P(2n+1, q^2)$ and

$$U(2n+1, q^2) = \coprod_{r=0}^{n} P\sigma_r Q. \tag{3.6}$$

Write, for each $r(0 \leq r \leq n)$ and σ_r as in (3.3),

$$A_r = A_r(q^2) = \{w \in P(2n+1, q^2) \mid \sigma_r w \sigma_r^{-1} \in P(2n+1, q^2)\}, \tag{3.7}$$

$$B_r = B_r(q^2) = \{w \in Q(2n+1, q^2) \mid \sigma_r w \sigma_r^{-1} \in P(2n+1, q^2)\}. \tag{3.8}$$

Then B_r is a subgroup of A_r of index $q+1$ and

$$\mid B_r \backslash Q \mid = \mid A_r \backslash P \mid . \tag{3.9}$$

Expressing $U(2n+1, q^2)$ as a disjoint union of right cosets of $P(2n+1, q^2)$, the decompositions in (3.2) and (3.6) can be rewritten as follows.

COROLLARY 3.2.

$$U(2n+1, q^2) = \coprod_{r=0}^{n} P\sigma_r(A_r \backslash P), \tag{3.10}$$

$$U(2n+1, q^2) = \coprod_{r=0}^{n} P\sigma_r(B_r \backslash Q), \tag{3.11}$$

where $P = P(2n+1, q^2)$, *and* σ_r, Q, A_r, B_r *are respectively as in* (3.3), (3.4), (3.7), (3.8).

Observing that $\det \sigma_r = (-1)^r$, we get from (3.11) the following decomposition for $SU(2n+1, q^2)$.

COROLLARY 3.3.

$$SU(2n+1, q^2) = \coprod_{\substack{0 \leq r \leq n \\ r \text{ even}}} Q\sigma_r(B_r \backslash Q) \coprod \Big(\coprod_{\substack{0 \leq r \leq n \\ r \text{ odd}}} Q^- \sigma_r(B_r \backslash Q)\Big), \tag{3.12}$$

where $Q^- = Q^-(2n+1, q^2)$ *is as in* (3.5).

Write $w \in P(2n+1, q^2)$ as

$$w = \begin{bmatrix} A & 0 & 0 \\ 0 & {}^*A^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^*h \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix}, \tag{3.13}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ {}^*A^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \tag{3.14}$$
$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \ h = [\, h_1 \quad h_2 \,].$$

Here A_{11}, A_{12}, A_{21}, and A_{22} are respectively of sizes $r \times r$, $r \times (n-r)$, $(n-r) \times r$, and $(n-r) \times (n-r)$, similarly for ${}^*A^{-1}$, B, and h_1 is of $1 \times r$.

Then, by multiplying out, we see that $\sigma_r w \sigma_r^{-1}$ is in $P(2n+1, q^2)$ if and only if $A_{11}B_{11} + A_{12}B_{21} = 0$, $A_{12} = 0$, $E_{21} = 0$, $A_{11}{}^*h_1 + A_{12}{}^*h_2 = 0$, $h_1 = 0$ if and only if $A_{12} = 0$, $B_{11} = 0$, $h_1 = 0$. Moreover, since $B + {}^*B + {}^*hh = 0$ in (3.13), we have $B_{21} = -{}^*B_{12}$ and $B_{22} + {}^*B_{22} + {}^*h_2 h_2 = 0$. Hence

$$| A_r(q^2) | = (q+1) g_r(q^2) g_{n-r}(q^2) q^{n(n+2)} q^{r(2n-3r-2)}, \tag{3.15}$$

where $g_n(q^2)$ is as in (2.14). Also,

$$| P(2n+1, q^2) | = (q+1) g_n(q^2) q^{n(n+2)}. \tag{3.16}$$

From (2.15), (3.15) and (3.16), we get

$$| A_r(q^2) \backslash P(2n+1, q^2) | = q^{r(r+2)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}. \tag{3.17}$$

This will be used later in Section 6 and 7. Also, from (2.14), (3.16) and (3.17), we have

$$| P(2n+1, q^2) |^2 | A_r(q^2) |^{-1} = (q+1) q^{2n^2+n} \prod_{j=1}^{n} (q^{2j} - 1) \tag{3.18}$$
$$\times q^{3r} (q^2)^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}.$$

The decomposition in (3.10) yields

$$| U(2n+1, q^2) | = \sum_{r=0}^{n} | P(2n+1, q^2) |^2 | A_r(q) |^{-1}. \tag{3.19}$$

Now, from (3.18) and (3.19) and applying the q-binomial theorem (2.17) with $x = -q^3$, we get the following well-known formula (3.20) for n odd. Although we provide a proof of that formula only for n odd, it is true also for n even.

THEOREM 3.4.

$$|U(n,q^2)| = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - (-1)^j). \tag{3.20}$$

Proof.

$$\begin{aligned}|U(2n+1,q^2)| =& (q+1)q^{2n^2+n} \prod_{j=1}^{n} (q^{2j}-1) \\ & \times \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} (q^2)^{\binom{r}{2}} q^{3r} \\ =& q^{2n^2+n}(q+1)(-q^3;q^2)_n \prod_{j=1}^{n}(q^{2j}-1) \\ =& q^{\binom{2n+1}{2}} \prod_{j=1}^{2n+1} (q^j - (-1)^j).\end{aligned}$$

□ □

4. Several propositions

The following proposition can be proved analogously to the corresponding Proposition 4.1 in [4].

PROPOSITION 4.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$, and let A be a nonsingular Hermitian matrix of size r with entries in $\mathbb{F}_{q^2}$. Then we have*

$$\sum_{y \in \mathbb{F}_{q^2}^{r\times 1}} \lambda({}^*yAy) = (-1)^r q^r, \tag{4.1}$$

where $\mathbb{F}_{q^2}^{r\times 1}$ denotes the set of all $r \times 1$ matrices over $\mathbb{F}_{q^2}$.

COROLLARY 4.2. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. For each positive integer r, let Ω_r be the set of all $r \times r$ nonsingular Hermitian matrices over $\mathbb{F}_{q^2}$. Then we have*

$$a_r(\lambda) = \sum_{A \in \Omega_r} \sum_{y \in \mathbb{F}_{q^2}^{r \times 1}} \lambda({}^*yAy) = (-1)^r q^{\binom{r+1}{2}} \prod_{j=1}^{r} (q^j + (-1)^j). \tag{4.2}$$

Proof. There is a transitive right action of $GL(r, q^2)$ on Ω_r given by $(A, z) \longmapsto {}^*zAz$. So (4.2) can be written as, for a fixed $A \in \Omega_r$,

$$\sum_{z \in U(r,q^2) \backslash GL(r,q^2)} \sum_{y \in \mathbb{F}_{q^2}^{r \times 1}} \lambda({}^*(zy)A(zy)) = |\, U(r, q^2) \backslash GL(r, q^2) \,| \sum_{y \in \mathbb{F}_{q^2}^{r \times 1}} \lambda({}^*yAy).$$

Now, the desired result follows from (2.14), (3.20) and (4.1). □

The following proposition can be shown by slightly modifying the proof of Theorem 5.30 in [8].

PROPOSITION 4.3. *Let λ be a nontrivial additive character of $\mathbb{F}_q$, η a multiplicative character of $\mathbb{F}_q$, and let ψ be a multiplicative character of $\mathbb{F}_q$ of order $d = (n, q-1)$. Then*

$$\sum_{\alpha \in \mathbb{F}_q^\times} \eta(\alpha^n)\lambda(\alpha^n) = \sum_{j=0}^{d-1} G(\eta\psi^j, \lambda), \tag{4.3}$$

where $G(\eta\psi^j, \lambda) = G(\eta\psi^j, \lambda : q)$ is the usual Gauss sum as in (2.10).

The next proposition can be proved just like Proposition 4.5 in [5], using the above proposition.

PROPOSITION 4.4. *Let λ' be the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$, and let χ be a multiplicative character of $\mathbb{F}_{q^2}$. Then*

$$\sum_{\alpha} \chi(\alpha)\lambda'(\alpha) = \frac{1}{q-1}\sum_{j=0}^{q-2} G(\chi\psi^j, \lambda'), \tag{4.4}$$

where the sum is over all $\alpha \in \mathbb{F}_{q^2}$ with $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}\alpha = 1$ (i.e., over $\alpha \in U(1, q^2)$), ψ is a multiplicative character of $\mathbb{F}_{q^2}$ of order $q-1$, and $G(\chi\psi^j, \lambda') = G(\chi\psi^j, \lambda' : q^2)$ is the Gauss sum as in (2.10).

REMARKS 4.5. (1) As we noted in the proof of Proposition 4.5 in [5],

$$\sum_{\alpha \in U(1,q^2)} \lambda'(\alpha) = \sum_{w \in SO^-(2,q)} \lambda(\mathrm{tr}\ w), \tag{4.5}$$

where, for $p > 2$, a fixed element $\varepsilon \in \mathbb{F}_q^\times \backslash \mathbb{F}_q^{\times^2}$, and $\delta_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -\varepsilon \end{bmatrix}$,

$$\begin{aligned} SO^-(2,q) =& \{w \in GL(2,q) \mid {}^t w \delta_\varepsilon w = \delta_\varepsilon, \det\ w = 1\} \\ =& \left\{ \begin{bmatrix} a & b\varepsilon \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{F}_q, a^2 - b^2\varepsilon = 1 \right\}. \end{aligned}$$

Here we may identify $\mathbb{F}_{q^2}$ with $\mathbb{F}_q(\sqrt{\varepsilon})$.
(2) Assume now that ψ is a multiplicative character of $\mathbb{F}_q$ of order $q-1$. Then $\psi' = \psi \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is a multiplicative character of $\mathbb{F}_{q^2}$ of order $q-1$ and $(\psi')^j = (\psi^j)'$. So, with χ trivial, (4.4) can be written as

$$\begin{aligned} \sum_{\alpha \in U(1,q^2)} \lambda'(\alpha) =& \frac{1}{q-1}\sum_{j=0}^{q-2} G((\psi^j)', \lambda') \\ =& -\frac{1}{q-1}\sum_{j=0}^{q-2} G(\psi^j, \lambda)^2 \text{ (Davenport - Hasse).} \end{aligned}$$

This was already noted in (4.13) of [5].

5. Certain exponential sums

Let λ' be a nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$. For a positive integer r and $a, b, c \in \mathbb{F}_{q^2}$, we define the exponential sum $F_r(\lambda'; a, b, c)$ as

(5.1)
$$F_r(\lambda'; a, b, c) := \sum_{\alpha_1, \cdots, \alpha_r \in \mathbb{F}_{q^2}^\times} \lambda'(a \sum_{j=1}^{r} \alpha_j + b \sum_{j=1}^{r} \alpha_j^{-1} + c \prod_{j=1}^{r} \alpha_j^{\tau - 1})$$

(cf. (2.2)).
Also, we put

$$F_0(\lambda'; a, b, c) := \lambda'(c). \tag{5.2}$$

Now, it is elementary to see that

$$\begin{aligned} &\sum_{\delta \in \mathbb{F}_{q^2}^\times} \lambda'(a\delta^{-1} + b\delta)\, F_r(\lambda'; a, b, c\delta^{1-\tau}) \\ =&F_{r+1}(\lambda'; a, b, c). \end{aligned} \tag{5.3}$$

In the next section, we will reduce the sum in (1.1) to an expression involving $S_t(\lambda'; a, b, c)$, which is defined just in below and can be expressed as a polynomial in q with coefficients involving exponential sums in (5.1) (cf. Theorem 5.1).

With λ' as above, and for a positive integer t and $a, b, c \in \mathbb{F}_{q^2}$, we define $S_t(\lambda'; a, b, c)$ as

$$\begin{aligned} &S_t(\lambda'; a, b, c) \\ :=& \sum_{w \in GL(t, q^2)} \lambda'(a \operatorname{tr} w + b \operatorname{tr} w^{-1} + c(\det w)^{\tau - 1}). \end{aligned} \tag{5.4}$$

If $c = 0$, then it is the Kloosterman sum $K_{GL(t,q^2)}(\lambda'; a, b)$ defined in (4.3) of [3]. Using the decomposition (4.4) in [3], a recursive relation for $K_{GL(t,q^2)}(\lambda'; a, b)$ can be obtained (cf. (4.18) in [3]) provided that $a, b \neq 0$ (it is trivial if a or b is zero).

The same decomposition of $GL(t, q^2)$ mentioned just above can be used in order to derive the following recursive relation : for $t \geq 2$, $a, b \in \mathbb{F}_{q^2}^{\times}$, $c \in \mathbb{F}_{q^2}$,

$$
\begin{aligned}
S_t(\lambda'; a, b, c) =& q^{2t-2} \sum_{\delta \in \mathbb{F}_{q^2}^{\times}} \lambda'(a\delta^{-1} + b\delta)\, S_{t-1}(\lambda'; a, b, c\delta^{1-\tau}) \\
&+ q^{4t-4}(q^{2t-2} - 1)\, S_{t-2}(\lambda'; a, b, c(a^{-1}b)^{\tau-1}),
\end{aligned} \tag{5.5}
$$

where we understand that, for $a, b, c \in \mathbb{F}_{q^2}$,

$$
S_0(\lambda'; a, b, c) := \lambda'(c). \tag{5.6}
$$

From the recursive relation in (5.5) and using (5.3), an explicit expression for (5.4) can be derived, by induction on t, in exactly the same manner as in the proof of Theorem 4.3 in [3].

THEOREM 5.1. *Let λ' be the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$. Then, for integers $t \geq 1, a, b \in \mathbb{F}_{q^2}^{\times}, c \in \mathbb{F}_{q^2}$, the exponential sum $S_t(\lambda'; a, b, c)$ defined in* (5.4) *is*

(5.7)

$$
\begin{aligned}
S_t(\lambda'; a, b, c) =& q^{(t+1)(t-2)} \sum_{l=1}^{[\frac{t+2}{2}]} q^{2l} F_{t+2-2l}(\lambda'; a, b, c((a^{-1}b)^{\tau-1})^{l-1}) \\
&\times \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1),
\end{aligned}
$$

where $F_r(\lambda'; a, b, c)$ is defined in (5.1) *and* (5.2), *and the inner sum is over all integers $j_1, \cdots, j_{l-1}$ satisfying $2l - 1 \leq j_{l-1} \leq \cdots \leq j_1 \leq t+1$. Here we understand that the inner sum in* (5.7) *is 1 for $l = 1$.*

REMARKS 5.2. (1) The inner sum in (5.7) is equivalently given by

$$
\sum \prod_{\nu=1}^{l-1} (q^{2j_\nu} - 1),
$$

where the sum runs over all integers $j_1, \cdots, j_{l-1}$ satisfying $2l - 3 \leq j_1 \leq t - 1$, $2l - 5 \leq j_2 \leq j_1 - 2, \cdots, 1 \leq j_{l-1} \leq j_{l-2} - 2$ (with the understanding that $j_0 = t + 1$ for $l = 2$).
(2) (5.7) is valid also for $t = 0$, in view of our definitions in (5.2) and (5.6).

6. SU(2n+1, q²) case

In this section, we will consider the sum in (1.1)

$$\sum_{w \in SU(2n+1,q^2)} \lambda'(\mathrm{tr}\ w)$$

and find an explicit expression for this, where λ' is the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$.

Using the decomposition in (3.12), the sum in (1.1) can be written as

$$\begin{aligned}(6.1)\qquad &\sum_{\substack{0 \le r \le n \\ r \text{ even}}} | B_r \backslash Q | \sum_{w \in Q} \lambda'(\mathrm{tr}\ w\sigma_r) \\ &+ \sum_{\substack{0 \le r \le n \\ r \text{ odd}}} | B_r \backslash Q | \sum_{w \in Q^-} \lambda'(\mathrm{tr}\ w\sigma_r),\end{aligned}$$

where $B_r = B_r(q^2), Q = Q(2n+1, q^2), Q^- = Q^-(2n+1, q^2), \sigma_r$ are respectively as in (3.8), (3.4), (3.5), (3.3). Here one should note that, for each $q \in Q$,

$$\begin{aligned}\sum_{w \in Q} \lambda'(\mathrm{tr}\ w\sigma_r q) &= \sum_{w \in Q} \lambda'(\mathrm{tr}\ qw\sigma_r) \\ &= \sum_{w \in Q} \lambda'(\mathrm{tr}\ w\sigma_r)\end{aligned}$$

and $qQ^- = Q^-$.

Write $w \in Q$ (cf. (3.4)) as

$$w = \begin{bmatrix} A & 0 & 0 \\ 0 & {}^*A^{-1} & 0 \\ 0 & 0 & (\det A)^{\tau - 1} \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^*h \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix},$$

with $A, {}^*A^{-1}, B, h$ as in (3.14). Recall here that $B + {}^*B + {}^*hh = 0$, which is equivalent to

$$\begin{aligned}(6.2)\qquad &B_{11} + {}^*B_{11} + {}^*h_1 h_1 = 0, \\ &B_{22} + {}^*B_{22} + {}^*h_2 h_2 = 0, \\ &B_{12} + {}^*B_{21} + {}^*h_1 h_2 = 0.\end{aligned}$$

Now, $w\sigma_r$ is $\begin{bmatrix} M & * & * \\ * & N & * \\ * & * & (\det A)^{\tau-1} \end{bmatrix}$ with

$$M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ B_{21} & 1_{n-r} \end{bmatrix}, N = \begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix}.$$

For any $r (0 \le r \le n)$,
(6.3)

$$\sum_{w \in Q} \lambda'(\text{tr } w\sigma_r)$$
$$= \sum \lambda'(\text{tr } A_{11}B_{11} + \text{tr } A_{12}B_{21} + \text{tr } A_{22} + \text{tr } E_{22} + (\det A)^{\tau-1}),$$

where the sum is over all $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, h = [\, h_1 \quad h_2 \,]$ subject to the conditions in (6.2).
For each fixed A, h, the subsum over B in (6.3) is

$$\sum \lambda'(\text{tr } A_{11}B_{11} + \text{tr } A_{12}B_{21}), \tag{6.4}$$

where the sum is over all B_{11}, B_{21}, B_{22} subject to the first and second coditions in (6.2). It is

$$q^{(n-r)^2} \sum_{B_{11}} \lambda'(\text{tr } A_{11}B_{11}) \sum_{B_{21}} \lambda'(\text{tr } A_{12}B_{21}), \tag{6.5}$$

since the summand is independent of B_{22}. The sum over B_{21} in (6.5) is nonzero if and only if $A_{12} = 0$, in which case it equals $q^{2r(n-r)}$. On the other hand, the sum over B_{11} in (6.5) is nonzero if and only if A_{11} is Hermitian, in which case it is $q^{r^2} \lambda(-h_1 A_{11}{}^* h_1)$. To see this, we need the following lemma whose proof we will omit (cf. Lemma 5.1 in [6]).

LEMMA 6.1. *Let λ' be the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$. Let $c \in \mathbb{F}_q, a \in \mathbb{F}_{q^2}$. Then*

$$\sum \lambda'(ab) = \begin{cases} q\lambda(ac), & \text{if } a \in \mathbb{F}_q, \\ 0, & \text{otherwise,} \end{cases} \tag{6.6}$$

where the sum is over all elements $b \in \mathbb{F}_{q^2}$ with $tr_{\mathbb{F}_{q^2}/\mathbb{F}_q} b = c$.

Now, let $A_{11} = (\alpha_{ij}), B_{11} = (\beta_{ij}), h_1 = [h_{11} \quad \ldots \quad h_{1r}]$. Then the condition $B_{11} + {}^*B_{11} + {}^*h_1 h_1 = 0$ is equivalent to

$$\begin{aligned} \beta_{ii} + \beta_{ii}^{\tau} &= -h_{1i}^{\tau} h_{1i}, \quad \text{for } 1 \le i \le r, \\ \beta_{ij} + \beta_{ji}^{\tau} &= -h_{1i}^{\tau} h_{1j}, \quad \text{for } 1 \le i < j \le r. \end{aligned} \tag{6.7}$$

Using these relations, it is not hard to see that

$$\begin{aligned} \lambda'(\text{tr } A_{11}B_{11}) = {} & \lambda'(-\sum_{1 \le i < j \le r} \alpha_{ji} h_{1i}^{\tau} h_{1j}) \lambda'(\sum_{i=1}^{r} \alpha_{ii}\beta_{ii}) \\ & \times \lambda'(\sum_{1 \le i < j \le r} (\alpha_{ij} - \alpha_{ji}^{\tau})\beta_{ji}). \end{aligned} \tag{6.8}$$

In view of (6.6), (6.7) and (6.8), the subsum over B_{11} in (6.5) is nonzero if and only if $\alpha_{ii} \in \mathbb{F}_q$ for $1 \le i \le r$ and $\alpha_{ij} = \alpha_{ji}^{\tau}$ for $1 \le i < j \le r$, i.e., A_{11} is Hermitian. Moreover, in that case it is

$$\begin{aligned} & q^{r^2} \lambda'(-\sum_{1 \le i < j \le r} \alpha_{ji} h_{1j} h_{1i}^{\tau}) \lambda(-\sum_{i=1}^{r} \alpha_{ii} h_{1i} h_{1i}^{\tau}) \\ & = q^{r^2} \lambda(-h_1 A_{11} {}^*h_1). \end{aligned}$$

So far we have shown that the sum in (6.4) is nonzero if and only if $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} nonsingular Hermitian. In addition, in that case it equals

$$\begin{aligned} & q^{(n-r)^2 + 2r(n-r) + r^2} \lambda(-h_1 A_{11} {}^*h_1) \\ = & q^{n^2} \lambda(-h_1 A_{11} {}^*h_1). \end{aligned}$$

For such an $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{-1} & * \\ 0 & {}^*A_{22}^{-1} \end{bmatrix}$ and $(\det A)^{\tau - 1} = (\det A_{22})^{\tau - 1}$.

The sum in (6.3) can be written as

(6.9)
$$q^{n^2} \sum_{A_{21},h_2} \sum_{A_{11},h_1} \lambda(-h_1 A_{11} {}^*h_1) \sum_{A_{22}} \lambda'(\text{tr } A_{22} + \text{tr } A_{22}^{-1} + (\det A_{22})^{\tau-1})$$
$$= q^{n^2+2(r+1)(n-r)} \sum_{A_{11},h_1} \lambda(-h_1 A_{11} {}^*h_1) \sum_{A_{22}} \lambda'(\text{tr } A_{22} + \text{tr } A_{22}^{-1} + (\det A_{22})^{\tau-1}),$$

where A_{11} is over the set Ω_r of all $r \times r$ nonsingular Hermitian matrices over $\mathbb{F}_{q^2}$, $h_1 \in \mathbb{F}_{q^2}^{1\times r}$, $A_{22} \in GL(n-r,q^2)$. Thus (6.9) equals

$$q^{n^2+2(r+1)(n-r)} a_r(\lambda) S_{n-r}(\lambda'; 1, 1, 1), \tag{6.10}$$

where we note that the first sum in (6.9) is the same as $a_r(\lambda)$ in (4.2) and $S_t(\lambda'; a, b, c)$ is as in (5.4) and (5.6).

From (3.9), (3.17), (4.2), and (6.10), the first sum in (6.1) equals

$$q^{n^2+2n} \sum_{\substack{0\le r\le n \\ r \text{ even}}} q^{\frac{1}{2}r(4n-r+1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \times S_{n-r}(\lambda'; 1, 1, 1). \tag{6.11}$$

On the other hand, glancing through the above argument, we see that, for any $r (0 \le r \le n)$,

$$\sum_{w\in Q^-} \lambda'(\text{tr } w\sigma_r)$$

is the same as (6.10) except that $S_{n-r}(\lambda'; 1, 1, 1)$ is now replaced by $S_{n-r}(\lambda'; 1, 1, -1)$. Thus the second sum in (6.1) is

$$-q^{n^2+2n} \sum_{\substack{0\le r\le n \\ r \text{ odd}}} q^{\frac{1}{2}r(4n-r+1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \times S_{n-r}(\lambda'; 1, 1, -1). \tag{6.12}$$

Finally, from (6.11), (6.12) and using the explicit expression of $S_t(\lambda'; a, b, c)$ in (5.7), we get the following main result of this section.

THEOREM 6.2. *Let λ' be the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$. Then the Gauss sum over $SU(2n+1, q^2)$*

$$\sum_{w \in SU(2n+1,q^2)} \lambda'(tr\ w)$$

is given by

$$q^{2n^2+n-2} \sum_{r=0}^{n} (-1)^r q^{\frac{1}{2}r(r+3)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j)$$
$$\times \sum_{l=1}^{[\frac{n-r+2}{2}]} q^{2l} F_{n-r-2l+2}(\lambda'; 1, 1, (-1)^r)$$
$$\times \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1),$$

where the innermost sum is over all integers $j_1, \cdots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le j_{l-2} \le \cdots \le j_1 \le n-r+1$, and $F_r(\lambda'; a, b, c)$ is as in (5.1) *and* (5.2).

7. U(2n+1, q²) case

Let χ be a multiplicative character of $\mathbb{F}_{q^2}$, and let λ' be the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$. Then we will consider the sum in (1.2)

$$\sum_{w \in U(2n+1,q^2)} \chi(\det w)\, \lambda'(\mathrm{tr}\ w)$$

and find an explicit expression for this.

Using the decomposition in (3.11), the sum in (1.2) can be written as

$$\sum_{r=0}^{n} \chi(-1)^r \mid B_r \backslash Q \mid \sum_{w \in P} \chi(\det w)\, \lambda'(\mathrm{tr}\ w\sigma_r). \tag{7.1}$$

Write $w \in P = P(2n+1,q^2)$ as in (3.13) with $A, {}^*A^{-1}, B, h$ as in (3.14). The inner sum in (7.1) is

$$\begin{aligned}(7.2)\qquad &\sum_i \chi(i)\lambda'(i) \sum_{A,h} \chi((\det A)^{1-\tau})\lambda'(\mathrm{tr}\ A_{22} + \mathrm{tr}\ E_{22}) \\ &\times \sum_B \lambda'(\mathrm{tr}\ A_{11}B_{11} + \mathrm{tr}\ A_{12}B_{21}),\end{aligned}$$

where i is over all $i \in \mathbb{F}_{q^2}$ with $N_{\mathbb{F}_{q^2}/\mathbb{F}_q} i = 1$ (i.e., $i \in U(1,q^2)$), $A \in GL(n,q^2)$, and B, h are subject to the condition $B + {}^*B + {}^*hh = 0$. We saw in Section 6 that, for each fixed A, h, the sum over B in (7.2) is nonzero if and only if $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} nonsingular Hermitian. Moreover, it is $q^{n^2}\lambda(-h_1A_{11}{}^*h_1)$ in that case. Also, for such an $A, (\det A)^{1-\tau} = (\det A_{22})^{1-\tau}$ and $E_{22} = {}^*A_{22}^{-1}$. Thus (7.2) equals

$$\begin{aligned}(7.3)\qquad &q^{n^2} \sum_i \chi(i)\lambda'(i) \sum_{A_{21},h_2} \sum_{A_{11},h_1} \lambda(-h_1A_{11}{}^*h_1) \\ &\times \sum_{A_{22}} \chi((\det A_{22})^{\tau-1})\lambda'(\mathrm{tr}\ A_{22} + \mathrm{tr}\ A_{22}^{-1}) \\ =&q^{n^2+2(r+1)(n-r)}a_r(\lambda) \sum_i \chi(i)\lambda'(i) \\ &\times \sum_{w \in GL(n-r,q^2)} \chi^{q-1}(\det w)\lambda'(\mathrm{tr}\ w - \mathrm{tr}\ w^{-1}),\end{aligned}$$

where $a_r(\lambda)$ is as in (4.2).

Combining (7.1) and (7.3), the sum in (1.2) is

$$\begin{aligned}(7.4)\qquad &q^{n^2+2n} \sum_i \chi(i)\lambda'(i) \sum_{r=0}^{n} \chi(-1)^r \mid B_r \backslash Q \mid \\ &\times q^{2r(n-r-1)}a_r(\lambda)K_{GL(n-r,q^2)}(\lambda', \chi^{q-1}; 1, 1),\end{aligned}$$

where in [6], for a nontrivial additive character λ of $\mathbb{F}_q, \psi$ a multiplicative character of $\mathbb{F}_q, a, b, \in \mathbb{F}_q, K_{GL(t,q)}(\lambda, \psi; a, b)$ is defined to be

$$(7.5)\qquad K_{GL(t,q)}(\lambda, \psi; a, b) = \sum_{w \in GL(t,q)} \psi(\det w)\, \lambda(a\mathrm{tr}\ w + b\mathrm{tr}\ w^{-1})$$

An explicit expression for this was obtained in [6].

THEOREM 7.1. [6, Theorem 4.2] *For integers $t \geq 1, a, b \in \mathbb{F}_q^\times$, the twisted Kloosterman sum $K_{GL(t,q)}(\lambda, \psi; a, b)$ defined in (7.5) is*

$$K_{GL(t,q)}(\lambda, \psi; a, b)$$

$$= q^{\frac{1}{2}(t-2)(t+1)} \sum_{l=1}^{[\frac{t+2}{2}]} \psi(-a^{-1}b)^{l-1} q^l K(\lambda, \psi; a, b : q)^{t+2-2l} \tag{7.6}$$

$$\sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),$$

where $K(\lambda, \psi; a, b : q)$ is the usual twisted Kloosterman sum as in (2.11), the inner sum is over all integers $j_1, \cdots, j_{l-1}$ satisfying $2l - 1 \leq j_{l-1} \leq j_{l-2} \leq \cdots \leq j_1 \leq t + 1$, and we agree that the inner sum in (7.6) is 1 for $l = 1$.

Now, we are ready to get the main theorem of this section from (4.4), (3.9), (3.17), (4.2), (7.6), and (7.4).

THEOREM 7.2. *Let χ be a multiplicative character of $\mathbb{F}_{q^2}$, and let λ' be the nontrivial additive character λ of $\mathbb{F}_q$ lifted to $\mathbb{F}_{q^2}$. Then the Gauss sum over $U(2n + 1, q^2)$*

$$\sum_{w \in U(2n+1,q^2)} \chi(\det w) \lambda'(tr\ w)$$

is given by

$$q^{2n^2+n-2} \left(\frac{1}{q-1} \sum_{j=0}^{q-2} G(\chi\psi^j, \lambda') \right)$$

$$\times \sum_{r=0}^{n} (-\chi(-1))^r q^{\frac{1}{2}r(r+3)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j)$$

$$\times \sum_{l=1}^{[\frac{n-r+2}{2}]} q^{2l} K(\lambda', \chi^{q-1}; 1, 1 : q^2)^{n-r-2l+2} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1),$$

where ψ is a multiplicative character of $\mathbb{F}_{q^2}$ of order $q-1$, $G(\chi\psi^j, \lambda') = G(\chi\psi^j, \lambda' : q^2)$ is the Gauss sum as in (2.10), $K(\lambda', \chi^{q-1}; 1, 1 : q^2)$ is *the twisted Kloosterman sum defined in* (2.11), *and the innermost sum is over all integers* $j_1, \cdots, j_{l-1}$ *satisfying* $2l-1 \leq j_{l-1} \leq j_{l-2} \leq \cdots \leq j_1 \leq n-r+1$.

8. Application to Hodges' Kloosterman sum

In [2], the generalized Kloosterman sum over nonsingular Hermitian matrices is defined as, for $t \times t$ Hermitian matrices A, B over $\mathbb{F}_{q^2}$

$$K_{Herm,t}(A,B) = \sum_g \lambda_1(\text{tr } (Ag + Bg^{-1})), \tag{8.1}$$

where g runs over the set of all nonsingular Hermitian matrices over $\mathbb{F}_{q^2}$ of size t. Here λ_1 is as in (2.3), and one should note that, for Hermitian matrices C, D over $\mathbb{F}_{q^2}$ of size t, tr $CD \in \mathbb{F}_q$.

In Theorem 6 of [2], we set $m = t = 2n+1, A = B = J$ with J as in (2.6), $X = a1_{2n+1}$ with $a \in \mathbb{F}_q^\times$. Then we get the following identity

$$\sum_{w \in U(2n+1,q^2)} \lambda'_a(\text{tr } w) = -K_{Herm,2n+1}(a^2J^{-1}, J).$$

This is summarized in the following.

THEOREM 8.1. *For* $a \in \mathbb{F}_q^\times$, *we have the identity:*

$$\begin{aligned} \sum_{w \in U(2n+1,q^2)} \lambda'_a(tr\ w) &= -K_{Herm,2n+1}(a^2J^{-1}, J) \\ &= -K_{Herm,2n+1}(a^2C^{-1}, C), \end{aligned} \tag{8.2}$$

where λ_a *is as in* (2.3) *and* C *is any nonsingular Hermitian matrix over* $\mathbb{F}_{q^2}$ *of size* $2n+1$.

REMARKS 8.2. (1) The second identity in (8.2) is clear from the definition of Kloosterman sum in (8.1).

(2) The whole discussion in [2] is valid even for $p = 2$ if the 'conjugate' of α in $\mathbb{F}_{q^2}$ means α^τ. So here we don't have to assume $q = p^d$ is a power of an odd prime.

Combining (8.2) and Theorem 7.2, we have the following result.

THEOREM 8.2. *Let $a \in \mathbb{F}_q^\times$, and let C be any nonsingular Hermitian matrix over $\mathbb{F}_{q^2}$ of size $2n+1$. Then the following generalized Kloosterman sum over nonsingular Hermitian matrices is the same for any such a C, and*

$$\begin{aligned}
&K_{Herm,2n+1}(a^2C^{-1},C)\\
=&q^{2n^2+n-2}\left(\frac{1}{q-1}\sum_{r=0}^{q-2}G(\psi^j,\lambda'_a)\right)\\
&\times\sum_{r=0}^{n}(-1)^{r+1}q^{\frac{1}{2}r(r+3)}\begin{bmatrix} n \\ r \end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j)\\
&\times\sum_{l=1}^{[\frac{n-r+2}{2}]}q^{2l}K(\lambda'_a;1,1:q^2)^{n-r-2l+2}\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1),
\end{aligned}$$

where the innermost sum is over all integers $j_1,\cdots,j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le j_{l-2} \le \cdots \le j_1 \le n-r+1, \psi$ is a multiplicative character of $\mathbb{F}_{q^2}$ of order $q-1, G(\psi^j,\lambda'_a) = G(\psi^j,\lambda'_a : q^2)$ is the Gauss sum as in (2.10), and $K(\lambda'_a;1,1 : q^2)$ is the Kloosterman sum as in (2.12).

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