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THE REPRESENTABILITY OF MODULAR FORMS BY CERTAIN THETA SERIES

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ABSTRACT. With the primitive orders in quaternion algebra, theta series associated with these orders are constructed. Here, we studied the space of modular forms generated by these theta series.

1. Introduction

Let N be a natural number and $\Gamma_0(N)$ the congruence modular group of level N, that is, $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z) | c \equiv 0 \mod N) \right\}$. There is a close connection between the theory of modular forms of weight $k \geq 2$ on $\Gamma_0(N)$ and the arithmetical theory of a rational quaternion algebra. This connection was first recognized by Hecke [6]. For example he conjectured that all cusp forms of weight 2 on $\Gamma_0(p)$, where p is a prime, are linear combination of certain theta series attached to the norm form of certain quaternion algebra. Hecke's oringinal conjecture was proven false. However, Eichler proved slightly weakened version of Hecke's conjecture [4]. He generalized this results and formulated the Basis problem in [5]. Recently, Hijikata, Pizer and Shemanske utilized, so called, special orders in quaternion divsion algebra and solved the Basis problem. Special orders in quaternion algebra are analogous to the order, $\begin{pmatrix} Z & Z \\ NZ & Z \end{pmatrix}$ in $M_{2\times 2}(Q)$.

In Eichler's thesis [3], he studied primitive orders in quaternion algebras over a number fields. An order M of a quaternion algebra A over a local field k is called primitive if it satisfies one of following conditions. If A is a division algebra, M contains the full ring of integers of a quadratic extension field of k. If A is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$,

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of a quadratic extension field of k. If A is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, then M contains a subset which is isomorphic either to $\mathcal{O}\oplus\mathcal{O}$ where $\mathcal{O}$ is the ring of integers in k or to the full ring of integers in a quadratic extension field of k. A special kind of primitive orders, called special orders, were studied in [8].

In [2], Brezinski studied primitive orders in quaternion algebra which is isomorphic to 2 by 2 matrix algebra over a local field and which contains the full ring of integers in a quadratic extension field of k. With theses primitive orders, we constructed theta series attached norm form of this quaternion algebra [10]. Here, we like to study what kind of subspace of modular forms are generated by these theta series.

The basic idea is to consider Brandt matrices $B(n)$ which occur in the theory of quaternion algebras and analogous to the Hecke operators $T(n)$. In fact they both generate semisimple commutative rings and Brandt matrices give representations of the Hecke operators on a space generated by theta series. In [11], we defined Brandt matrices associated with the primitive orders in rational quaternion algebra and we calculated their traces. Theorem 4 below gives a relation between the trace of the Brandt matrices and the trace of Hecke operators. From this, we obtain several results on the representability of modular forms by theta series.

2. Primitive orders and Brandt matrices

In this section, we summerize the properties of quaternion algebra and its order.

2.1 Let A be a rational quaternion algebra ramified precisely at the odd prime q and ∞. That is, $A_q = A\otimes Q_q$ and $A_\infty = A\otimes R$ are division algebras. Otherwise, $A_p = A\otimes Q_p$ is isomorphic to $M_{2\times 2}(Q_p)$ for a finite prime $p\neq q$ (See [11]).

Fix an odd prime p and let L be a quadratic extension field of Q_p. In [10], we have proved that $\left\{\begin{pmatrix}\alpha & \bar\beta\\ \beta & \bar\alpha\end{pmatrix} \mid \alpha,\beta\in L\right\}$ is a quaternion algebra over Q_p. Let $A_p = \left\{\begin{pmatrix}\alpha & \bar\beta\\ \beta & \bar\alpha\end{pmatrix} \mid \alpha,\beta\in L\right\} = L+\xi L$ where $\xi = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Hence, we can define the norm of an element in A as its determinant.

2.2 Let P_L be the prime ideal of $\mathcal{O}_L$, the ring of integers in L. In [10], we have proved that the possibilities of an order, R, of A_p containing $\mathcal{O}_L$ are

$$R = \begin{cases} R_{2\nu}(L) = \mathcal{O}_L + \xi P_L^{\nu} & \text{if } L \text{ is unramified} \\ R_{\nu}(L) = \mathcal{O}_L + (1+\xi)P_L^{\nu-1} & \text{if } L \text{ is ramified} \\ \overline{R_0(L)} = \mathcal{O}_L + (1-\xi)P_L^{-1} & \text{if } L \text{ is ramified} \end{cases}$$

for some nonnegative integer ν, where $\xi \in A_p$ and $\xi \notin \mathcal{O}_L$.

We now define the level of order M of A.

DEFINITION 2.1. Let A be a rational quaternion algebra ramified precisely at one finite prime q and ∞. For finite odd primes, $p_1, p_2, \cdots p_d \neq q$, an order M has level $(q; L(p_1), \nu(p_1); \cdots; L(p_d), \nu(p_d))$ if

(i) M_q is the maximal order of A_q,
(ii) for a prime $p \neq q$, there exists a quadratic extension field $L(p)$ of Q_p and nonnegative integer $\nu(p)$ (which is even if $L(p)$ is unramified) such that $M_p = R_{\nu(p)}(L(p))$,
(iii) $\nu(p_i) > 0$ for $i = 1, 2 \cdots d$ and $\nu(p) = 0$ for $p \neq q, p_1 \cdots, p_d$. (i.e M_p is a maximal order of A_p if $p \neq p_1, p_2, \cdots p_d$).

2.3 In the rest of this paper, let A be a rational quaternion algebra ramified precisely at the odd prime q and ∞ and we will restrict ourselves with the primitive orders R in a quaternion algebra which has level $N' = (q; L(p), \nu(p))$ with $\nu(p) > 1$ where $L(p)$ is the unramified extension field of Q_p. By the definition 2.1, $\nu(p)$ is always even number. So for the notational convenience, let $\nu(p) = 2\nu$.

DEFINITION 2.2. Let $\mathcal{O}$ be an order of level N' in A. A left $\mathcal{O}$ ideal I is a lattice on A such that $I_p = \mathcal{O}_p a_p$ (for some $a_p \in A_p^{\times}$) for all $p < \infty$. Two left $\mathcal{O}$ ideals I and J are said to belong to the same class if $I = Ja$ for some $a \in A^{\times}$. One has the analogous definition for right $\mathcal{O}$ ideals.

DEFINITION 2.3. The class number of left ideals for any order $\mathcal{O}$ of level N' is the number of distint classes of such ideals.

DEFINITION 2.4. The norm of an ideal, denoted by $N(I)$, is the positive rational number which generates the fractional ideal of Q generated by $\{N(a)|a \in I\}$. The conjugate of an ideal I, denoted by $\bar{I}$, is given by $\bar{I} = \{\bar{a}|a \in I\}$. The inverse of an ideal, denoted by I^{-1}, is given by $I^{-1} = \{a \in A|IaI \subset I\}$.

PROPOSITION 2.5. *Let $\mathcal{O}$ be an order of level N' in A. Let $I_1, I_2, I_3, \cdots, I_H$ be the complete set of representatives of all the distint left $\mathcal{O}$ ideal classes. Let $\mathcal{O}_j$ be the right order of I_j, $j = 1, 2, \cdots, H$. Then $I_j^{-1}I_1, \cdots, I_j^{-1}I_H$ is a complete set of representatives of all the distint left $\mathcal{O}$ ideal classes (for $j = 1, 2, \cdots, H$).*

Proof. See Proposition 2.13 and 2.15 of [14]. □

Fixing a set of representatives of the (left) R-ideal classes, we define generalized Brandt matrices $B(n) = B(n; N')$ in exactly the same manner as Eichler (See [5], equation 15 and 15a on the page 105). Here n is nonnegative integer. With these Brandt matrices, we constructed a set of theta series which are modular forms of weight 2 on $\Gamma_0(qp^{2\nu})$ (See [11]). In this section we will modify the Brandt matrices to study the subspace of theta series generated by the quaternion theory.

In this section, we will explain how to construct the Brandt matrix series and theta series briefly. For the details, see [5], [8], [11] or [14].

2.4 Let R be an order of A with level N' and let $I_1, I_2, \cdots, I_H$ be a representatives of (left) ideal classes of R, where H is the class number. Then let e_j be the number of unit elements in $I_j^{-1}I_i$. Now, we define $b_{ij}(n) = \frac{1}{e_j}$ the number of elements in $I_j^{-1}I_i$ with norm $nN(I_i)/N(I_j)$ for $n \geq 1$. If $n = 0$, $b_{ij}(0) = \frac{1}{e_j}$. Then $B(n, N') = (b_{ij}(n))$ is called a Brandt matrix, which is a $H \times H$ matrix. Further, let

$$\Theta(\tau, N') = (\theta_{ij}(\tau)) = \sum_{n=0}^{\infty} B(n; N')e^{2\pi i n\tau}. \tag{2-1}$$

Then the entries of $\Theta(\tau, N')$, $\theta_{ij}(\tau)$, are modular forms of weight 2 on $\Gamma_0(N)$ where $N = qp^{2\nu}$ (See Theorem 3.4 [11]).

To study the theta series constructed by primitive orders, we need to modify the Brandt matrices analogously as in [14].

PROPOSITION 2.6. *Let $B(n) = B(n, L(p), 2\nu)$. Then the entries of the matrix series $\sum_{n=0}^{\infty} B(n)exp(2\pi i n\tau)$ are modular forms of weight 2 on $\Gamma_0(N)$, $N = qp^{2\nu}$.*

Proof. See Theorem 3.4 [11]. □

LEMMA 2.7. *Let M be an order of level N' and let I, J be left M ideals. Let $\Theta_I = \sum_\alpha exp(\tau N(\alpha)/N(I))$ be the theta series attached to I and similarly for J. Then $\Theta_I - \Theta_J$ is a cusp form of weight 2 on $\Gamma_0(N)$, $N = qp^{2\nu}$.*

Proof. For a finite prime l, let $I_l = M_l a$ and $J_l = M_l b$ for some $a, b \in A_l^\times$. There exists $u \in Z_l$ such that

$$uN(a)/N(I) = N(b)/N(J).$$

Since M_l is an order containing the ring of integers of unramified quadratic extension field of Q_l, there exists a unit $v \in M_l$ such that $N(v) = u$ (See [13] page 188). So quadratic forms $N(x)/N(I)$ for $x \in I$ and $N(x)/N(J)$ for $x \in J$ are locally equivalent for a finite prime l. If $l = \infty$, then it is clear. Thus Θ_I and Θ_J have same genus. It is classical result that the difference, $\Theta_I - \Theta_J$ is a cusp form [17].□

COROLLARY 2.8. *The difference of two theta series appearing in the same column of the matrix series $\sum_{n=0}^{\infty} B(n)exp(2\pi i n\tau)$ is a cusp form.*

Proof. This is immediate from Proposition 2.6 and Lemma 2.7. □

REMARK. Recall that $b_{ij}(n)$ is that $\frac{1}{e_j}$ times the number of elements, α,in $I_j^{-1}I_i$ with $N(\alpha) = nN(I_i)/N(I_j)$ and e_j is the number of elements of norm 1 in $I_j^{-1}I_j$.

LEMMA 2.9. *Let $B(n) = (b_{ij}(n))$ where $1 \le i \le H$ and $1 \le j \le H$. Then we have*

(a) *$e_j b_{ij}(n) = e_i b_{ji}(n)$ for all $i, j, 1 \le i, j \le H$ and all $n \ge 0$*

(b) *$\sum_{j=1}^{H} b_{ij}(n) = b(n)$ are independent of i.*

Proof. See lemma 2.18 in [15]. □

LEMMA 2.10. *Let the notation be as above. Let* $B(n) = B(n, L(p), 2\nu) = (b_{ij}(n))$. *Consider the matrix*

$$A = \begin{pmatrix} 1 & e_1e_2^{-1} & \cdots & e_1e_H^{-1} \\ 1 & -1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 1 & 0 & \cdots & -1 \end{pmatrix}$$

that is, $A = (a_{ij})$ *where* $a_{i1} = 1$ *for* $i = 1, \cdots, H; a_{1j} = e_1e_j^{-1}$ *for* $j = 1, \cdots, H,\quad a_{ii} = -1$ *for* $i = 2, \cdots, H$ *and all other* $a_{ij} = 0 (i \neq 1, j \neq 1, i \neq j)$. *Then* $AB(n)A^{-1} = B'(n)$ *for all* $n \geq 0$ *where* $B'(n) = (b'_{ij}(n))$ *and* $b'_{11}(n) = b(n) = \sum_{j=1}^{H} b_{ij}(n)$*(independent of* i *by Lemma 2.5);* $b'_{1i}(n) = b'_{i1}(n) = 0$ *for* $i = 2, \cdots, H$ *and* $b'_{ij}(n) = b_{ij}(n) - b_{1j}(n)$ *for* $2 \leq i, j \leq H$.

Proof. See the proof of Lemma 2.19 in [15]. □

2.5 We are now able to consider the case of cusp forms of weight 2.

$$AB(n)A^{-1} = \begin{pmatrix} b(n) & 0 \\ 0 & B'(n, L(p), 2\nu) \end{pmatrix} \tag{2-2}$$

where $B'(n, L(p), 2\nu)$ is a $H - 1 \times H - 1$ matrix. From now we will denote $B'(n)$ for $AB(n)A^{-1}$.

THEOREM 2.10. *Let* $B'(n, L(p), 2\nu)$ *and* $B'(n)$ *be as above. Then the entries of modified Brandt matrices*

$$\Theta'(\tau, N') = \sum_{n=0}^{\infty} B'(n, L(p), 2\nu) exp(2\pi i n\tau)$$

are cusp forms of weight 2 on $\Gamma_0(N)$ *where* $N = qp^{2\nu}$.

Proof. Each entry of $\Theta'(\tau, N')$ is $\sum_{n=0}^{\infty}(b_{ij}(n) - b_{1j}(n))e^{2\pi i n\tau}$. By Corollary 2.8, this is a cusp form of weight 2 on $\Gamma_0(N)$. □

PROPOSITION 2.11. *Fix* p, q *and* $N' = (L(p), 2\nu)$ *as above. Then the* $B(n)$ *with* $(n, qp^{2\nu}) = 1$ *generates a commutative semi simple ring. Similary, the* $B'(n, L(p), 2\nu)$ *with* $(n, qp^{2\nu}) = 1$ *generates a commutative semi simple ring.*

Proof. By the Theorem 2 on page 106 of Eichler[5], the $B(n) = B(n, N')$ with $(n, qp^{2\nu}) = 1$ generates a commutative ring. Thus it follows from the above that $B(n)$ with $(n, N) = 1$ generates a commutative ring and clearly, so does the $B'(n)$ with $(n, qp^{2\nu}) = 1$. □

REMARK. By Proposition 2.11, there exists a $H \times H$ matrix E such that $EB(n)E^{-1}$ is simultaneously diagonal matrix for all n with $(n, qp^{2\nu}) = 1$. Similarly, there exists a $H-1 \times H-1$ matrix E' such that $E'B'(n, L(p), 2\nu)E'^{-1}$ is simultaneously diagonal matrix for all n with $(n, qp^{2\nu}) = 1$.

3. Traces formular

Let $tr_N T_2(n)$ be the trace of Hecke operator $T(n)$ acting on the space of cusp forms $S_2(N)$. Hijikata has computed $tr_N T_2(n)$, which is given by [7].

THEOREM 3.1.

$$tr_N T_2(n) = -\sum_s a_k(s) \sum_f b(s,f) \prod_{p|N} c'(s,f,p) + \delta(\sqrt{n}) \frac{k-1}{12} \cdot N \cdot \prod_{p|N} (1 + \frac{1}{p}) + \delta(k) deg T_2(n),$$

where $\delta(\sqrt{n}) = \begin{cases} 1 & \textit{if } n \textit{ is a perfect square} \\ 0 & \textit{otherwise} \end{cases}$.

REMARK. The meaning of $s, a(s), f$ and $b(s,f)$ are as follows. Let s run over all integers such that $s^2 - 4n$ has one of followings.

$$s^2 - 4n = \begin{cases} 0 \\ t^2 \\ t^2 m & 0 > m \equiv 1 \mod 4 \\ t^2 4m & 0 > m \equiv 2,3 \mod 4. \end{cases}$$

Let $\phi(X) = X^2 - sX + n$ and let x, y be roots of $\phi(X)$ in C. Then

$$a(s) = \begin{cases} |x| \cdot (4N)^{-1} & \text{if } s^2 - 4n = 0 \\ \min(|x|, |y|)/(|x-y|) & \text{if } s^2 - 4n = t^2 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

For each fixed s, $a(s)$ is corresponding to its classification.

Let f run over the following set.

$$f = \begin{cases} 1, 2, \cdots, N & \text{if } s^2 - 4n = 0 \\ \text{all positive divisors of t} & \text{otherwise} \end{cases}$$

and

$$b(s,f) = \begin{cases} 1 & \text{if } s^2 - 4n = 0 \\ \frac{1}{2}\varphi(\sqrt{s^2 - 4n})/f) & \text{if } s^2 - 4n = t^2 \\ h((s^2 - 4n)/f^2)/\omega((s^2 - 4n)/f^2) & \text{otherwise} \end{cases}$$

where φ is Euler φ-function, $h(d)$ (resp. $\omega(d)$) denotes the class number of locally principal ideals (resp. $\frac{1}{2}$ the cardinality of the unit group) of the order $Q(\sqrt{d})$ with discriminant d.

Proof. See [7]. □

THEOREM 3.2. *The trace of the Brandt matrix is given by*

$$trB(n, L(p), qp^{2\nu}) = \sum_s \sum_f b(s,f) \prod_{l | qp^{2\nu}} c(s,f,l) + \delta(\sqrt{n}) \frac{1}{12}(q-1)(p^{2\nu} - p^{2\nu-1})$$

where $\nu \geq 1$ *and* $\delta(\sqrt{n}) = \begin{cases} 1 & \textit{if } n \textit{ is a perfect square} \\ 0 & \textit{otherwise} \end{cases}$.

REMARK. The meaning of $s, f, b(s,f)$ and $c(s,f,l)$ are as follows. Let s run over all integers such that $s^2 - 4n$ is negative. Hence with some positive integer t and square free integer m, we can classify $s^2 - 4n$ by

$$s^2 - 4n = \begin{cases} t^2 m & m \equiv 1 \mod 4 \\ t^2 4m & m \equiv 2, 3 \mod 4. \end{cases}$$

For each s, let f run over all positive divisors of t. Let $L = Q[x]/(\Psi_s(x))$ where $\Psi_s(x) = x^2 - sx + n$ and ξ is the canonical image of x in L. Then L is an imaginary quadratic number field and ξ generates the order $Z + Z\xi$ of L. For each f, there is uniquely determined order $\mathcal{O}_f$

containing $Z+Z\xi$ as a submodule of index f. Let $\Delta(\mathcal{O}_f)=s^2-4n/f^2$. Let $h(\Delta(\mathcal{O}_f))$(resp. $\omega(\Delta(\mathcal{O}_f))$) denote the number of locally principal $\mathcal{O}_f$ ideals(resp. $\frac{1}{2}|U(\mathcal{O}_f)|$). Then $b(s,f)=\frac{h(\Delta(\mathcal{O}_f))}{\omega(\Delta(\mathcal{O}_f))}$.

Let M be an order of level N' of A. Then $c(s,f,l)$ is the number of $M_l^\times=(M\otimes Z_l)^\times$ equivalence classes of optimal embeddings of $\mathcal{O}_f\otimes Z_l$ into $M\otimes Z_l$. In other words, let $Z+Z\alpha$ be the maximal order of L, then $\mathcal{O}_f\otimes Z_l=Z_l+Z_l l^m\alpha$ and $(s^2-4n)/f^2\equiv l^{2m}\Delta(\alpha) \mod (Z_l^\times)^2$. So it is easy to calculate $c(s,f,l)$, the number of $M_l^\times=R_{\nu(l)}^\times(L(l))$ (See Definition 2.1) equivalent classes of optimal embeddings of $l^m\alpha$ $(=Z_l+Z_l l^m\alpha)$ into $M_l=R_{\nu(l)}(L(l))$, in Theorem 3.3 in [2], if s,f and n are given.

Proof. See the proof of Theorem 3.8 [10]. □

THEOREM 3.3. *For all positive integer n with $(n,qp)=1$ and for all $\nu\geq 2$ we have*

(3-1)
$$\begin{aligned}&trB(n,L(p),qp^{2\nu})-trB(n,L(p),qp^{2\nu-2})\\&=tr_{qp^{2\nu}}T(n)-2tr_{p^{2\nu}}T(n)-2(tr_{qp^{2\nu-1}}T(n)-2tr_{p^{2\nu-1}}T(n))\\&\quad+tr_{qp^{2\nu-2}}T(n)-2tr_{p^{2\nu-2}}T(n).\end{aligned}$$

Proof. To prove this identity we will compare with term by term.
First, the degree term of R.H.S of (3-1) is

$$\deg T(n)-2\deg T(n)-2(\deg T(n)-2\deg T(n))+\deg T(n)-2\deg T(n)=0$$

and the degree terms of L.H.S do not occur.

Second, we consider mass terms. By Theorem 3.6 in [10],mass term of L.H.S is

$$(q-1)(p-1)p^{2\nu-1}-(q-1)(p-1)p^{2\nu-3}=(q-1)(p-1)^2(p+1)p^{2\nu-3}.$$

On the other hand, mass term of R.H.S is

$$\begin{aligned}&(q+1)(p+1)p^{2\nu-1}-2(p+1)p^{2\nu-1}-2((q+1)(p+1)p^{2\nu-2}\\&-2(p+1)p^{2\nu-2})+(q+1)(p+1)p^{2\nu-3}-2(p+1)p^{2\nu-3}\\=&(q-1)(p-1)^2(p+1)p^{2\nu-3}.\end{aligned}$$

Finally we will check the main part of (3-1).

Suppose $s^2 - 4n = 0$ or t^2 for some integer t.
Then M_q is the maximal order of A_q which is a division ring and $c'(s,f,q)_q$ is the number of $M_q^\times = (M \otimes Z_q)^\times$ equivalence classes of optimal embeddings of $\mathcal{O} \otimes Z_q$ with discriminant 0 or t^2 into $M \otimes Z_q$ which is independent from s, f (See Table following Theorem 2.1 of [12]). Since there are no optimal embeddings into the division algebra, L.H.S is 0. On the other hand, R.H.S is split into two parts,

$$(3\text{-}2) \qquad tr_{qp^{2\nu}}T(n) - 2(tr_{qp^{2\nu-1}}T(n)) + tr_{qp^{2\nu-2}}T(n)$$

and

$$(3\text{-}3) \qquad 2(tr_{p^{2\nu}}T(n) - 2tr_{p^{2\nu-1}}T(n) - tr_{p^{2\nu-2}}T(n)).$$

If $s^2 - 4n = 0$, then (3-2) is

$$|\frac{s}{2}|\frac{1}{4}c'(s,f,q) \cdot (\frac{1}{qp^{2\nu}} \sum_{1}^{qp^{2\nu}} c'(s,f,p)_{p^{2\nu}}$$
$$- 2\frac{1}{qp^{2\nu-1}} \sum_{1}^{qp^{2\nu-1}} c'(s,f,p)_{p^{2\nu-1}} + \frac{1}{qp^{2\nu-2}} \sum_{1}^{qp^{2\nu-2}} c'(s,f,p)_{p^{2\nu-2}})$$
$$=|\frac{s}{2}|\frac{1}{4}c'(s,f,q)(2-4+2) = 0.$$

Similary (3-2) is 0.

Next, if $s^2 - 4n = t^2$, then by the Remark following Theorem 3.1, $a(s)$ and $b(s,f)$ are independent from level $qp^{2\nu}$, $qp^{2\nu-1}$ and $qp^{2\nu-2}$. So it suffices to compare with $c'(s,f,p)$. Now, (3-1) is $c'(s,f,p)_{p^{2\nu}} - 2c'(s,f,p)_{p^{2\nu-1}} + c'(s,f,p)_{p^{2\nu-2}} = 0$. By the similar way, (3-2)=0. Hence R.H.S is 0.

We now have

$$\Delta = \frac{(s^2 - 4n)}{f^2} = p^a q^b d$$

where $(pq, d) = 1$.

We now need the tables of the number of inequivalent optimal embeddings.

Here, u is a quadratic nonresidue mod p.

Table.

$\Delta = p^{2m}$

	$\nu < m$	$\nu = m$	$\nu > m$
$c'(s,f,p)_{p^{2\nu+1}}$	$2p^{\nu}$	$2p^m + 2p^{m-1}$	$2p^m + 2p^{m-1}$
$c'(s,f,p)_{p^{2\nu}}$	$p^{\nu} + p^{\nu-1}$	$p^m + 2p^{m-1}$	$2p^m + 2p^{m-1}$

$\Delta = p^{2m}u$

	$\nu < m$	$\nu = m$	$\nu > m$
$c'(s,f,p)_{p^{2\nu+1}}$	$2p^{\nu}$	0	0
$c'(s,f,p)_{p^{2\nu}}$	$p^{\nu} + p^{\nu-1}$	p^m	0

$\Delta = p^{2m+1}a$ where $a = 1$ or $a = u$.

	$\nu < m$	$\nu = m$	$\nu > m$
$c'(s,f,p)_{2p^{\nu+1}}$	$2p^{\nu}$	p^m	0
$c'(s,f,p)_{2p^{\nu}}$	$p^{\nu} + p^{\nu-1}$	$p^m + p^{m-1}$	0

Suppose that $b \geq 2$. Then $c'(s,f,q)_q = 2$ and $c(s,f,q)_q = 0$. Since $b(s,f)$ is fixed if s and f is given,

$$\begin{aligned}
&\text{L.H.S} \\
&= \sum_s \sum_f b(s,f) c(s,f,q)_{q^b} c(s,f,p)_{p^{2\nu}} \\
&\quad - \sum_s sum_f b(s,f) c(s,f,q)_{q^b} c(s,f,p)_{p^{2\nu-2}} \\
&= \sum_s \sum_f b(s,f) \cdot 0 \cdot c(s,f,p)_{p^{2\nu}} - \sum_s \sum_f b(s,f) \cdot 0 \cdot c(s,f,p)_{p^{2\nu-2}} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
&\text{R.H.S} \\
&= \sum_s \sum_f b(s,f) c'(s,f,q)_{q^b} c'(s,f,p)_{p^{2\nu}} - 2 \sum_s \sum_f b(s,f) c'(s,f,p)_{p^{2\nu}} \\
&\quad - 2\Big(\sum_s \sum_f b(s,f) c'(s,f,q)_{q^b} c'(s,f,p)_{p^{2\nu-1}} - 2 \sum_s \sum_f b(s,f) c'(s,f,p)_{p^{2\nu-1}}\Big) \\
&\quad + \sum_s \sum_f b(s,f) c'(s,f,q)_{q^b} c'(s,f,p)_{p^{2\nu-2}} - 2 \sum_f b(s,f) c'(s,f,p)_{p^{2\nu-2}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_s \sum_f b(s,f) \cdot 2 \cdot c'(s,f,p)_{p^{2\nu}} - 2\sum_s \sum_f b(s,f) c'(s,f,p)_{p^{2\nu}} \\
&\quad - 2(\sum_s \sum_f b(s,f) \cdot 2 \cdot c'(s,f,p)_{p^{2\nu-1}} - 2\sum_s \sum_f b(s,f) c'(s,f,p)_{p^{2\nu-1}}) \\
&\quad + \sum_s \sum_f b(s,f) \cdot 2 \cdot c'(s,f,p)_{p^{2\nu-2}} - 2\sum_f b(s,f) c'(s,f,p)_{p^{2\nu-2}} \\
&=0.
\end{aligned}
$$

Hence, we need to check cases that $\frac{s^2-4n}{f^2}$ is $p^a d$ or $p^a qd$.

CASE A. $\frac{s^2-4n}{f^2} = p^a d$.

If $a = 2m$ and $(\frac{d}{q}) = 1$, then $c(s,f,q)_q = 0$ and $c'(s,f,q)_q = 2$. So it is clear.

If $a = 2m$ and $(\frac{d}{q}) = -1$, then $c(s,f,q)_q = 2$ and $c'(s,f,q)_q = 0$. So (3-1) becomes

$$
\begin{aligned}
&2(\sum_{s,f} b(s,f) c(s,f,p)_{p^{2\nu}} - \sum_{s,f} b(s,f) c(s,f,p)_{p^{2\nu-2}}) \\
= &-2\sum_{s,f} b(s,f) c'(s,f,p)_{p^{2\nu}} + 4\sum_{s,f} b(s,f) c'(s,f,p)_{p^{2\nu-1}} \\
&-2\sum_{s,f} b(s,f) c'(s,f,p)_{p^{2\nu-2}}.
\end{aligned}
$$

Hence, to check (3-1), it suffices to compare

$$
\begin{aligned}
&c(s,f,p)_{p^{2\nu}} - c(s,f,p)_{p^{2\nu-2}} \\
&= c'(s,f,p)_{p^{2\nu}} - 2c'(s,f,p)_{p^{2\nu-1}} + c(s,f,p)_{p^{2\nu-2}}.
\end{aligned}
\tag{3-2}
$$

In the following table, we will read $p^{-1} = p^{-2} = 0$. For the L.H.S. of (3-2), let $g(\nu) = c(s,f,p)_{p^{2\nu}} - c(s,f,p)_{p^{2\nu-2}}$.

	$c(s,f,p)_{p^{2\nu}}$	$c(s,f,p)_{p^{2\nu-2}}$	$g(\nu)$
$\nu > m+1$	$2p^m - 2p^{m-1}$	$2p^m - 2p^{m-1}$	0
$\nu = m+1$	$2p^m - 2p^{m-1}$	$p^m - 2p^{m-1}$	p^m
$\nu = m$	$p^m - 2p^{m-1}$	$p^{m-1} - p^{m-2}$	$p^m - 3p^{m-1} + p^{m-2}$
$\nu < m$	$p^\nu - p^{\nu-1}$	$p^{\nu-1} - 2p^{\nu-2}$	$p^\nu - 2p^{\nu-1} + p^{\nu-2}$

For the R.H.S of (3-2), let $t(\nu) = c'(s,f,p)_{p^{2\nu}} - 2c'(s,f,p)_{p^{2\nu-1}} + c'(s,f,p)_{p^{2\nu-2}}$.

	$c'(s,f,p)_{p^{2\nu}}$	$c'(s,f,p)_{p^{2\nu-1}}$	$c'(s,f,p)_{p^{2\nu-2}}$	$t(\nu)$
$\nu > m+1$	0	0	0	0
$\nu = m+1$	0	0	p^m	p^m
$\nu = m$	p^m	$2p^{m-1}$	$p^{m-1}+p^{m-2}$	$p^m - 3p^{m-1} + p^{m-2}$
$\nu < m$	$p^\nu - p^{\nu-1}$	$2p^{\nu-1}$	$p^{\nu-1} - p^{\nu-2}$	$p^\nu - 2p^{\nu-1} + p^{\nu-2}$

If $a = 2m+1$, as in the above case it suffices to check $(\frac{pd}{q}) = -1$.

	$c(s,f,p)_{p^{2\nu}}$	$c(s,f,p)_{p^{2\nu-2}}$	$g(\nu)$
$\nu > m+1$	0	0	0
$\nu = m+1$	0	$p^m - p^{m-1}$	$-p^m + p^{m-1}$
$\nu = m$	$p^m - p^{m-1}$	$p^{m-1} - p^{m-2}$	$p^m - 2p^{m-1} + p^{m-2}$
$\nu < m$	$p^\nu - p^{\nu-1}$	$p^{\nu-1} - 2p^{\nu-2}$	$p^\nu - 2p^{\nu-1} + p^{\nu-2}$

For the R.H.S of (3-2), let $t(\nu) = c'(s,f,p)_{p^{2\nu}} - 2c'(s,f,p)_{p^{2\nu-1}} + c'(s,f,p)_{p^{2\nu-2}}$.

	$c'(s,f,p)_{p^{2\nu}}$	$c'(s,f,p)_{p^{2\nu-1}}$	$c'(s,f,p)_{p^{2\nu-2}}$	$t(\nu)$
$\nu > m+1$	0	0	0	0
$\nu = m+1$	0	p^m	$p^m + p^{m-1}$	$-p^m + p^{m-1}$
$\nu = m$	$p^m + p^{m+1}$	$2p^{m-1}$	$p^{m-1} + p^{m-2}$	$p^m - 2p^{m-1} + p^{m-2}$
$\nu < m$	$p^\nu - p^{\nu-1}$	$2p^{\nu-1}$	$p^{\nu-1} - p^{\nu-2}$	$p^\nu - 2p^{\nu-1} + p^{\nu-2}$

CASE B. $\frac{s^2-4n}{f^2} = p^a qd$.

If $a = 2m$ and $(\frac{d}{q}) = 1$, then $c(s,f,q) = 1$ and $c'(s,f,q) = 1$. This case becomes Case A with $a = 2m$.

If $a = 2m+1$ and $(\frac{pd}{q}) = 1$, then $c(s,f,q) = 0$ and $c'(s,f,q) = 2$. This is clear from Case A with $a = 2m+1$. □

LEMMA 3.4.

$$trB(n, L(p), q) = tr_q T(n) - 2tr_1 T(n) + degT(n)$$

Note that the Brandt matrix $B(n, L(p), q)$ is independent of which quadratic extension $L(p)$ of Q_p we use since in all cases the order R is simply a maximal order of A_p.

Proof. See Lemma 6.5 in [9]. □

4. Consequences

In the previous section, trace identity between Brandt matrix and Hecke operators is proved. We are now in position to determine to what subspace is generated by the set of entries in the Brandt matrix series
$(\theta'_{ij}(\tau)) = \sum_{n=0}^{\infty} B'(n, L(p), qp^{2\nu})e^{2\pi i n\tau}$. As in section 2, we will use $B'(n, L(p), 2\nu)$ instead of $B'(n, L(p), qp^{2\nu})$ for the convenience.

4.1 By proposition 2.11, there exists a matrix E' such that $E'B'(n, L(p), 2\nu)E'^{-1}$ is a diagonal matrix for all $(n, qp^{\nu}) = 1$. Let $\{\theta_1(\tau), \cdots, \theta_d(\tau)\}$ be the set of forms appearing on the diagonal of the diagonalized matrix series
$\sum_{n=0}^{\infty} E'B'(n, L(p), 2\nu)E'^{-1}\exp(2\pi i n\tau)$. It is well known that two repesentatives on commutative semi simple algebra are identical if their traces are equal. Thus by Theorem 3.3, the Brandt matrices give the action of the Hecke operators on the space of forms, $(\theta'_{ij}(\tau))$. The forms appearing in the diagonal of $\sum_{n=0}^{\infty} E'B'(n, L(p), 2\nu)E'^{-1}\exp(2\pi i n\tau)$ are the eigen forms for the action of Hecke operators $T(n)$. Further, the action of the $T(n)$ is given by the diagonalized Brandt matrix $B'(n, L(p), 2\nu)$.

4.2 Let $S_2^0(N)$ be the set of all new forms in $S_2(N)$ (See [1]). An important result from the theory of new forms is the decomposition

$$S_2(N) = \oplus_{a|N}\delta(N/a)S_2^0(a) \tag{4-1}$$

where $\oplus_{a|N}$ means the direct sum over all positive integers a and $a|N$. Here $\delta(s)$ denotes the number of positive integers dividing s and $2A = A \oplus A$ (See Lemma 15 and Theorem 5 of [1]).

THEOREM 4.1. *Let $\{\theta_1(\tau), \cdots, \theta_d(\tau)\}$ be as above in 4.1 and let $< \theta_i(\tau) >$ denote the 1-dimensional (complex) vector space generated by $\theta_i(\tau)$. Then*

$$\begin{aligned} & S_2(qp^{2\nu}) \oplus 2\sum_{s=1}^{\nu} S_2(qp^{2s-1}) \oplus 4\sum_{s=1}^{\nu-1} S_2(p^{2s-2}) \oplus S_2(q) \\ \simeq\ & < \theta_1 > \oplus < \theta_2 > \oplus \cdots < \theta_d > \oplus 2\sum_{s=1}^{\nu} S_2(qp^{2s-1}) \\ & \oplus 2S_2(p^{2\nu}) \oplus 4\sum_{s=1}^{\nu} S_2(p^{2s-2}) \oplus 2S_2(1) \end{aligned} \tag{4-2}$$

where the isomorphism is a module for the Hecke algebra H generated by the $T(n)$ with $(n,N)=1$. Here $2S_2(qp^{2s+1}) = S_2(qp^{2s+1}) \oplus S_2(qp^{2s+1})$ etc.

Proof. As H is a semi-simple ring, we need only check that the traces of the transformations induced by $T(n)$ on both sides of (4-2) are equal. By (2-2), $\mathrm{tr}B'(n,L(p),2\nu) = \mathrm{tr}B(n,L(p),2\nu) - b(n) = \mathrm{tr}B(n,L(p),2\nu) - \mathrm{deg}T(n)$ for $(n,qp^{2\nu})=1$, where $b(n)=\mathrm{deg}T(n)$ for $(n,qp)=1$ (See p94 [5]). Hence, by theorem 3.3, $trB(n,L(p),2\nu) - trB(n,L(p),0)$ equals

$$\begin{aligned}
&\sum_{k=1}^{\nu}\{tr_{qp^{2k}}T(n) - 2tr_{p^{2k}}T(n) - 2(tr_{qp^{2k-1}}T(n) - 2tr_{p^{2k-1}}T(n)) \\
&+ tr_{qp^{2k-2}}T(n) - 2tr_{p^{2k-2}}T(n)\} \\
= {}& tr_{qp^{2\nu}}T(n) + 2\sum_{s=2}^{\nu} tr_{qp^{2s-2}}T(n) + 4\sum_{s=1}^{\nu-1} tr_{p^{2s-1}}T(n) + tr_qT(n) \\
&- \{2tr_{p^{2\nu}}T(n) + 2\sum_{s=1}^{\nu} tr_{qp^{2s-1}}T(n) + 4\sum_{s=2}^{\nu} tr_{p^{2s-2}}T(n) + 2tr_1T(n)\}.
\end{aligned}$$

By Lemma 3.4, $trB(n,L(p),0) = tr_qT(n) - 2tr_1T(n)$. Thus if $\{\theta_1,\theta_2,\cdots,\theta_d\}$ are the set of forms appearing on the diagonal of the diagonalized matrix series, $\sum_{n=0}^{\infty} E'B'(n,L(p),2\nu)E'^{-1}\exp(in\tau)$, then (4-2) is given. □

COROLLARY 4.2. *Let $\{\theta_1,\cdots,\theta_d\}$ be as in Theorem 4.1. $<\theta_1> \oplus <\theta_2> \oplus \cdots <\theta_d> \simeq S_2^0(qp^{2k}) \oplus S_2^0(qp^{2k-2}) \oplus \cdots \oplus S_2^0(qp^2) \oplus S_2^0(q)$, where the isomorphism is a module for the Hecke algebra H generated by the $T(n)$ with $(n,qp)=1$.*

Proof. From (4-1), we have

$$\begin{aligned}
S_2(qp^{2\nu}) &= \sum_{k=0}^{2\nu}(2\nu-k+1)S_2^0(qp^k) + 2\sum_{k=0}^{2\nu}(2\nu-k+1)S_2^0(p^k) \\
S_2(p^{2\nu}) &= \sum_{k=0}^{2\nu}(2\nu-k+1)S_2^0(p^k).
\end{aligned}$$

Then (4-2) can be replaced with the newforms. Thus we conclude that $< \theta_1 > \oplus < \theta_2 > \oplus \cdots < \theta_d > \simeq \sum_{k=0}^{2\nu} S_2^0(qp^{2k})$. □

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