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HOMOGENEOUS C^*-ALGEBRAS OVER A SPHERE

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ABSTRACT. It is shown that for $A_{k,m}$ a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ such that no non-trivial matrix algebra can be factored out of $A_{k,m}$ and $A_{k,m} \otimes M_l(\mathbb{C})$ has a non-trivial bundle structure for any positive integer l, we construct an $A_{k,m}$-$C(S^{2n-1} \times S^1) \otimes M_k(\mathbb{C})$-equivalence bimodule to show that every k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ is strongly Morita equivalent to $C(S^{2n-1} \times S^1)$. Moreover, we prove that the tensor product of the k-homogeneous C^*-algebra $A_{k,m}$ with a UHF-algebra of type p^∞ has the trivial bundle structure if and only if the set of prime factors of k is a subset of the set of prime factors of p.

1. Introduction

In [9], the authors showed that the group $[S^{2n}, BPU(k)]$ of homotopy classes of continuous maps of S^{2n} into the classifying space $BPU(k)$ of $PU(k)$ classifies k-homogeneous C^*-algebras over S^{2n}, and that the set $[S^{2n-1} \times S^1, BPU(k)]$ classifies k-homogeneous C^*-algebras over $S^{2n-1} \times S^1$. k-homogeneous C^*-algebras over S^n were constructed in [5], and k-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ were constructed in [7].

It is well-known that for M a finite dimensional CW-complex, the Čech cohomology group $H^3(M, \mathbb{Z})$ classifies tensor products of k-homogeneous C^*-algebras over M with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$, and (cf. [1]) that the Čech cohomology group $H^3(M, \mathbb{Z})$ is isomorphic to the singular cohomology

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group $H^3(M,\mathbb{Z})$ if M is *triangularizable*. The Čech(=singular) cohomology group $H^3(S^{2n},\mathbb{Z}) \cong \{0\}$, and the Čech(=singular) cohomology group $H^3(S^{2n-1} \times S^1,\mathbb{Z}) \cong \{0\}$ if $n \neq 2$. And the Čech(=singular) cohomology group $H^3(S^3 \times S^1,\mathbb{Z}) \cong H^3(S^3,\mathbb{Z}) \cong \mathbb{Z}$, which corresponds to the set of tensor products of $\mathcal{K}(\mathcal{H})$-homogeneous C^*-algebras over S^3 with $C(S^1)$ for $\mathcal{H}$ a separable Hilbert space. Thus the tensor products of k-homogeneous C^*-algebras over S^{2n} (resp. $S^{2n-1} \times S^1$) with $\mathcal{K}(\mathcal{H})$ are isomorphic to $C(S^{2n}) \otimes \mathcal{K}(\mathcal{H})$ (resp. $C(S^{2n-1} \times S^1) \otimes \mathcal{K}(\mathcal{H})$).

In [3], the authors proved that $A \otimes \mathcal{K}(\mathcal{H})$ is isomorphic to $B \otimes \mathcal{K}(\mathcal{H})$ if and only if there exists an A-B-equivalence bimodule defined in the first section. In this case, we say that A and B are strongly Morita equivalent. In [2], M. Brabanter constructed an $A_{\frac{m}{k}}$-$C(\mathbb{T}^2)$-equivalence bimodule to prove that every rational rotation algebra is strongly Morita equivalent to $C(\mathbb{T}^2)$.

We are going to construct an equivalence bimodule to show that every k-homogeneous C^*-algebra over S^{2n} (resp. $S^{2n-1} \times S^1$) is strongly Morita equivalent to $C(S^{2n})$ (resp. $C(S^{2n-1} \times S^1)$).

Thomsen computed in [10] the homotopy group of equivalence classes of maps of S^n into the classifying space of the automorphism group of an infinite dimensional UHF-algebra: let $U = \bigotimes_{i=1}^{\infty} M_{p_i}(\mathbb{C})$ be a UHF-algebra where $\{p_i \mid i \in \mathbb{N}\}$ is a sequence of prime numbers. Let P denote all the prime numbers. For each $p \in P$, we define

$$f(p) = \#\{i \in \mathbb{N} \mid p = p_i\}.$$

Then for n even

$$[S^n,\, B(\mathrm{Aut}(U))] = \Pi_{p \in P_F} \mathbb{Z}_{p^{f(p)}},$$

where $P_F = \{p \in P \mid 0 < f(p) < \infty\}$ (interpreted to mean the trivial group if $P_F = \emptyset$). For n odd, $[S^n,\, B(\mathrm{Aut}(U))] = \{0\}$. This result clarifies the bundle structure of k-homogeneous C^*-algebras over S^n with a UHF-algebra of type p^∞.

By a different trick, we are going to investigate the bundle structure of k-homogeneous C^*-algebras over S^{2n} with a UHF-algebra of type p^∞. Furthermore, we are going to study the bundle structure of k-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ with a UHF-algebra of type p^∞.

2. Strong Morita equivalence for k-homogeneous C^*-algebras over S^{2n} and $S^{2n-1} \times S^1$

For a separable Hilbert space $\mathcal{H}$, $[S^n, BPU(\mathcal{H})] = [S^{n-1}, PU(\mathcal{H})]$ and $PU(\mathcal{H})$ is a $K(\mathbb{Z}, 2)$-space, so every $\mathcal{K}(\mathcal{H})$-homogeneous C^*-algebra over S^n has the trivial bundle structure if $n \neq 3$. And Woodward [11] proved that $[S^3, BPU(k)]$ injects to $H^2(S^3, \mathbb{Z}_k) \oplus H^4(S^3, \mathbb{Z}) \cong \{0\}$, so every k-homogeneous C^*-algebra over S^3 is isomorphic to $C(S^3) \otimes M_k(\mathbb{C})$. Thus every k-homogeneous C^*-algebra over S^n is stably isomorphic to $C(S^n)$, i.e., every k-homogeneous C^*-algebra over S^n is strongly Morita equivalent to $C(S^n)$.

DEFINITION 2.1 [8]. Let A and B be C^*-algebras. By an A-B-*equivalence bimodule* is meant an A-B-bimodule on which are defined an A-valued and a B-valued inner product such that

(1) $\langle x, y\rangle_A z = x\langle y, z\rangle_B, \quad \forall x, y, z \in X$,
(2) the representation of A on X is a continuous $*$-representation by operators which are defined for $\langle \cdot, \cdot\rangle_B$, and similarly for the right representation of B,
(3) the linear span of $\langle X, X\rangle_B$, which is an ideal in B, is dense in B, and similarly for $\langle X, X\rangle_A$.

We say that two C^*-algebras A and B are *strongly Morita equivalent* if there exists an A-B-equivalence bimodule. It is shown in [3] that if they are separable, then they are strongly Morita equivalent if and only if $A \otimes \mathcal{K}(\mathcal{H})$ is isomorphic to $B \otimes \mathcal{K}(\mathcal{H})$.

Note that Brabanter constructed an $A_{\frac{m}{k}}$-$C(\mathbb{T}^2)$-equivalence bimodule using the structure of the rational rotation algebra $A_{\frac{m}{k}}$.

Since the condition over the boundaries S^1 on sections of a k-homogeneous C^*-algebra over S^2 is the same as the condition over the boundaries S^1 on sections of a suitable k-homogeneous C^*-algebra over $\mathbb{T}^2$, one can modify the construction given by Brabanter to construct a $B_{\frac{m}{k}}$-$C(S^2)$-equivalence bimodule. But we are going to construct a $B_{\frac{m}{k}}$-$C(S^2)$-equivalence bimodule using a slightly different trick.

For n a positive integer $(n > 1)$, $[S^{2n}, BPU(k)] = [S^{2n-1}, PU(k)] \cong \mathbb{Z}$, which is a cyclic group. So the group has a generator, and there is a unitary $U(z) \in PU(k)$ such that the generating k-homogeneous C^*-algebra over S^{2n} can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra over S^{2n} characterized by the inner automorphism of the unitary $U(z) \in PU(k)$ over the boundaries S^{2n-1}. Here $PU(k) = \mathrm{Aut}(M_k(\mathbb{C})) = \mathrm{Inn}(M_k(\mathbb{C}))$. If $(k, m) = p(|p| > 1)$, then we consider the k-homogeneous C^*-algebra over S^{2n} as the tensor product of $M_p(\mathbb{C})$ with a $\frac{k}{p}$-homogeneous C^*-algebra over S^{2n}, which is characterized by the inner automorphism of the unitary $U(z)^{\frac{m}{p}} \in PU(\frac{k}{p})$. Consider $U(z)^m$ as $U(z)^{\frac{m}{p}} \otimes I_p \in PU(k)$. Then every k-homogeneous C^*-algebra over S^{2n} can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra over S^{2n} characterized by the inner automorphism of the unitary $U(z)^m \in PU(k)$ over the boundaries S^{2n-1}. See [5] for details.

Throughout this paper, we will denote the n-dimensional northern hemisphere (resp. the n-dimensional southern hemisphere) by e^n_+ (resp. e^n_-).

PROPOSITION 2.2 [5]. *Every k-homogeneous C^*-algebra over S^{2n} is isomorphic to one of the following C^*-algebras $B_{k,m} = C_{g_m}(e^{2n}_+ \amalg e^{2n}_-, M_k(\mathbb{C}))$, $m \in \mathbb{Z}$, given as follows: if $f \in B_{k,m}$, then the following condition is satisfied*

$$f_+(z) = U(z)^{-m} f_-(z) U(z)^m$$

for all $z \in S^{2n-1}$, where $U(z) \in PU(k) = \mathrm{Inn}(M_k(\mathbb{C}))$ is a suitable projective unitary.

$[S^{2n}, BPU(\mathcal{H})] \cong [S^{2n-1}, PU(\mathcal{H})] \cong \{0\}$, so every k-homogeneous C^*-algebra over S^{2n} is strongly Morita equivalent to $C(S^{2n})$. However, we don't have any equivalence bimodule constructed before.

THEOREM 2.3. *The k-homogeneous C^*-algebra $B_{k,m}$ over S^{2n} is strongly Morita equivalent to $C(S^{2n})$.*

Proof. We are going to construct a $B_{k,m}$-$C(S^{2n}) \otimes M_k(\mathbb{C})$-equivalence bimodule.

Let $X = \{f \in C(e^{2n}_+ \amalg e^{2n}_-, M_k(\mathbb{C})) \mid f_+(z) = U(z)^{-m} f_-(z)\}$, where $U(z)$ is given as above. The space X is a left $B_{k,m}$-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = (f_{ij})^k_{i,j=1} \in B_{k,m}$ and $x \in X$. If $f \in C(S^{2n}) \otimes M_k(\mathbb{C})$ and $x \in X$, then $x \cdot f$ defines a right $C(S^{2n}) \otimes M_k(\mathbb{C})$-module structure on X. We define $B_{k,m}$ and $C(S^{2n}) \otimes M_k(\mathbb{C})$-valued inner products $\langle \cdot, \cdot \rangle_{B_{k,m}}$ and $\langle \cdot, \cdot \rangle_{C(S^{2n}) \otimes M_k(\mathbb{C})}$ on X by

$$\langle x, y \rangle_{B_{k,m}} = x \cdot y^* \quad \& \quad \langle x, y \rangle_{C(S^{2n}) \otimes M_k(\mathbb{C})} = x^* \cdot y$$

if $x, y \in X$ and we have matrix multiplication on the right. Equipped with this structure, X becomes a $B_{k,m}$-$C(S^{2n}) \otimes M_k(\mathbb{C})$-equivalence bimodule. So $B_{k,m}$ is strongly Morita equivalent to $C(S^{2n}) \otimes M_k(\mathbb{C})$, which is strongly Morita equivalent to $C(S^{2n})$.

Therefore, $B_{k,m}$ is strongly Morita equivalent to $C(S^{2n})$. □

We have obtained that every k-homogeneous C^*-algebra over S^{2n} is strongly Morita equivalent to $C(S^{2n})$.

In [7], the author showed that every k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ is isomorphic to one of the following C^*-algebras; k-homogeneous C^*-algebras whose restrictions to $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ are k'-homogeneous C^*-algebras over S^{2n-1} $(k'|k)$, and which correspond to k-homogeneous C^*-algebras over S^{2n} with respect to the boundaries S^{2n-1} of $S^{2n-1} \times [0,1]$ and $e^{2n}_+ \amalg e^{2n}_-$. See [7] for details. It is well-known (cf. [5, T]) that for a k'-homogeneous C^*-algebra A over S^{2n-1}, there is a matrix algebra $M_l(\mathbb{C})$ such that $A \otimes M_l(\mathbb{C})$ has the trivial bundle structure. So for a k-homogeneous C^*-algebra A over $S^{2n-1} \times S^1$, there are a k-homogeneous C^*-algebra $A_{k,m}$ over $S^{2n-1} \times S^1$ and a matrix algebra $M_l(\mathbb{C})$ such that $A \otimes M_l(\mathbb{C})$ is isomorphic to $A_{k,m} \otimes M_l(\mathbb{C})$, where $A_{k,m}$ is a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ whose restriction to S^{2n-1} has the trivial bundle structure.

From now on we will only consider the k-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ whose restrictions to S^{2n-1} have the trivial bundle structures.

PROPOSITION 2.4 [7]. *Every k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$, whose restriction to S^{2n-1} has a trivial bundle structure, is isomorphic to one of the following C^*-algebras $A_{k,m} = C_{g_m}(S^{2n-1} \times [0,1], M_k(\mathbb{C}))$, $m \in \mathbb{Z}$, given as follows: if $f \in A_{k,m}$, then the following condition is satisfied*

$$f(z,1) = U(z)^{-m} f(z,0) U(z)^m$$

for all $z \in S^{2n-1}$, where $U(z) \in PU(k)$ is given in the statement of Proposition 2.2.

Since the condition over the boundaries S^{2n-1} of $S^{2n-1} \times [0,1]$ on sections of a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ is the same as the condition over the boundaries S^{2n-1} of e_+^{2n} and e_-^{2n} on sections of a suitable k-homogeneous C^*-algebra over S^{2n}, one can modify the construction given in the proof of Theorem 2.3 to construct an $A_{k,m}$-$C(S^{2n-1} \times S^1) \otimes M_k(\mathbb{C})$-equivalence bimodule, and one can show that the unitaries $U(z)^m$ represent the $B_{k,m}$.

THEOREM 2.5. *The k-homogeneous C^*-algebra $A_{k,m}$ over $S^{2n-1} \times S^1$ is strongly Morita equivalent to $C(S^{2n-1} \times S^1)$.*

We have obtained that every k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ is strongly Morita equivalent to $C(S^{2n-1} \times S^1)$.

3. The K-theory of k-homogeneous C^*-algebras over a sphere

Since every k-homogeneous C^*-algebra over S^{2n} is strongly Morita equivalent to $C(S^{2n})$, the K-theory of a k-homogeneous C^*-algebra $B_{k,m}$ over S^{2n} is isomorphic to the K-theory of $C(S^{2n})$. Thus $K_0(B_{k,m}) \cong \mathbb{Z}^2$ and $K_1(B_{k,m}) \cong \{0\}$ since $K_0(C(S^{2n})) \cong \mathbb{Z}^2$ and $K_1(C(S^{2n})) \cong \{0\}$.

LEMMA 3.1. *Let $B_{k,m}$ be a k-homogeneous C^*-algebra over S^{2n} of which no non-trivial matrix algebra can be factored out. $K_0(B_{k,m}) \cong K_0(C(S^{2n})) \cong \mathbb{Z}^2$ and $K_1(B_{k,m}) \cong K_1(C(S^{2n})) \cong \{0\}$. The class $[1_{B_{k,m}}]$ of the unit $1_{B_{k,m}}$ is a primitive element of $K_0(B_{k,m})$.*

Proof. We know that $B_{k,m}$ is strongly Morita equivalent to $C(S^{2n})$. So $K_0(B_{k,m}) \cong K_0(C(S^{2n})) \cong \mathbb{Z}^2$ and $K_1(B_{k,m}) \cong K_1(C(S^{2n})) \cong \{0\}$.

It is enough to show that the class $[1_{B_{k,m}}] \in K_0(B_{k,m})$ is primitive. The $C(S^{2n})$ can be embedded into the $B_{k,m}$, and the embedding of the $C(S^{2n})$ into the $B_{k,m}$ induces an isomorphism of $K_0(C(S^{2n}))$ to $K_0(B_{k,m})$. By the isomorphism $[1_{C(S^{2n})}]$ maps to $[1_{B_{k,m}}]$ or $[1_{C(S^{2n})}]$. If $[1_{C(S^{2n})}]$ maps to $[1_{A_{k,m}}]$, then $[1_{B_{k,m}}]$ is primitive, since $[1_{C(S^{2n})}] \in K_0(C(S^{2n}))$ is primitive (see [6]). If $[1_{C(S^{2n})}]$ maps to $[1_{C(S^{2n})}]$, then $[1_{B_{k,m}}] = k[1_{C(S^{2n})}]$, and so there is a suitable matrix algebra $M_p(\mathbb{C})$

such that $1_{B_{k,m}} \otimes J_p = 1_{C(S^{2n})} \otimes I_k \otimes J_p$ in the $B_{k,m} \otimes M_p(\mathbb{C})$, since $k[1_{C(S^{2n})}]$ can be represented by the projection $1_{C(S^{2n})} \otimes I_k \in C(S^{2n}) \otimes M_k(\mathbb{C})$, where I_k denotes the $k \times k$ identity matrix and J_p denotes the $p \times p$ matrix with components 0 except for the (1,1)-component 1. Then

$$(1_{B_{k,m}} \otimes J_p)|_{S^{2n-1}} = (1_{C(S^{2n})} \otimes I_k \otimes J_p)|_{S^{2n-1}}.$$

But $(1_{B_{k,m}} \otimes J_p)|_{S^{2n-1}} = 1_{C(S^{2n-1})} \otimes J_p$, and $(1_{C(S^{2n})} \otimes I_k \otimes J_p)|_{S^{2n-1}} = 1_{C(S^{2n-1})} \otimes I_k \otimes J_p$. However, Thomsen [10] computed

$$\pi_{2n-1}(\mathrm{Aut}(M_{kp}(\mathbb{C}) \otimes M_{q^\infty})) \cong \mathbb{Z}/(kp\mathbb{Z}),$$

when kp and q are relatively prime. This means that $B_{k,m} \otimes M_p(\mathbb{C})$ is not isomorphic to $C(S^{2n}) \otimes M_{kp}$, since if it is, then $\pi_{2n-1}(\mathrm{Aut}(M_{kp}(\mathbb{C}) \otimes M_{q^\infty}))$ is trivial. But the $B_{k,m}$ corresponds to the $A_{k,m}$ with respect to the conditions on sections over the boundaries S^{2n-1}, and $A_{k,m}|_{S^{2n-1}} \cong C(S^{2n-1})$, since if $A_{k,m}|_{S^{2n-1}} \cong C(S^{2n-1}) \otimes M_k(\mathbb{C})$, then $A_{k,m}$ is isomorphic to $C(S^{2n-1}) \otimes C(S^1) \otimes M_k(\mathbb{C})$, which is a contradiction. So $B_{k,m}|_{S^{2n-1}} \cong C(S^{2n-1})$. Thus

$$(1_{B_{k,m}} \otimes J_p)|_{S^{2n-1}} \neq (1_{C(S^{2n})} \otimes I_k \otimes J_p)|_{S^{2n-1}}.$$

Hence $1_{B_{k,m}} \in K_0(B_{k,m})$ is primitive.

Therefore, $K_0(B_{k,m}) \cong \mathbb{Z}^2$, $K_1(B_{k,m}) \cong \{0\}$, and the class $[1_{B_{k,m}}]$ of the unit $1_{B_{k,m}}$ is a primitive element of $K_0(B_{k,m})$. □

COROLLARY 3.2. *Let l be a positive integer. $B_{k,m} \otimes M_l(\mathbb{C})$ is not isomorphic to $C(S^{2n}) \otimes M_{kl}(\mathbb{C})$.*

Proof. Assume $B_{k,m} \otimes M_l(\mathbb{C})$ is isomorphic to $C(S^{2n}) \otimes M_{kl}(\mathbb{C})$. Then the unit $1_{B_{k,m}} \otimes I_l$ maps to the unit $1_{C(S^{2n})} \otimes I_{kl}$. So

$$[1_{B_{k,m}} \otimes I_l] = [1_{C(S^{2n})} \otimes I_{kl}].$$

Thus there is a projection $p \in B_{k,m}$ such that $l[1_{B_{k,m}}] = (kl)[p]$. But $K_0(B_{k,m}) \cong \mathbb{Z}^2$ is torsion-free, so $[1_{B_{k,m}}] = k[p]$. This is a contradiction.

Therefore, $B_{k,m} \otimes M_l(\mathbb{C})$ is not isomorphic to $C(S^{2n}) \otimes M_{kl}(\mathbb{C})$. □

Let $A_{k,m}$ be a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ of which no non-trivial matrix algebra can be factored out, i.e., k and m are relatively prime. The k-homogeneous C^*-algebras $A_{k,m}$ over $S^{2n-1} \times S^1$ are strongly Morita equivalent to $C(S^{2n-1} \times S^1)$, so $K_0(A_{k,m}) \cong K_0(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$ and $K_1(A_{k,m}) \cong K_1(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$.

We are going to show that the class $[1_{A_{k,m}}] \in K_0(A_{k,m})$ is primitive.

LEMMA 3.3. *Let $A_{k,m}$ be a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ given in the statement of Proposition 2.4 whose restriction to S^{2n-1} is isomorphic to $C(S^{2n-1})$. Then $K_0(A_{k,m}) \cong K_1(A_{k,m}) \cong \mathbb{Z}^2$. And the class $[1_{A_{k,m}}] \in K_0(A_{k,m})$ is primitive.*

Proof. We know that $A_{k,m}$ is strongly Morita equivalent to $C(S^{2n-1} \times S^1)$. So $K_0(A_{k,m}) \cong K_0(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$ and $K_1(A_{k,m}) \cong K_1(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$.

So it is enough to show that the class $[1_{A_{k,m}}] \in K_0(A_{k,m})$ is primitive. The $C(S^{2n-1} \times S^1)$ can be embedded into the $A_{k,m}$, and the embedding of the $C(S^{2n-1} \times S^1)$ into the $A_{k,m}$ induces an isomorphism of $K_0(C(S^{2n-1} \times S^1))$ to $K_0(A_{k,m})$. By the isomorphism $[1_{C(S^{2n-1} \times S^1)}$ maps to $[1_{A_{k,m}}]$ or $[1_{C(S^{2n-1} \times S^1)}]$. If $[1_{C(S^{2n-1} \times S^1)}]$ maps to $[1_{A_{k,m}}]$, then $[1_{A_{k,m}}]$ is primitive, since $[1_{C(S^{2n-1} \times S^1)}] \in K_0(C(S^{2n-1} \times S^1))$ is primitive (see [6]). If $[1_{C(S^{2n-1} \times S^1)}]$ maps to $[1_{C(S^{2n-1} \times S^1)}]$, then $[1_{A_{k,m}}] = k[1_{C(S^{2n-1})}]$, and so there is a suitable matrix algebra $M_p(\mathbb{C})$ such that $1_{A_{k,m}} \otimes J_p = 1_{C(S^{2n-1} \times S^1)} \otimes I_k \otimes J_p$ in the $A_{k,m} \otimes M_p(\mathbb{C})$, since $k[1_{C(S^{2n-1} \times S^1)}]$ can be represented by the projection $1_{C(S^{2n-1} \times S^1)} \otimes I_k \in C(S^{2n-1} \times S^1) \otimes M_k(\mathbb{C})$. Then

$$(1_{A_{k,m}} \otimes J_p)|_{S^{2n-1}} = (1_{C(S^{2n-1} \times S^1)} \otimes I_k \otimes J_p)|_{S^{2n-1}}.$$

But $(1_{A_{k,m}} \otimes J_p)|_{S^{2n-1}} = 1_{C(S^{2n-1})} \otimes J_p$, and $(1_{C(S^{2n-1} \times S^1)} \otimes I_k \otimes J_p)|_{S^{2n-1}} = 1_{C(S^{2n-1})} \otimes I_k \otimes J_p$. However, $A_{k,m} \otimes M_p(\mathbb{C})$ has a non-trivial bundle structure for $M_p(\mathbb{C})$ any matrix algebra, since if it has not, then the same is true for the k'-homogeneous C^*-algebra $B_{k,m}$ over S^{2n} corresponding to the $A_{k,m}$ with respect to the conditions on sections. Thomsen result says that $B_{k,m} \otimes M_p(\mathbb{C})$ is not isomorphic to $C(S^{2n}) \otimes M_{kp}$. Thus

$$(1_{A_{k,m}} \otimes J_p)|_{S^{2n-1}} \neq (1_{C(S^{2n-1} \times S^1)} \otimes I_k \otimes J_p)|_{S^{2n-1}}.$$

Hence $1_{A_{k,m}} \in K_0(A_{k,m})$ is primitive.

Therefore, $K_0(A_{k,m}) \cong \mathbb{Z}^2$, $K_1(A_{k,m}) \cong \mathbb{Z}^2$, and the class $[1_{A_{k,m}}]$ of the unit $1_{A_{k,m}}$ is a primitive element of $K_0(A_{k,m})$. □

COROLLARY 3.4. *Let l be a positive integer. $A_{k,m} \otimes M_l(\mathbb{C})$ is not isomorphic to $C(S^{2n-1} \times S^1) \otimes M_{kl}(\mathbb{C})$.*

The proof is similar to the proof of Corollary 3.2.

We have obtained that the class $[1_{A_{k,m}}] \in K_0(A_{k,m})$ and the class $[1_{B_{k,m}}] \in K_0(B_{k,m})$ are primitive.

4. The tensor product of a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ and S^{2n} with a UHF-algebra

In this section, we investigate the bundle structure of the tensor product of a k-homogeneous C^*-algebra $B_{k,m}$ over S^{2n} with a UHF-algebra M_{p^∞} of type p^∞ and the bundle structure of the tensor product of a k-homogeneous C^*-algebra $A_{k,m}$ over $S^{2n-1} \times S^1$ with M_{p^∞}.

The following is useful.

THEOREM 4.1 [4]. *Suppose there exists an intertwining of the sequence of C^*-algebra homomorphisms $A_1 \to A_2 \to \cdots$ and $B_1 \to B_2 \to \cdots$. Then the inductive limit C^*-algebras $\lim A_i$ and $\lim B_i$ are isomorphic.*

Let $B_{k,m}$ be a k-homogeneous C^*-algebra over S^{2n} of which no nontrivial matrix algebra can be factored out, i.e., k and m are relatively prime.

THEOREM 4.2. *Let $B_{k,m}$ be a k-homogeneous C^*-algebra over S^{2n} with $(k,m) = 1$. Let M_{p^∞} be a UHF-algebra of type p^∞. Then $B_{k,m} \otimes M_{p^\infty}$ is isomorphic to $C(S^{2n}) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.*

Proof. Assume the set of prime factors of k is a subset of the set of prime factors of p. To show that $B_{k,m} \otimes M_{p^\infty}$ is isomorphic to $C(S^{2n}) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $B_{k,m} \otimes M_{k^\infty}$ is isomorphic to $C(S^{2n}) \otimes M_{k^\infty}$. But there exist the canonical C^*-algebra homomorphisms:

$$B_{k,m} \hookrightarrow C(S^{2n}) \otimes M_k(\mathbb{C}) \hookrightarrow B_{k,m} \otimes M_k(\mathbb{C}) \hookrightarrow C(S^{2n}) \otimes M_{k^2}(\mathbb{C}) \hookrightarrow \cdots.$$

The inductive limit of the odd terms

$$\cdots \to B_{k,m} \otimes M_{k^d}(\mathbb{C}) \to B_{k,m} \otimes M_{k^{d+1}}(\mathbb{C}) \to \cdots$$

is $B_{k,m} \otimes M_{k^\infty}$, and the inductive limit of the even terms

$$\cdots \to C(S^{2n}) \otimes M_{k^d}(\mathbb{C}) \to C(S^{2n}) \otimes M_{k^{d+1}}(\mathbb{C}) \to \cdots$$

is $C(S^{2n}) \otimes M_{k^\infty}$. Thus by Theorem 4.1, $B_{k,m} \otimes M_{k^\infty}$ is isomorphic to $C(S^{2n}) \otimes M_{k^\infty}$.

Conversely, assume $B_{k,m} \otimes M_{p^\infty}$ is isomorphic to $C(S^{2n}) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$. Then the unit $1_{B_{k,m}} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(S^{2n})} \otimes 1_{M_{p^\infty}} \otimes I_k$. So

$$[1_{B_{k,m}} \otimes 1_{M_{p^\infty}}] = [1_{C(S^{2n})} \otimes 1_{M_{p^\infty}} \otimes I_k].$$

And $[1_{B_{k,m}} \otimes 1_{M_{p^\infty}}] = [1_{B_{k,m}}] \otimes [1_{M_{p^\infty}}]$ and $[1_{C(S^{2n})} \otimes 1_{M_{p^\infty}} \otimes I_k] = k([1_{C(S^{2n})}] \otimes [1_{M_{p^\infty}}])$. But $K_0(B_{k,m} \otimes M_{p^\infty}) \cong [\frac{1}{p}](K_0(B_{k,m}))$ and $K_0(C(S^{2n}) \otimes M_{p^\infty} \otimes M_k(\mathbb{C}) \cong k[\frac{1}{p}](K_0(C(S^{2n})))$. If there is a prime factor q of k such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[p_\infty]$ for p_∞ a projection in M_{p^∞} under the assumption that the unit $1_{B_{k,m}} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(S^{2n})} \otimes 1_{M_{p^\infty}} \otimes I_k$. So there is a projection $p_1 \in B_{k,m}$ such that $[1_{B_{k,m}}] = q[p_1]$. This contradicts Theorem 3.3. Thus the set of prime factors of k is a subset of the set of prime factors of p.

Therefore, $B_{k,m} \otimes M_{p^\infty}$ is isomorphic to $C(S^{2n}) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p. □

Next, let $A_{k,m}$ be a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ whose restriction to S^{2n-1} is isomorphic to $C(S^{2n-1})$, i.e., k and m are relatively prime.

THEOREM 4.3. *Let $A_{k,m}$ be a k-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ with $(k,m) = 1$. Let M_{p^∞} be a UHF-algebra of type p^∞. Then $A_{k,m} \otimes M_{p^\infty}$ is isomorphic to $C(S^{2n-1} \times S^1) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the prime factors of p.*

The proof is similar to the proof of Theorem 4.2.

We have obtained that $A_{k,m} \otimes M_{p^\infty}$ has the trivial bundle structure if and only if the set of prime factors of k is a subset of the set of prime factors of p, and that $B_{k,m} \otimes M_{p^\infty}$ has a trivial bundle structure if and only if the set of prime factors of k is a subset of the set of prime factors of p.

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