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ANALYSIS AND COMPUTATIONS OF OPTIMAL AND FEEDBACK CONTROL PROBLEMS FOR NAVIER-STOKES EQUATIONS

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ABSTRACT. We present analysis and some computational methods for boundary optimal and feedback control problems for Navier-Stokes equations. We use one example to illustrate our methodology and ideas which are applicable to general control problems for Navier-Stokes equations. First, we discuss the existence of optimal solutions and derive an optimality system of equations from which an optimal solution may be computed. Then we present a gradient type iterative method. Finally, we present some numerical results.

1. Introduction

The optimal feedback control of fluid flows has emerged as an important area of technological and scientific research. It has many important applications in engineering sciences, both in military situations and in industrial processes.

Boundary optimal control problems for Navier-Stokes equations has been studied recently in papers, e.g. [5], [7], [8], [10] and the references therein. The intrinsic difficulties caused, among other things, by the complex nonlinearity of the governing equations, lead to difficult computational and theoretical problems. Thus, very little attention has been given to the question of controlling the Navier-Stokes equations. As a first step of studying the general control problems for Naver-Stokes equations, we formulate and solve computationally the steady control

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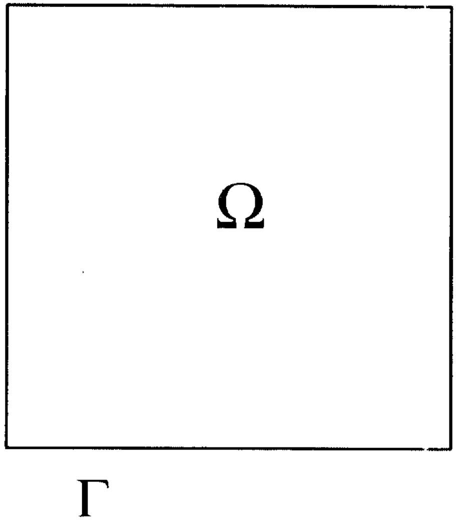

FIGURE 1. Driven cavity

problem for driven cavity flow that has been investigated extensively in numerical computations. To improve the performance of the numerical optimization algorithm we apply the feedback control algorithm and the simple gradient algorithm which were introduced in [1], [4], [9].

We consider the two dimensional motion of fluid modeled by the (stationary) Navier-Stokes equation,

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma, \tag{1.3}$$

confined in a square cavity Ω, depicted in Fig. 1. Here $\mathbf{u}$, p and $\mathbf{f}$, denote velocity field, pressure and a given body force, respectively. ν is the kinematic viscosity of the fluid ($\nu = 1/\mathrm{Re}$, where Re is the Reynolds number). We denote the boundary of the domain Ω by Γ.

This paper is organized as follows. In section 2 the function spaces and the basic theory of Navier-Stokes equation are given. In section 3 the optimal control problem is described precisely and the existence and first-order optimality condition for optimal control problem are established. In section 4 the numerical feedback control algorithm and the simple gradient algorithm are described. Finally numerical results are given in the section 5.

2. Function spaces; a weak formulation of the Navier-Stokes equations

We introduce some function spaces and their norms, along with some related notation used in subsequent sections (for details see [2]). Throughout, C will denote a positive constant whose meaning and value changes with context. We define the Sobolev space $H^m(\Omega)$ for nonnegative integer m by

$$H^m(\Omega) = \{T \in L^2(\Omega) | D^\alpha T \in L^2(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$$

where $D^\alpha T$ is the weak (or distributional) partial derivative, α is a multi-index, $|\alpha| = \sum_i \alpha_i$. Clearly, $H^0(\Omega) = L^2(\Omega)$. The norm associated with $H^m(\Omega)$ that we use is $||\cdot||_m$, given by

$$||T||_m = \left\{ \sum_{0\leq|\alpha|\leq m} ||D^\alpha T||_0^2 \right\}^{\frac{1}{2}}.$$

One particular subspace is given as

$$H_0^1(\Omega) = \left\{ S \in H^1(\Omega) : S = 0 \text{ on } \Gamma \right\}. \tag{2.1}$$

The usual inner product associated with $H^m(\Omega)$ will be denoted by $(\cdot,\cdot)_m$. For vector valued functions, we define the Sobolev space $\mathbf{H}^m(\Omega)$ by

$$\mathbf{H}^m(\Omega) := \{\mathbf{u} \mid u_i \in H^m(\Omega), i = 1, \text{ and } 2 \}, \tag{2.2}$$

where $\mathbf{u} = \{u_1, u_2\}$, and its associated norm $||\cdot||$ is given by

$$||\mathbf{u}||_m = \left\{ \sum_{i=1}^{2} ||u_i||_m^2 \right\}^{\frac{1}{2}}. \tag{2.3}$$

We also define a subspace of $L^2(\Omega)$

$$L_0^2(\Omega) := \{q \in L^2(\Omega) | \int_\Omega q \, d\Omega = 0\}. \tag{2.4}$$

In all subspaces, we use norms induced by the original spaces. We also make use of the well-known space $\mathbf{L}^4(\Omega)$ equipped with the norm $||\cdot||_{\mathbf{L}^4(\Omega)}$.

Certain trace spaces are also needed. In particular,

$$\mathbf{H}^{1/2}(\Gamma) := \{\mathbf{u}|_\Gamma \mid u_i|_\Gamma \in H^{1/2}(\Gamma)\} \tag{2.5}$$

$$\mathbf{H}^{1/2}(\Gamma) := \{\mathbf{u}|_\Gamma \mid u_i|_\Gamma \in H^{1/2}(\Gamma)\} \quad \text{and} \quad \mathbf{H}^{-1/2}(\Gamma) := (\mathbf{H}^{1/2}(\Gamma))^* \tag{2.6}$$

which are equipped with the usual norms

$$||\mathbf{h}||_{1/2,\Gamma} := \inf_{\mathbf{u}\in \mathbf{H}^1(\Omega), \mathbf{u}|_\Gamma = \mathbf{h}} ||\mathbf{u}||_1, \tag{2.7}$$

and

$$||\mathbf{g}||_{-1/2,\Gamma} := \sup_{\mathbf{h}\in \mathbf{H}^{1/2}(\Gamma), \mathbf{h}\neq \mathbf{0}} \frac{< \mathbf{g}, \mathbf{h} >_\Gamma}{||\mathbf{h}||_{1/2,\Gamma}}, \tag{2.8}$$

respectively. Here $< \cdot, \cdot >$ denotes the duality pairing.

We also define the solenoidal spaces

$$\mathbf{V} := \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) | \nabla \cdot \mathbf{u} = 0\}.$$

If Ω is bounded and has a Lipschitz continuous boundary (these are kinds of domains under consideration here), Sobolev's embedding theorem yields that $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$, where $\hookrightarrow\hookrightarrow$ denotes compact embedding, *i.e.* a constant C exists such that

$$||u||_{L^4(\Omega)} \leq C||u||_1. \tag{2.9}$$

Obviously a similar result holds for the spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^4(\Omega)$.

The matrix $\partial u_i/\partial x_j$ will be denoted $\nabla \mathbf{u}$. Let $a(\mathbf{u}, \mathbf{v}) : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to R$ be the symmetric sesquilinear for defined by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.10}$$

and define the bilinear form $b(\mathbf{u}, p) : \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \to R$ by

$$b(\mathbf{v}, q) = -\int_\Omega q \nabla \cdot \mathbf{v} \, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \ \forall q \in L^2_0(\Omega), \tag{2.11}$$

The trilinear form c on $\mathbf{H}^1(\Omega)$ that corresponds to the convective term in (2.1) is defined by

$$c(\mathbf{u}, \mathbf{w}, \mathbf{v}) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} \, d\Omega \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega), \tag{2.12}$$

These forms are continuous in the sense that there exist constants C_a, C_b, and $C_c > 0$ such that

$$|a(\mathbf{u},\mathbf{v})| \leq C_a||\mathbf{u}||_1||\mathbf{v}||_1 \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.13}$$

$$|b(\mathbf{u},q)| \leq C_b||\mathbf{u}||_1||q||_0 \quad \mathbf{u} \in \mathbf{H}^1(\Omega) \text{ and } q \in L^2_0(\Omega) \tag{2.14}$$

and

$$|c(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C_c||\mathbf{u}||_1||\mathbf{v}||_1||\mathbf{w}||_1 \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega). \tag{2.15}$$

Moreover, we have the coercivity properties

$$a(\mathbf{u},\mathbf{u}) \geq c_a||\mathbf{u}||_1^2 \quad \forall \mathbf{u} \in \mathbf{H}^1_0(\Omega) \tag{2.16}$$

and

$$\sup_{0\neq \mathbf{u}\in \mathbf{H}^1_0(\Omega)} \frac{b(\mathbf{u},q)}{||\mathbf{u}||_1} \geq c_b||q||_0 \quad \forall q \in L^2_0(\Omega), \tag{2.17}$$

for some constant c_a and $c_b > 0$.

The weak form of the constraint equations (1.1)-(??) with boundary condition is then given as follows: seek $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $p \in L^2_0(\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = <\mathbf{f},\mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{2.18}$$

$$b(\mathbf{u},q) = 0 \quad \forall q \in L^2_0(\Omega), \tag{2.19}$$

and

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma. \tag{2.20}$$

THEOREM 2.1. *Let Ω be a bounded domain of R^2 with a Lipschitz continuous boundary Γ. Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ satisfying*

$$\int_\Gamma \mathbf{g}\, d\Gamma = 0, \tag{2.21}$$

there exists at least one pair $(\mathbf{u},p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ solution of (2.18)-(2.20).

Proof. See [6]. □

3. Optimal control problems

We consider the following optimal boundary control problem: Given the bottom velocity U_{bot}, find the top velocity U_{top} such that the separation of flow occurs at a desired horizontal line location Γ_L. We cast the problem as a minimization of cost functional defined for U_{top}

$$\mathcal{J}(U_{top}) = \int_{\Gamma_L} |u_2|^2 \, ds, \tag{3.1}$$

subject to (2.18)-(2.20), where u_2 is the vertical component of the velocity field $\mathbf{u}$ and $u_2 = 0$ on Γ_L implies that no flow crosses the horizontal line Γ_L.

Now, our optimal control problem can be formulated as a finite dimensional constrained minimization in a Hilbert space

Find $(\mathbf{u}, p, \alpha) \in \mathbf{H}^1(\Omega) \times L_0^2 \times U$ which minimizes $\mathcal{J}(\mathbf{u})$ subject to

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ &b(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \\ &\mathbf{u} = \mathbf{g}_0 + \alpha \mathbf{g}_1 \quad \text{on } \Gamma, \\ &\alpha \in U, \end{aligned} \tag{3.2}$$

where $\mathbf{g}_0, \mathbf{g}_1 \in \mathbf{H}^{1/2}(\Gamma)$ with $\int_\Gamma \mathbf{g}_0 \cdot \mathbf{n} \, ds = \int_\Gamma \mathbf{g}_1 \cdot \mathbf{n} \, ds = 0$ and U is a closed set in R.

DEFINITION 3.1. The *admissibility set* $\mathcal{U}_{ad}$ is defined by $\mathcal{U}_{ad} = \{(\mathbf{u}, \alpha)$ $\mathbf{H}^1(\Omega) \times U$: there exists $p \in L_0^2(\Omega)$ and so that $(\mathbf{u}, p)$ satisfies the constrained equations in the problem (3.2) $\}$.

DEFINITION 3.2. We define an optimal solution $(\hat{\mathbf{u}}, \hat{\alpha})$ to be one for which $\mathcal{J}(\hat{\mathbf{u}}) \leq \mathcal{J}(\mathbf{u})$ for any $(\mathbf{u}, \alpha) \in \mathcal{U}_{ad}$.

DEFINITION 3.3. We define a local optimal solution $(\hat{\mathbf{u}}, \hat{\alpha})$ to be one for which there exists $\epsilon > 0$ such that $\mathcal{J}(\hat{\mathbf{u}}) \leq \mathcal{J}(\mathbf{u})$ for any $(\mathbf{u}, \alpha) \in \mathcal{U}_{ad}$ with $||\mathbf{u} - \hat{\mathbf{u}}||_1 < \epsilon$.

We now show that an optimal solution exists.

THEOREM 3.1. *There exists an optimal solution* $(\hat{\mathbf{u}}, \hat{\alpha}) \in \mathcal{U}_{ad}$ *for the control problem* (3.2).

Proof. We first claim that $\mathcal{U}_{ad}$ is not empty. Let $\alpha = 0$ and then let $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathbf{H}^1(\Omega) \times L_0^1(\Omega)$ be a solution to the constrained equations of the control problem (3.2); note that with $\alpha = 0$, the constrained equations of the problem (3.2) is equivalent to

$$a(\tilde{\mathbf{u}}, \mathbf{v}) + c(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.3}$$

$$b(\tilde{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{3.4}$$

$$\tilde{\mathbf{u}} = \mathbf{g}_0 \quad \text{on } \Gamma, \tag{3.5}$$

Since $\mathbf{g}_0 \in \mathbf{H}^{1/2}(\Gamma)$, $(\tilde{\mathbf{u}}, \tilde{p})$ exists from Theorem 2.1. Moreover, we have $\mathcal{J}(\tilde{\mathbf{u}}) \leq ||\tilde{\mathbf{u}}||_1^2 < \infty$. Thus $(\tilde{\mathbf{u}}, 0) \in \mathcal{U}_{ad}$.

Now let $(\mathbf{u}_n, \alpha_n)$ be a minimizing sequence. Since U is compact, it is easy to show that the admissibility set $\mathcal{U}_{ad}$ is bounded in $\mathbf{H}^1(\Omega)$. Thus there is a constant C_u such that $||\mathbf{u}_n||_1 \leq C_u$ for all n. We therefore have a subsequence $(\mathbf{u}_n, \alpha_n)$ such that $\mathbf{u}_n \to \mathbf{u}^*$ weakly in $\mathbf{H}^1(\Omega)$ and $\alpha_n \to \alpha^*$ in U. Note that $(\mathbf{u}_n, \alpha_n) \in \mathbf{H}^1(\Omega) \times U$ satisfies

$$a(\mathbf{u}_n, \mathbf{v}) + c(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + b(\mathbf{v}, p_n) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.6}$$

$$b(\mathbf{u}_n, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{3.7}$$

$$\mathbf{u}_n = \mathbf{g}_0 + \alpha_n \mathbf{g}_1 \quad \text{on } \Gamma, \tag{3.8}$$

$$\alpha_n \in U, \tag{3.9}$$

for some $p_n \in L_0^2(\Omega)$.

Now we show that the limiting flow is incompressible. We define the operator $B : \mathbf{H}^1(\Omega) \to L_0^2(\Omega)$ by

$$< B\mathbf{u}, p >= b(\mathbf{u}, p) \quad \mathbf{u} \in \mathbf{H}^1(\Omega),\ p \in L_0^2(\Omega). \tag{3.10}$$

The condition

$$b(\mathbf{u}_n, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{3.11}$$

means that $\mathbf{u}_n \in (ker)(B)$ for all n. But since $\ker(B)$ is a closed subspace, it is weakly closed, and so

$$\mathbf{u}^* \in \ker(B) \Rightarrow b(\mathbf{u}^*, q) = 0 \quad \forall q \in L_0^2(\Omega). \tag{3.12}$$

It is clear that $a(\mathbf{u}_n, \mathbf{v}) \to a(\mathbf{u}^*, \mathbf{v})$ for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$.

We also have that $||c(\mathbf{u}, \mathbf{v}, \mathbf{w})|| \leq M||\mathbf{u}||_{\mathbf{L}^4(\Omega)}||\mathbf{v}||_1||\mathbf{w}||_1$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$. Since $\mathbf{H}^1(\Omega)$ is compactly embedded in $\mathbf{L}^4(\Omega)$, $||\mathbf{u}_n - \mathbf{u}^*||_{L^4} \to 0$

and therefore $(\mathbf{u}^*, \alpha^*)$ satisfies the constrained equations of the control problem (3.2) *i.e.* $(\mathbf{u}^*, \alpha^*) \in \mathcal{U}_{ad}$. Hence if the cost functional $J : \mathbf{H}^1(\Omega) \to R$ is weakly, sequentially lower semicontinuous, then $(\mathbf{u}^*, \alpha^*)$ minimizes J. It is not difficult to show that the cost functional (3.1) is weakly, sequentially lower semicontinuous. In fact, the claim follows from the fact that trace operator of $\mathbf{H}^1(\Omega)$ on Γ_L is compact in $\mathbf{L}^2(\Gamma_L)$. Hence the control problem has an solution. □

Next we discuss the first-order necessary optimality condition. We wish to use the method of Lagrange multipliers to turn the constrained optimization problem (3.2) into an unconstrained one.

Let $\mathbf{u} = \mathbf{w} + \mathbf{u}_\epsilon^{(0)} + \alpha \mathbf{u}_\epsilon^{(1)}$ where $\mathbf{w} \in \mathbf{V}$ and $\mathbf{u}_\epsilon^{(0)}, \mathbf{u}_\epsilon^{(1)} \in \mathbf{H}^1(\Omega)$ are Hopf's functions[6] such that $\nabla \cdot \mathbf{u}_\epsilon^{(i)} = 0$ and $\mathbf{u}|_\Gamma = \mathbf{g}_i$ for $i = 1, 2$. Then the nonhomogeneous boundary value problem which is the constrained equations of the problem (3.2) can be transformed into the one with homogeneous boundary condition; find $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ such that

$$a(\mathbf{w}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = <\bar{\mathbf{f}}, \mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.13}$$

$$b(\mathbf{w}, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{3.14}$$

where $<\bar{\mathbf{f}}, \mathbf{v}> = -a(\mathbf{u}_\epsilon, \mathbf{v})$ and $\mathbf{u}_\epsilon = \mathbf{u}_\epsilon^{(0)} + \alpha \mathbf{u}_\epsilon^{(1)}$. The problem (3.2) can also be equivalently written as

Find $(\mathbf{w}, p, \alpha) \in \mathbf{H}_0^1(\Omega) \times L_0^2 \times U$ which minimizes $\mathcal{J}(\mathbf{u})$
subject to

$$a(\mathbf{w}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = <\bar{\mathbf{f}}, \mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.15}$$

$$b(\mathbf{w}, q) = 0 \quad \forall q \in L_0^2(\Omega),$$

where $\mathbf{u} = \mathbf{w} + \mathbf{u}_\epsilon^{(0)} + \alpha \mathbf{u}_\epsilon^{(1)}$. Here not only is the boundary control problem transformed into the distributed control problem but also the control α appears directly in the cost functional $\mathcal{J}$.

Assume that $\mathbf{u}^* = (\mathbf{w}^*, \alpha^*) \in \mathbf{H}_0^1(\Omega) \times U$ is a local optimal solution of (3.15) and that $\alpha^* \in \text{int}(U)$. Let the nonlinear mapping $M : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U \to (\mathbf{H}_0^1(\Omega))^* \times L_0^2(\Omega)$ be defined by $M(\mathbf{w}, p, \alpha) = (\bar{\mathbf{f}}, z)$ if and only if

$$a(\mathbf{w}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = <\bar{\mathbf{f}}, \mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.16}$$

$$b(\mathbf{w}, q) = <z, q> \quad \forall q \in L_0^2(\Omega). \tag{3.17}$$

Thus the constraints in the problem (3.15) can be expressed as $M(\mathbf{w}, p, \alpha) = (\bar{\mathbf{f}}, 0)$. In what follows we identify $\mathbf{u}$ with the pair $(\mathbf{w}, \alpha)$ whenever $\mathbf{u} = \mathbf{w} + \mathbf{u}_\epsilon^{(0)} + \alpha \mathbf{u}_\epsilon^{(1)}$.
Given $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the operator $M'(\mathbf{u}) \in \mathcal{L}(\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U; (\mathbf{H}_0^1(\Omega))^* \times L_0^2(\Omega))$ may be defined as follows: $M'(\mathbf{u}) \cdot (\mathbf{y}, r, \beta) = (\tilde{\mathbf{f}}, \tilde{z})$ if and only if

$$\begin{aligned} a(\mathbf{y}, \mathbf{v}) + c(\mathbf{y}, \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{y}, \mathbf{v}) + \beta(c(\mathbf{u}_\epsilon^{(1)}, \mathbf{u}, \mathbf{v}) \\ + c(\mathbf{u}, \mathbf{u}_\epsilon^{(1)}, \mathbf{v})) + b(\mathbf{v}, p) \\ = < \tilde{\mathbf{f}}, \mathbf{v} > - \beta a(\mathbf{u}_\epsilon^{(1)}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \end{aligned} \tag{3.18}$$

$$b(\mathbf{y}, q) = < \tilde{z}, q > \qquad \forall q \in L_0^2(\Omega), \tag{3.19}$$

where $< \tilde{\mathbf{f}}, v > = -a(\mathbf{u}_\epsilon^{(0)}, \mathbf{v})$. It follows from [11] that if the $M'(\mathbf{u}^*)$ which is the Fréchet derivative of M is onto, then the regular point conditions is satisfied and hence there exists a Lagrange multiplier $(\mathbf{d}, \phi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ satisfying the Euler equations

$$\begin{aligned} \mathcal{J}'(\mathbf{u}^*) \cdot (\mathbf{y}, r, \beta) + < (\mathbf{d}, \phi), M'(\mathbf{u}^*) \cdot (\mathbf{y}, r, \beta) > = 0 \\ \forall (\mathbf{y}, r, \beta) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U. \end{aligned} \tag{3.20}$$

LEMMA 3.2. *For* $\mathbf{u} \in \mathbf{H}^1(\Omega)$, *the operator* $M'(\mathbf{u})$ *from* $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U$ *into* $\mathbf{H}_0^1(\Omega))^* \times L_0^2(\Omega)$ *has closed range.*

Proof. For $\mathbf{u} \in \mathbf{H}^1(\Omega)$, it is easily seen that $M'(\mathbf{u})$ is a compact perturbation of the operator $S(\mathbf{u}) \in \mathcal{L}(\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U; (\mathbf{H}_0^1(\Omega))^* \times L_0^2(\Omega))$, where S is defined as follows : $S(\mathbf{u}) \cdot (\mathbf{y}, r, \beta) = (\tilde{\mathbf{f}}, \tilde{z})$ if and only if

$$\begin{aligned} a(\mathbf{y}, \mathbf{v}) + \beta(c(\mathbf{u}_\epsilon^{(1)}, \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}_\epsilon^{(1)}, \mathbf{v})) + b(\mathbf{v}, p) \\ = < \tilde{\mathbf{f}}, \mathbf{v} > - \beta a(\mathbf{u}_\epsilon^{(1)}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \end{aligned} \tag{3.21}$$

$$b(\mathbf{y}, q) = < \tilde{z}, q > \qquad \forall q \in L_0^2(\Omega), \tag{3.22}$$

The adjoint operator to $S(\mathbf{u})$ can be shown to be a semi-Fredholm operator, i.e., to have a closed range and a finite-dimensional kernel. Then it follows that $S(\mathbf{u})$ itself, and any compact perturbation of $S(\mathbf{u})$, has closed range; see [12]. □

By the similar technique in [5] and [7], we can easily have the following result.

LEMMA 3.3. *For* $\mathbf{u} \in \mathbf{H}^1(\Omega)$. *the operator* $M'(\mathbf{u})$ *from* $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U$ *into* $\mathbf{H}_0^1(\Omega))^* \times L_0^2(\Omega)$ *is onto.*

Under the condition in Lemma 3.3 we obtain the first-order necessary conditions for optimality:

$$a(\mathbf{w},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\bar{\mathbf{f}},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.23}$$

$$b(\mathbf{w},q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{3.24}$$

(3.25)
$$a(\mathbf{d},\mathbf{e}) + c(\mathbf{e},\mathbf{u},\mathbf{d}) + c(\mathbf{u},\mathbf{e},\mathbf{d}) + b(\mathbf{e},\psi) + J'(\mathbf{u})(\mathbf{e}) = 0 \quad \forall \mathbf{e} \in \mathbf{H}_0^1(\Omega),$$

$$b(\mathbf{d},r) = 0 \quad \forall r \in L_0^2(\Omega), \tag{3.26}$$

and

(3.27)
$$a(\mathbf{u}_\epsilon^{(1)},\mathbf{d}) + c(\mathbf{u}_\epsilon^{(1)},\mathbf{u},\mathbf{d}) + c(\mathbf{u},\mathbf{u}_\epsilon^{(1)},\mathbf{d}) + b(\mathbf{u}_\epsilon^{(1)},\psi) + J'(\mathbf{u})(\mathbf{u}_\epsilon^{(1)}) = 0.$$

where $\mathbf{u}_\epsilon^{(1)}$ is Hopf's function.

4. Feedback control procedure

Let us consider the boundary optimal control problem:
Find $(\mathbf{u},p,\mathbf{g}) \in \mathbf{H}^1(\Omega) \times L_0^2 \times U$ which minimizes cost functional $\mathcal{J}(\mathbf{u})$ subject to

$$a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

$$b(\mathbf{u},q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{4.1}$$

$$\mathbf{u} = \mathbf{g}_0 + \mathbf{g} \quad \text{on } \Gamma,$$

where $\mathbf{g}_0 \in \mathbf{H}^{1/2}$ is given. The best optimal control $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ can be obtained throught the adjoint state and some algorithms. Feedback theory involves constructing $\mathbf{g}$ as a function of $(\mathbf{u},p)$ or some observation of $(\mathbf{u},p)$(see, e.g.,[4], [10]). Althought feedback schemes are mainly relevant to time-dependent problems, we formulate such a scheme here. Using the theory of feedback control, we can transform the infinite dimensional optimal control problem to the finite dimensional one.

Finding for the best feedback in a prescribed class of feedbacks, we reduce the problem to a parameter optimization one. For instance, we could look for

$$\mathbf{g} = \sum_{i=1}^{n} \alpha_i \mathbf{g}_i(\mathbf{u}). \tag{4.2}$$

where $\mathbf{g}_1, \ldots, \mathbf{g}_n$, are the shape functions. Actually the control problem (3.2) is this kind one. The shape functions $\mathbf{g}_i$ can be obtained from physical intuition or experience and α_i are determined through a control algorithm and thus have an implicit dependence on $(\mathbf{u}, p)$.

Now let us consider the problem (3.2). The gradient algorithm consists of constructing sequence α^k, reculsively defined by

$$\alpha^{k+1} - \alpha^k = -\rho^k \frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\alpha}(\alpha^k) \tag{4.3}$$

Note that the parameters of decent $\rho^k > 0$ are chosen differently in the iteration (see, e.g. [9]).

The gradient algorithm with variable step lengths ρ^k is given as follows (Note we will use a weak form instead of a strong form; the former can be conveniently discretized using finite element methods.)

1) choose an initial guess $\alpha^{(1)}$;

2) for each $n \geq 0$,

solve for $(\mathbf{u}^{(n)}, p^{(n)}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ from

$$a(\mathbf{u}^{(n)}, \mathbf{v}) + c(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v}) + b(\mathbf{v}, p^{(n)}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{4.4}$$

$$b(\mathbf{u}^{(n)}, q) = 0 \quad \forall q \in L^2_0(\Omega), \tag{4.5}$$

and

$$\mathbf{u}^{(n)} = \mathbf{g}_0 + \alpha^{(n)} \mathbf{g}_1 \quad \text{on } \Gamma, \tag{4.6}$$

solve for $(\mathbf{d}^{(n)}, \psi^{(n)}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ from

$$\begin{aligned} a(\mathbf{d}^{(n)}, \mathbf{e}) + c(\mathbf{e}, \mathbf{u}^{(n)}, \mathbf{d}^{(n)}) + c(\mathbf{u}^{(n)}, \mathbf{e}, \mathbf{d}^{(n)}) + b(\mathbf{e}, \psi^{(n)}) \\ + J'(\mathbf{u}^{(n)})(\mathbf{e}) = 0 \quad \forall \mathbf{e} \in \mathbf{H}^1_0(\Omega), \end{aligned} \tag{4.7}$$

$$b(\mathbf{d}^{(n)}, r) = 0 \quad \forall r \in L^2_0(\Omega), \tag{4.8}$$

and

$$\mathbf{d}^{(n)} = \mathbf{0} \quad \text{on } \Gamma, \tag{4.9}$$

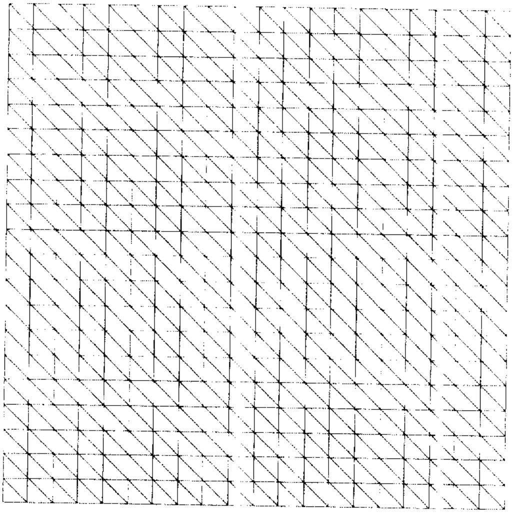

FIGURE 2. Triangulation for simulating flow in a cavity

and solve for $\alpha^{(n+1)} \in U$ from

$$\alpha^{(n+1)} = \alpha^{(n)} + \rho^{(n)}(2\alpha^{(n)}c(\mathbf{u}_\epsilon^{(1)}, \mathbf{u}_\epsilon^{(1)}, \mathbf{d}) + c(\mathbf{u}_\epsilon^{(1)}, \mathbf{w} + \mathbf{u}_\epsilon^{(0)}, \mathbf{d})$$

$$+c(\mathbf{w} + \mathbf{u}_\epsilon^{(0)}, \mathbf{u}_\epsilon^{(1)}, \mathbf{d}) + a(\mathbf{u}_\epsilon^{(1)}, \mathbf{d}) + b(\mathbf{u}_\epsilon^{(1)}, \psi^{(n)})). \tag{4.10}$$

5. Numerical results

In this section we discuss numerical solution of the optimal control problem formulated in section 3. The solution to the boundary optimal control problem (3.2) is obtained using the gradient algorithm formulated in section 4. To carry out the computation we discretized the problem using the mixed finite element method [6]. In our calculation the piecewise quadratic element for the velocity and the linear element for the pressure over the triangular grid with mesh size $h = 0.05$ are used (see Fig. 2).

Let $\{\mathbf{s}_h^i\}$, $\{\chi_h^i\}$ be the linearly independent basis functions of the finite-dimensional subspaces $\mathbf{V}_h \subset \mathbf{H}^1(\Omega)$ and $S_h \subset L^2(\Omega)$ for velocity and pressure, respectively. Then the approximation problem for (3.23)-(3.27) is given as follows.

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in S_h$ and $\alpha \in U$ such that

$$a(\mathbf{w}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\bar{\mathbf{f}}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h(\Omega), \tag{5.1}$$

$$b(\mathbf{w}_h, q_h) = 0 \quad \forall q \in S_h(\Omega), \tag{5.2}$$

$$\begin{aligned} a(\mathbf{d}_h, \mathbf{e}_h) \ &+ c(\mathbf{e}_h, \mathbf{u}_h, \mathbf{d}_h) + c(\mathbf{u}_h, \mathbf{e}_h, \mathbf{d}_h) + b(\mathbf{e}_h, \psi_h) \\ &+ J'(\mathbf{u}_h)(\mathbf{e}_h) = 0 \qquad \forall \mathbf{e}_h \in \mathbf{V}_h(\Omega), \end{aligned} \tag{5.3}$$

$$b(\mathbf{d}_h, r_h) = 0 \quad \forall r_h \in S_h, \tag{5.4}$$

and

$$\begin{aligned} a(\mathbf{u}_\epsilon^{(1)}, \mathbf{d}_h) \ &+ \ c(\mathbf{u}_\epsilon^{(1)}, \mathbf{u}, \mathbf{d}_h) + c(\mathbf{u}_h, \mathbf{u}_\epsilon^{(1)}, \mathbf{d}_h) + b(\mathbf{u}_\epsilon^{(1)}, \psi) \\ &+ \ J'(\mathbf{u}_h)(\mathbf{u}_\epsilon^{(1)}) = 0. \end{aligned} \tag{5.5}$$

where $(\bar{\mathbf{f}}_h, \mathbf{v}_h) = -a(\mathbf{u}_\epsilon^{(0)} + \alpha \mathbf{u}_\epsilon^{(1)}, \mathbf{v}_h)$ and $\mathbf{u}_\epsilon^{(1)}$ is Hopf's function.

We performed two numerical experiments. First, we take the Reynolds number to be 50 ($\nu = 1/50$) and Γ_L to be the horizontal line 0.4 unit from the bottom. Given the bottom velocity as 0.5 we obtained the top velocity $U_{top}^{opt} = 0.98$. Directional velocity plots for the $\mathbf{u}_\epsilon^{(0)}$ and $\mathbf{u}_\epsilon^{(1)}$ which may be obtained from the two auxiliary Stokes problems are given Fig. 3. Second, we take the Reynolds number to be 10 ($\nu = 1/10$) and Γ_L to be the horizontal line 0.4 unit from the bottom. Given the bottom velocity as 0.5 we obtained the top velocity $U_{top}^{opt} = 0.99$. The resulting flow fields obtained using the optimal control method are shown in Fig. 4. The costs are shown in Fig. 5 for $Re = 50$ and $Re = 10$. From Fig. 5 one can know that separation occurs at a fixed horizontal line for relatively low values of Re and so one-dimensional Dirichlet control may be adequate for low Reynolds number which agrees with [3]. We suggest that for high Reynolds number we need multidimensional Dirichlet boundary control. Such optimal control problems are the subject of a forthcoming paper.

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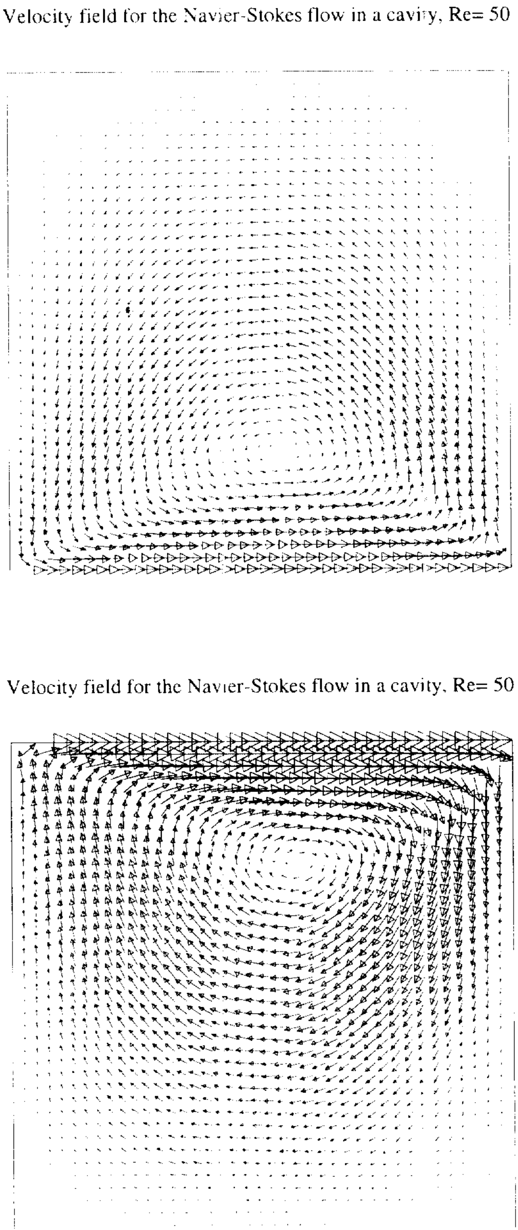
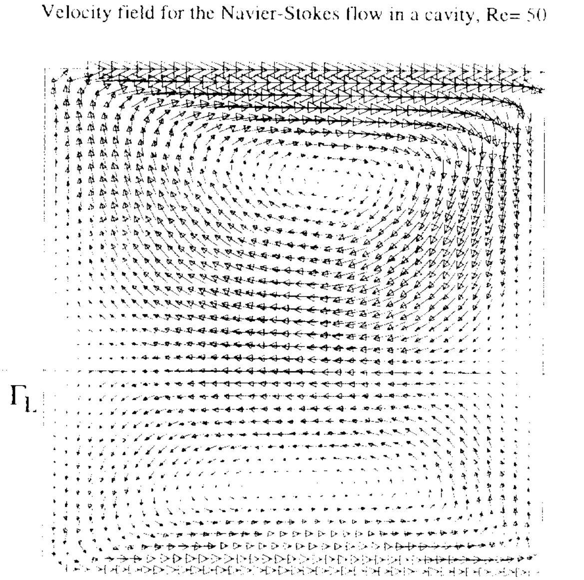


FIGURE 3. Velocity field for auxiliary Stokes problems : top for $\mathbf{u}_\epsilon^{(0)}$, bottom for $\mathbf{u}_\epsilon^{(1)}$



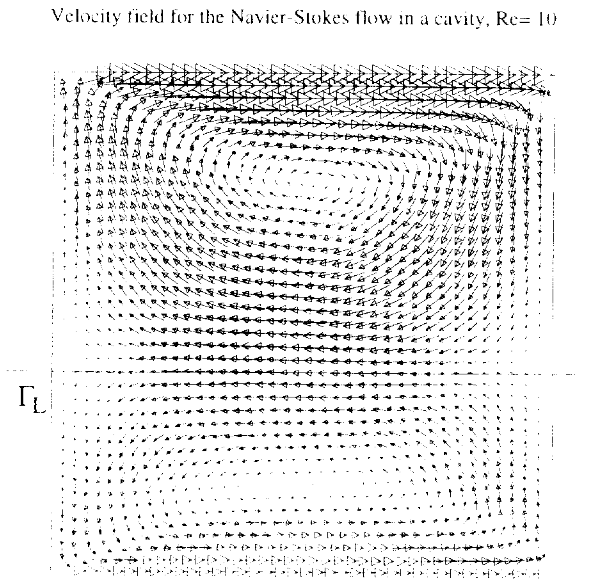


FIGURE 4. Locating separation line at $y = 0.4$: top for $Re = 50$ and bottom for $Re = 10$

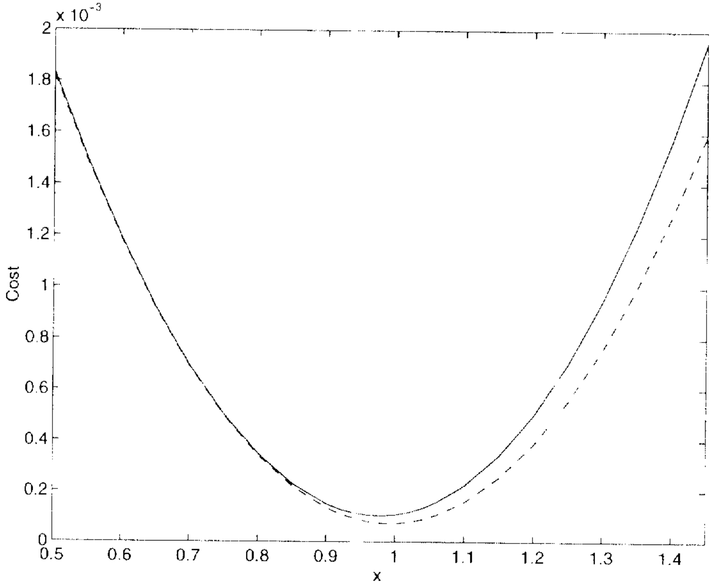


FIGURE 5. Cost : — for $Re = 50$ and - - for $Re = 10$

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