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ON THE MINIMUM PERMANENTS RELATED WITH CERTAIN BARYCENTRIC MATRICES

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ABSTRACT. The permanent function on certain faces of the polytope of doubly stochastic matrices are studied. These faces are shown to be barycentric, and the minimum values of the permanent are determined.

1. Introduction and preliminaries

Let Ω_n be the polyhedron of $n \times n$ doubly stochastic matrices, that is, the n by n nonnegative matrices whose row and column sums are all equal to 1. Let per(A) be the permanent of matrix A and let $J_{r,s}$ denote the $r \times s$ matrix all of whose entries are 1. In 1981 Egorycev [3] and Falikman [4] proved the van der Waerden permanent conjecture: If $A \in \Omega_n$, then

$$\mathrm{per}(A) \geq \mathrm{per}(\frac{1}{n} J_{n,n}).$$

The techniques of Egorycev have been used, with some success, for determination of minimum permanents in various faces of Ω_n(See [7] -[10]). The key technique is replacing rows(or columns) of a matrix with minimum permanent by their average without altering its permanent. Unfortunately, the presence of fixed zeros restricts the use of this technique. Indeed this tool is not available at all in the case of faces which consist of matrices with at least one fixed zero in each row and column. In this paper we use this technique in some parts of proofs.

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Let $D = [d_{ij}]$ be an n-square nonnegative matrix, and let

$$\Omega(D) = \{X = [x_{ij}] \in \Omega_n | x_{ij} = 0 \text{ whenever } d_{ij} = 0\}.$$

Then $\Omega(D)$ is a face of Ω_n, and since it is compact, $\Omega(D)$ contains a minimizing matrix A such that $\text{per}(A) \leq \text{per}(X)$ for all $X \in \Omega(D)$.

Brualdi [1] defined an n-square (0,1) matrix D to be *cohesive* if there is a matrix Z in the interior of $\Omega(D)$ for which

$$\text{per}(Z) = \min\{\text{per}(X) | X \in \Omega(D)\}.$$

And he defined an n-square (0,1) matrix D to be *barycentric* if

$$\text{per}(b(D)) = \min\{\text{per}(X) | X \in \Omega(D)\},$$

where the *barycenter* $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\text{per}(D)} \sum_{P \leq D} P,$$

where the summation extends over the set of all permutation matrices P with $P \leq D$ and $\text{per}(D)$ is their number.

Let I_n denote the identity matrix of order n and 0_k the $k \times k$ zero matrix.

In [10], Song considered a matrix $V_{m,n} = \begin{bmatrix} J_{m,m} & J_{m,n} \\ J_{n,m} & I_n \end{bmatrix}$ and suggested determining the minimum permanents and minimizing matrices on $\Omega(V_{m,n})$ for $m \geq 2$, and $n \geq 3$. This face $\Omega(V_{m,n})$ is extended one of $\Omega(W_n)$ in Theorem 5 in [1].

Brualdi [1] determined the minimum permanent and minimizing matrix on $\Omega(V_{1,n-1})$. Song determined the minimum permanents on $\Omega(V_{2,n})$ in [8] and on $\Omega(V_{m,3})$ in [9], respectively.

In this paper, we consider the faces $\Omega(V_{3,n})$ and $\Omega(U_{3,n})$,where $U_{m,n} = \begin{bmatrix} 0_m & J_{m,n} \\ J_{n,m} & I_n \end{bmatrix}$. We show that $U_{3,n}$ is both cohesive and barycentric and determine the minimum permanents and minimizing matrices on the face $\Omega(V_{3,n})$ of Ω_{3+n} for $n \geq 5$.

Recall that an n-square nonnegative matrix is said to be *fully indecomposable* if it contains no $k \times (n-k)$ zero submatrix for $k = 1, \cdots, n-1$.

We use the following well-known Lemma([5] or [6]).

LEMMA 1.1. *Let $D = [d_{ij}]$ be an n-square fully indecomposable $(0,1)$ matrix, and $A = [a_{ij}]$ be a minimizing matrix on $\Omega(D)$. Then A is fully indecomposable, and for (i,j) such that $d_{ij} = 1$,*

$$\begin{aligned} \operatorname{per}(A(i|j)) &= \operatorname{per}(A) \quad \text{if } a_{ij} > 0, \\ \operatorname{per}(A(i|j)) &\geq \operatorname{per}(A) \quad \text{if } a_{ij} = 0. \end{aligned}$$

2. The cohesiveness of $\mathbf{U}_{3,n}$

In this section, we consider the $(3+n)$-square $(0,1)$ matrix $U_{3,n}$ and we show that this matrix is cohesive for $n \geq 5$.

If a column j of an $n \times n$ matrix A contains exactly k nonzero entries $(2 \leq k \leq n)$, say in rows $r_1, \ldots, r_k$, then the $(n-1)$ square matrix $GC(A)$ obtained from A by replacing rows $r_2, \ldots, r_k$ with $\frac{1}{k-1}(r_1 + r_2 + \ldots + r_k)$ and deleting row r_1 and column j is called a *generalized contraction* of A.

LEMMA 2.1. *If an $n \times n$ nonnegative matrix A is fully indecomposable, so is $GC(A)$.*

Proof. It suffices to consider the case where $GC(A)$ is the generalized contraction of A on column 1 relative to rows $1, 2, \ldots, k$. Thus A and $GC(A)$ have the form

$$A = \begin{bmatrix} a_{11} & \alpha \\ a_{21} & \beta \\ \vdots & \vdots \\ a_{k1} & \gamma \\ 0 & C \end{bmatrix}, \quad GC(A) = \begin{bmatrix} \frac{1}{k-1}(\alpha + \beta + \ldots + \gamma) \\ \vdots \\ \vdots \\ \frac{1}{k-1}(\alpha + \beta + \ldots + \gamma) \\ C \end{bmatrix}.$$

where $a_{j1} \neq 0$ for $j = 1, 2, \ldots, k$. Suppose $GC(A)$ is not fully indecomposable. Then there exists an $r \times s$ zero submatrix $0_{r,s}$ of $GC(A)$ where $r + s = n - 1$. If $0_{r,s}$ is a submatrix of C, then clearly A has an $r \times (s+1)$ zero submatrix where $r + (s+1) = n$. Hence in this case A is not fully indecomposable. Suppose $0_{r,s}$ is not a submatrix of C. Since a_{i1}'s are positive while $\alpha, \beta, \ldots, \gamma$ are nonnegative, A has an $(r+1) \times s$ zero submatrix where $(r+1) + s = n$. Therefore A is not fully indecomposable. This contradiction implies that $GC(A)$ is fully indecomposable. □

LEMMA 2.2. *Suppose $D \in \Omega_n$ is fully indecomposable and has a column (row) with exactly k positive entries and those k rows (columns) have the same zero pattern. Let A be a minimizing matrix on $\Omega(D)$. Then there is a minimizing matrix $\overline{GC(A)} \in \Omega(GC(A))$ satisfying*

$$\operatorname{per}(A) = (\frac{k-1}{k})^{k-1}\operatorname{per}(GC(A)) \geq (\frac{k-1}{k})^{k-1}\operatorname{per}(\overline{GC(A)}).$$

Proof. It suffices to consider the case where $GC(A)$ is the generalized contraction of A on column 1 relative to rows $1, 2, \ldots, k$. Using the averaging method (see [3] or [4]) on the first k rows of A, A has the following form

$$A = \begin{bmatrix} x & \mathbf{a} \\ \vdots & \vdots \\ x & \mathbf{a} \\ 0 & \\ \vdots & B \\ 0 & \end{bmatrix},$$

where x is not zero, $\mathbf{a}$ is a $1 \times (n-1)$ matrix and B is an $(n-k) \times (n-1)$ matrix. Hence

$$\begin{aligned} \operatorname{per}(A) &= \operatorname{per}(A(1|1)) \\ &= \operatorname{per}(\begin{bmatrix} \mathbf{a} \\ \vdots \\ \mathbf{a} \\ B \end{bmatrix}) = (\frac{k-1}{k})^{k-1}\operatorname{per}(\begin{bmatrix} \frac{1}{k-1}(\mathbf{a} + \ldots + \mathbf{a}) \\ \vdots \\ \frac{1}{k-1}(\mathbf{a} + \ldots + \mathbf{a}) \\ B \end{bmatrix}) \\ &= (\frac{k-1}{k})^{k-1}\operatorname{per}(GC(A)) \geq (\frac{k-1}{k})^{k-1}\operatorname{per}(\overline{GC(A)}). \end{aligned}$$

□

LEMMA 2.3. ([7]) *For $n \geq 3$, $U_{2,n} = \begin{bmatrix} 0_2 & J_{2,n} \\ J_{n,2} & I_n \end{bmatrix}$ is barycentric and the minimum permanent on $\Omega(U_{2,n})$ is $\frac{2(n-1)(n-2)^{n-2}}{n^{n+1}}$.*

THEOREM 2.4. *For $n \geq 5$, $U_{3,n}$ is a cohesive matrix.*

Proof. Let $B_{3,n}$ be a minimizing matrix on $\Omega(U_{3,n})$. Since the first 3 columns and first 3 rows of $U_{3,n}$ are the same, we can use the averaging method on those columns and rows of $B_{3,n}$, respectively. Thus we can write $B_{3,n}$ as follows:

$$B_{3,n} = \left[\begin{array}{c|cccc} 0_3 & \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \dots & \bar{\mathbf{b}}_n \\ \hline \bar{\mathbf{b}}_1^t & x_1 & & & 0 \\ \bar{\mathbf{b}}_2^t & & x_2 & & \\ \vdots & 0 & & \ddots & \\ \bar{\mathbf{b}}_n^t & & & & x_n \end{array}\right], \tag{2.1}$$

where $\bar{\mathbf{b}}_i(\bar{\mathbf{b}}_i^t)$ is a column (row) vector with b_i as all its entries for $i = 1, 2, \cdots, n$.

Since the permanent value is invariant under the interchange of rows (and columns, respectively), we may assume that

$$b_{i+1} \leq b_i \quad (\text{i.e.} \quad x_{i+1} \geq x_i) \tag{2.2}$$

for $i = 1, 2, \cdots, n-1$, without loss of generality. Since $U_{3,n}$ is fully indecomposable, each

$$b_i \neq 0 \tag{2.3}$$

for $i = 1, \cdots, n$. Suppose $x_1 = 0$. Then the fourth row and column of $B_{3,n}$ have exactly 3 nonzero entries. Thus we can obtain a generalized contraction $GC(B_{3,n})$ of $B_{3,n}$. Since the third row of $GC(B_{3,n})$ has exactly 3 nonzero entries, we can obtain its generalized contraction $GC(GC(B_{3,n}))$, which is contained in $\Omega(U_{2,n-1})$. Using Lemmas 2.2 and 2.3, we have

$$\begin{aligned} \text{per}(B_{3,n}) &= (\frac{2}{3})^2 \text{per}(GC(B_{3,n})) \\ &\geq (\frac{2}{3})^2 \{(\frac{2}{3})^2 b(U_{2,n-1})\} = (\frac{2}{3})^4 \cdot \frac{2(n-2)(n-3)^{n-3}}{(n-1)^n}. \end{aligned} \tag{2.4}$$

But $\text{per}(B_{3,n})$ is less than or equal to the permanent of the barycenter $b(U_{3,n})$. That is,

$$
\begin{aligned}
&\text{per}(B_{3,n}) \leq \text{per}(b(U_{3,n})) \\
(2.5) \qquad &= 3 \cdot \frac{1}{n}[4(\frac{1}{n})^2(\frac{1}{n})^2(\frac{n-3}{n})^{n-3} \times \frac{(n-1)(n-2)}{2}] \\
&= 6 \cdot \frac{(n-3)^{n-3} \cdot (n-1)(n-2)}{n^{n+2}}.
\end{aligned}
$$

If we divide the value in (2.5) by the value in (2.4), then we can show that the result is less than 1 by a direct calculation for $5 \leq n \leq 15$ and from $(\frac{3^4 \cdot 3}{2^4 \cdot n})(\frac{n-1}{n})^{n+1} < 1 \cdot (\frac{n-1}{n})^{n+1} < 1$ for $n \geq 16$. Thus we have a contradiction from the inequalities in (2.4) and (2.5). Hence x_1 is not zero. By (2.2), each

$$(2.6) \qquad x_i \neq 0$$

for $i = 1, \cdots, n$. Hence $U_{3,n}$ is cohesive by (2.3) and (2.6). □

3. Minimum permanents on $\Omega(\mathbf{U}_{3,n})$

In this section, we show that the face $\Omega(U_3, n)$ is barycentric for $n \geq 5$.

For a matrix A, let $A(i, j, \cdots, k|l, m, \cdots, n)$ denote the submatrix obtained from A by deleting rows $i, j, \cdots, k$, and columns $l, m, \cdots, n$. In particular, we simplify the notation $A(i, j, \cdots, k|i, j, \cdots, k)$ to $A(i, j, \cdots, k)$.

THEOREM 3.1. *For $n \geq 5$, the minimum permanent on $\Omega(U_{3,n})$ is*

$$(3.1) \qquad 6 \cdot \frac{(n-3)^{n-3} \cdot (n-1) \cdot (n-2)}{n^{n+2}},$$

which occurs at the barycenter.

Proof. Let $B_{3,n}$ be a minimizing matrix on $\Omega(U_{3,n})$. Then $B_{3,n}$ has the form of $B_{3,n}$ in (2.1) as the proof of Theorem 2.4. Without loss of generality, we also assume

$$(3.2) \qquad b_{i+1} \leq b_i \ \text{ (i.e. } x_{i+1} \geq x_i)$$

for $i = 1, 2, \ldots, n-1$. As the sum of b_i's is 1 and the b_i's are positive, at most k of the b_i's are greater than or equal to $\frac{1}{k}$. Thus $b_k < \frac{1}{k}$ for $k = 1, 2, \cdots, n-1$, and $b_n \le \frac{1}{n}$. Hence we have

$$x_k > (k-3)b_k \tag{3.3}$$

for all k with $4 \le k \le n-1$, and

$$x_n \ge (n-3)b_n. \tag{3.4}$$

Since b_4 and b_n are positive, we have

(3.5)

$$\begin{aligned}
0 &= \mathrm{per}(B_{3,n}(1|7)) - \mathrm{per}(B_{3,n}(1|n+3)) \\
&= 3b_4\{4b_n^2\mathrm{per}(B_{3,n}(1,7,n+3,2)) + x_n\mathrm{per}(B_{3,n}(1,7,n+3))\} \\
&\quad - 3b_n\{4b_4^2\mathrm{per}(B_{3,n}(1,n+3,7,2)) + x_4\mathrm{per}(B_{3,n}(1,n+3,7))\} \\
&= 12b_4b_n(b_n - b_4)\mathrm{per}(B_{3,n}(1,2,7,n+3)) \\
&\quad + 3(b_4x_n - b_nx_4)\mathrm{per}(B_{3,n}(1,7,n+3)) \\
&= 12b_4b_n(b_n - b_4)\Big(\sum_{\substack{i=1 \\ i\neq 4}}^{n-1} b_i^2 x_1x_2x_3x_5x_6\cdots x_{n-1}/x_i\Big) \\
&\quad + 3\{b_4(1-3b_n) - b_n(1-3b_4)\} \\
&\quad \Big\{\sum_{\substack{i=1 \\ i\neq 4}}^{n-1} 2b_i^2\mathrm{per}(B_{3,n}(1,2,7,i+3,n+3))\Big\} \\
&= 6(b_n - b_4)\Big[2b_4b_n\Big(\sum_{\substack{i=1 \\ i\neq 4}}^{n-1} b_i^2 x_1x_2x_3x_5x_6\cdots x_{n-1}/x_i\Big) \\
&\quad - \sum_{\substack{i=1 \\ i\neq 4}}^{n-1} b_i^2\Big(\sum_{\substack{j=1 \\ j\neq 4,i}}^{n-1} b_j^2 x_1x_2x_3x_5x_6\cdots x_{n-1}/(x_ix_j)\Big)\Big].
\end{aligned}$$

Since each x_i is positive by Theorem 2.4, the quantity in the large bracket in (3.5) is less than

$$\begin{aligned}
&b_1^2\{(b_4b_nx_2 - b_2^2)x_3x_5x_6\cdots x_{n-1} + (b_4b_nx_3 - b_3^2)x_2x_5x_6\cdots x_{n-1}\} \\
&+ b_2^2\{(b_4b_nx_1 - b_1^2)x_3x_5x_6\cdots x_{n-1} + (b_4b_nx_3 - b_3^2)x_1x_5x_6\cdots x_{n-1}\} \\
&+ \sum_{\substack{i=3\\ i\neq 4}}^{n-1} b_i^2\{(b_4b_nx_1 - b_1^2)x_2x_3x_5x_6\cdots x_{n-1}/x_i \\
&+ (b_4b_nx_2 - b_2^2)x_1x_3x_5x_6\cdots x_{n-1}/x_i\},
\end{aligned}$$

which is negative because

$$b_4b_nx_i - b_i^2 < b_ib_i\cdot 1 - b_i^2 = 0$$

for $i = 1, 2$ and 3, where the inequality comes from (3.2) and the fact that $x_i < 1$. Hence we have $b_4 = b_n$ from (3.5). Using (3.2), we have that

$$b_i = b_4 \quad (\text{and hence} \quad x_i = x_n) \tag{3.6}$$

for all i with $4 \leq i \leq n$.

Suppose to the contrary that

$$(\frac{2}{n-3})b_1 > b_4. \tag{3.7}$$

Since x_1 and x_4 are positive, we have

(3.8)

$$\begin{aligned}
0 &= \mathrm{per}(B_{3,n}(4|4)) - \mathrm{per}(B_{3,n}(7|7)) \\
&= (3b_4)^2\mathrm{per}(B(4,7,1)) + x_4\mathrm{per}(B(4,7)) \\
&\quad - \{(3b_1)^2\mathrm{per}(B(7,4,1)) + x_1\mathrm{per}(B(7,4))\} \\
&= 9(b_4^2 - b_1^2)\mathrm{per}(B(1,4,7)) + \{(1-3b_4) - (1-3b_1)\}\mathrm{per}(B(4,7)) \\
&= 3(b_4 - b_1)\{3(b_1+b_4)\mathrm{per}(B(1,4,7)) - \mathrm{per}(B(4,7))\}.
\end{aligned}$$

Using (3.6) in the calculation for the value of the braces in (3.8), we have

$$
\begin{aligned}
&3(b_1+b_4)\mathrm{per}(B(1,4,7)) - \mathrm{per}(B(4,7))\\
&\quad = 3(b_1+b_4)\{4b_2^2b_3^2x_4^{n-4} + 4(n-4)b_2^2b_4^2x_3x_4^{n-5}\\
&\qquad + 4(n-4)b_3^2b_4^2x_2x_4^{n-5} + 2(n-5)(n-4)b_4^4x_2x_3x_4^{n-6}\}\\
&\qquad - 6(n-4)\{6b_2^2b_3^2b_4^2x_4^{n-5} + 3(n-5)b_2^2b_4^4x_3x_4^{n-6}\\
&\qquad + 3(n-5)b_3^2b_4^2x_2x_4^{n-6} + (n-5)(n-6)b_4^6x_2x_3x_4^{n-7}\}\\
(3.9)\quad &\quad = x_4^{n-7}[12b_2^2b_3^2x_4^2\{(b_1+b_4)x_4 - 3(n-4)b_4^2\}\\
&\qquad + 12(n-4)b_2^2b_4^2x_3x_4\{(b_1+b_4)x_4 - \frac{3}{2}(n-5)b_4^2\}\\
&\qquad + 12(n-4)b_3^2b_4^2x_2x_4\{(b_1+b_4)x_4 - \frac{3}{2}(n-5)b_4^2\}\\
&\qquad + 6(n-5)(n-4)b_4^4x_2x_3\{(b_1+b_4)x_4 - (n-6)b_4^2\}].
\end{aligned}
$$

But
(3.10)

$$
\begin{aligned}
\{(b_1+b_4)x_4 - 3(n-4)b_4^2\} &> \{(\frac{n-3}{2})b_4 + b_4\}x_4 - 3(n-4)b_4^2\\
&\geq b_4\{(\frac{n-1}{2})(n-3)b_4 - 3(n-4)b_4\}\\
&> 0
\end{aligned}
$$

for $n \geq 5$, where the first inequality comes from (3.7) and the second comes from (3.4) and (3.6). Thus the four braces in (3.9) are positive by (3.10). This implies that $b_4 = b_1$ in (3.8), which contradicts (3.7). Therefore we have

$$
(3.11)\qquad \frac{2}{n-3}b_1 \leq b_4.
$$

Then the similar method as (3.5) gives

$$
\begin{aligned}
&0 = \mathrm{per}(B_{3,n}(1|4)) - \mathrm{per}(B_{3,n}(1|n+3))\\
(3.12)\quad &\quad = 6(b_n - b_1)[2b_1b_n(\sum_{i=2}^{n-1} b_i^2x_2x_3\cdots x_{n-1}/x_i)
\end{aligned}
$$

$$- \sum_{i=2}^{n-1} b_i^2 \{ \sum_{\substack{j=2 \\ j\neq i}}^{n-1} b_j^2 x_2 x_3 \cdots x_{n-1}/(x_i x_j)\}]$$

$$= 6(b_n - b_1)[\sum_{i=2}^{n-1} b_i^2 \{ \sum_{\substack{j=2 \\ j\neq i}}^{n-1} (\frac{2}{n-3} b_1 b_n x_j - b_j^2) x_2 x_3 \cdots x_{n-1}/(x_i x_j)\}].$$

But the quantity in the second parenthesis in (3.12) is negative because

$$\frac{2}{n-3} b_1 b_n x_j - b_j^2 \leq b_4 b_n x_j - b_j^2 < b_j b_j 1 - b_j^2 = 0$$

for all $j = 2, \cdots, n-1$, where the first inequality comes from (3.11) and the second comes from (3.2), (3.6) and the fact that $x_j < 1$. Thus the quantity in the large bracket in (3.12) is negative, which implies that $b_1 = b_n$. From (3.2), all b_i and x_i are the same, respectively. Thus $B_{3,n}$ with each $b_i = \frac{1}{n}$ and $x_i = \frac{n-3}{n}$ is a minimizing matrix on $\Omega(U_{3,n})$, and it is the barycenter of $\Omega(U_{3,n})$. Moreover, the minimum permanent is

$$\begin{aligned} \text{per}(b(\Omega(U_{3,n}))) &= \text{per}(B_{3,n}(1|4)) \\ &= 3 \cdot \frac{1}{n}[4(\frac{1}{n})^2(\frac{1}{n})^2(\frac{n-3}{n})^{n-3} \times \frac{(n-1)(n-2)}{2}] \\ &= 6 \cdot \frac{(n-3)^{n-3} \cdot (n-1)(n-2)}{n^{n+2}}, \end{aligned}$$

as required in (3.1). □

4. Minimum permanents on $\Omega(V_{3,n})$

In this section, we determine the minimum permanents and minimizing matrices on the faces $\Omega(V_{3,n})$ for $n \geq 5$. For our purpose, we use the faces $\Omega(U_{3,n})$ in section III.

THEOREM 4.1. *For $n \geq 6$, the minimum permanent on $\Omega(V_{3,n})$ is the value in (3.1), which occurs at the barycenter of $\Omega(U_{3,n})$.*

Proof. Let

$$(4.1) \qquad A_{3,n} = \left[\begin{array}{c|cccc} aJ_{3,3} & \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \dots & \bar{\mathbf{b}}_n \\ \hline \bar{\mathbf{b}}_1^t & x_1 & & & 0 \\ \bar{\mathbf{b}}_2^t & & x_2 & & \\ \vdots & 0 & & \ddots & \\ \bar{\mathbf{b}}_n^t & & & & x_n \end{array}\right]$$

be a minimizing matrix on $\Omega(V_{3,n})$. Without loss of generality, we may assume that

$$(4.2) \qquad b_{i+1} \le b_i \ \ (\text{i.e. } x_{i+1} \ge x_i)$$

for $i = 1, \dots, n-1$. Then we have

$$(4.3) \qquad x_n \ge (n-3)b_n$$

by the similar method as (3.4). Since $V_{3,n}$ is fully indecomposable, each b_i and each x_i are positive.

Suppose to the contrary that

$$a \ne 0.$$

Then we have

(4.4)

$$\begin{aligned} 0 =& \operatorname{per}(A_{3,n}(1|1)) - \operatorname{per}(A_{3,n}(1|n+3)) \\ =& \{2b_n \operatorname{per}(A_{3,n}(1,2|1,n+3)) + x_n \operatorname{per}(A_{3,n}(1,n+3|1,n+3))\} \\ & - 3b_n \operatorname{per}(A_{3,n}(1,n+3|1,n+3)) \\ =& (2b_n)^2 \operatorname{per}(A_{3,n}(1,2,n+3)) + (x_n - 3b_n)\operatorname{per}(A_{3,n}(1,n+3)). \end{aligned}$$

Case 1) $n = 6$. From (4.3), we have $x_6 \ge 3b_6$. Since $(2b_6)^2$ and $\operatorname{per}(A_{3,6}(1,6))$ are positive, we must have $\operatorname{per}(A_{3,6}(1,2,6)) = 0$ and

$(x_6 - 3b_6) = 0$ from (4.4). Thus $x_6 = 3b_6 = \frac{1}{2}$. But then we have a contradiction as follows:

$$1 = \sum_{i=1}^{6} b_i + 3a \geq 6b_6 + 3a = 1 + 3a > 1.$$

This contradiction implies that $a = 0$.
Case 2) $n \geq 7$. From (4.3), we have $x_n \geq (n-3)b_n > 3b_n$. Since $\mathrm{per}(A_{3,n}(1, n+3)) > 0$, the last term in (4.4) is positive. Then we have a contradiction in (4.4), which implies that $a = 0$.

Thus, for $n \geq 6$, the minimizing matrix on $\Omega(V_{3,n})$ becomes the matrix $A_{3,n}$ with $a = 0$ in (4.1). Therefore a minimizing matrix on $\Omega(V_{3,n})$ is the barycenter $b(U_{3,n})$ of $\Omega(U_{3,n})$ by Theorem 3.1, and the minimum permanent on $\Omega(V_{3,n})$ is the value in (3.1), as required. □

THEOREM 4.2. *The minimum permanent on $\Omega(V_{3,5})$ is the value in (3.1) with $n = 5$, which occurs at the barycenter of $\Omega(U_{3,5})$.*

Proof. Let $A_{3,5}$ be a minimizing matrix on $\Omega(V_{3,5})$. Then $A_{3,5}$ is the form in (4,1) with n=5. Without loss of generality, we may assume that

$$b_{i+1} \leq b_i \ \ (\text{i.e. } x_{i+1} \geq x_i) \tag{4.5}$$

for $i = 1, \cdots, 4$. Then we have

$$x_5 \geq 2b_5 \tag{4.6}$$

by the similar method as (3.4). Since $V_{3,5}$ is fully indecomposable, b_6 (and hence all b_i) and x_3 (and hence x_4, x_5) are positive.

Assume that a is not zero. Then we have

$$\begin{aligned} 0 &= \mathrm{per}(A_{3,5}(1|1)) - \mathrm{per}(A_{3,5}(1|8)) \\ &= (2b_5)^2 \mathrm{per}(A_{3,5}(1,2,8)) + (x_5 - 3b_5)\mathrm{per}(A_{3,5}(1,8)). \end{aligned} \tag{4.7}$$

Since

$$\begin{aligned} \mathrm{per}(A_{3,5}(1,8)) &= (2b_4)^2 \mathrm{per}(A_{3,5}(1,2,7,8)) + x_4 \mathrm{per}(A_{3,5}(1,7,8)) \\ &= x_4(4b_1^2)b_2^2 x_3 + (\text{other terms}) > 0, \end{aligned}$$

we have a contradiction in (4.7) if $x_5 > 3b_5$.

For the case $x_5 = 3b_5$, in order to hold the equation (4.7), we must have $x_1 = x_2 = 0$ from $\mathrm{per}(A_{3,5}(1,2,8)) = 0$. Then $b_1 = b_2 = \frac{1}{3}$ and $b_5 = \frac{1}{6}$, which implies a contradiction as follows:

$$1 = \sum b_i + 3a \geq \frac{1}{3} + \frac{1}{3} + 3 \cdot \frac{1}{6} + 3a = 1 + 3a > 1.$$

Thus we have $x_5 < 3b_5$. From (4.5), we have

$$x_i < 3b_i \tag{4.8}$$

for $i = 1, \cdots, 5$. Now, consider

$$\begin{aligned}
0 =& \mathrm{per}(A_{3,5}(1|4)) - \mathrm{per}(A_{3,5}(1|8)) \\
=& 3(b_5 - b_1)[4b_1b_5(ax_2x_3x_4 + b_2^2x_3x_4 + b_3^2x_2x_4 + b_4^2x_2x_3) \\
& - \{2a(ax_2x_3x_4 + b_2^2x_3x_4 + b_3^2x_2x_4 + b_4^2x_2x_3) \\
& + 2b_2^2(ax_3x_4 + b_3^2x_4 + b_4^2x_3) + 2b_3^2(ax_2x_4 + b_2^2x_4 + b_4^2x_2) \\
& + 2b_4^2(ax_2x_3 + b_2^2x_3 + b_3^2x_2)\}] \\
=& 3(b_5 - b_1)[4ax_2x_3(b_1b_5x_4 - b_4^2) \\
& + 4b_2^2x_4(b_1b_5x_3 - b_3^2) + 4b_3^2x_2(b_1b_5x_4 - b_4^2) \\
& + 4b_4^2x_3(b_1b_5x_2 - b_2^2) - 2ax_4(ax_2x_3 + 2b_2^2x_3 + b_3^2x_2)].
\end{aligned} \tag{4.9}$$

The quantity in the large bracket in (4.9) is negative because

$$\begin{aligned}
b_1b_5x_i - b_i^2 &< \frac{1}{3} \cdot b_5 \cdot (3b_i) - b_i^2 \\
&= b_i(b_5 - b_i) \leq 0
\end{aligned}$$

for $i = 2, 3$ and 4, where the first inequality comes from (4.8) and the fact that $b_1 \leq \frac{1}{3}$. Thus we have $b_1 = b_5$ from (4.9). That is, all b_i(and x_i) are equal for $i = 1, \cdots, 5$.

Letting $b = b_i$ and $x = x_i$ for $i = 1, \cdots, 5$, we have $2b \leq x < 3b$ (i.e., $\frac{1}{6} < b \leq \frac{1}{5}$) from (4.6) and (4.8), and $0 < a < \frac{1}{18} < \frac{1}{3}b$ from

$1 = 3a + 5b$. Using these facts, we have a contradiction as follows:

$$\begin{aligned}
0 =& \mathrm{per}(A_{3,5}(1|1)) - \mathrm{per}(A_{3,5}(1|8)) \\
=& \{2a(ax^5 + 5b^2x^4) + (5b)(2b)(ax^4 + 4b^2x^3)\} \\
& - 3b\{2a(ax^4 + 2b^2x^3) + (4b)(2b)(ax^3 + 3b^2x^2)\} \\
=& (2a^2x^5 + 20ab^2x^4 + 40b^4x^3) - (6a^2bx^4 + 48ab^3x^3 + 72b^5x^2) \\
=& 2a^2x^4(x - 2b) + 2abx^4(b - a) + 18ab^2x^3(x - 2b) \\
& + 4b^3x^3(b - 3a) + 36b^4x^2(x - 2b) \\
>& 0.
\end{aligned}$$

This contradiction implies that $a = 0$. Thus the minimizing matrix on $\Omega(V_{3,5})$ becomes the matrix $A_{3,5}$ with $a = 0$ in (4.1), which is contained in the face $\Omega(U_{3,5})$. Therefore the minimum permanent on $\Omega(V_{3,5})$ is the value in (3.1), which occurs at the barycenter $b(U_{3,5})$ of $\Omega(U_{3,5})$, as required. □

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