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ON STABILITY OF A TRANSMISSION PROBLEM

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ABSTRACT. We investigate the behavior of the gradient of solutions to the refraction equation $\operatorname{div}((1+(k-1)\chi_D)\nabla u) = 0$ under perturbation of domain D. If u and u_h are solutions to the refraction equation corresponding to subdomains D and D_h of a domain Ω in 2 dimensional plane with the same Neumann data on $\partial\Omega$, respectively, we prove that $\|\nabla(u-u_h)\|_{L^2(\Omega)} \le C\sqrt{\operatorname{dist}(D, D_h)}$ where $\operatorname{dist}(D, D_h)$ is the Hausdorff distance between D and D_h. We also show that this is the best possible result.

1. Introduction and statement of results

Let Ω be a simply connected bounded domain in $\mathbb{R}^n$ ($n \ge 2$) with the C^2 smooth boundary and let D be a simply connected C^2 subdomain of Ω with closure in Ω. Let $k \neq 1$ be a positive number. Consider the following Neumann problem

$$P[D,g]\begin{cases} \operatorname{div}((1+(k-1)\chi_D)\nabla u) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega, & \displaystyle\int_{\partial\Omega} g = 0, \quad g \in L^2(\partial\Omega) \\ \displaystyle\int_{\Omega} u = 0, & \end{cases}$$

where $\nu(x)$ is the unit normal to the boundary, $\frac{\partial u}{\partial \nu} = \nu \cdot \nabla u$, and χ_D is the characteristic function of D.

In this paper we study stability of the solution to the transmission problem $P[D,g]$ under the perturbation of D. This study is motivated in relation to the inverse problem to $P[D,g]$, namely the inverse conductivity problem.

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Let D_h be a bounded simply connected subdomain with C^2 smooth boundary defined by

$$\partial D_h : f(s) + h\omega_h(s)\nu(s) \qquad (\partial D : x = f(s))$$

where s is 1 dimensional local parameter, $\omega_h(s)$ is a C^1 function on ∂D whose C^1 norm is bounded uniformly in h, and $\nu(s)$ is the outward unit normal to ∂D. Let u and u_h be solutions to $P[D, g]$ and $P[D_h, g]$, respectively. In [2] and [1], it is proved that

$$\|u - u_h\|_{L^p(\Omega)} \leq Ch \tag{1.1}$$

if $p < 4$ and ∂D and ∂D_h are only $C^{1,1}$. (This result is for n-dimension ($n = 2, 3$).) In [7], (1.1) is proved for $p = \infty$. Both results were used strongly in the study of the inverse problem (see [2, 1, 7]).

In this paper we investigate the behaviour of ∇u under perturbation of D. We prove that

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C\sqrt{h}.$$

We also prove that $\sqrt{h}$ is the best possible result one can expect. To be precise, we have the following theorem:

THEOREM 1.1. *Let Ω be a simply connected bounded domain in $\mathbb{R}^2$ and D and D_h be as above. Let $D\Delta D_h$ be the symmetric difference of D and D_h. Then, there exists a constant C such that*

$$\lim_{h\to 0} \frac{1}{h} \int_{\Omega\setminus D\Delta D_h} |\nabla(u - u_h)|^2 dx = 0, \tag{1.2}$$

(1.3)

$$\limsup_{h\to 0} \frac{1}{h} \int_{D\Delta D_h} |\nabla(u - u_h)|^2 dx \leq C \int_{\partial D} \left| \frac{\partial u}{\partial \nu^{\pm}} \right|^2 d\sigma.$$

Here, $d\sigma$ is the line element on ∂D and

$$\frac{\partial u}{\partial \nu^{\pm}}(P) = \lim_{t\to 0^+} \langle \nabla u(P \pm t\nu(P)), \nu(P)\rangle.$$

Moreover, if ω_h converges to ω uniformly as $h \to 0$, then

$$\lim_{h\to 0} \frac{1}{h} \int_{D\Delta D_h} |\nabla(u - u_h)|^2 dx = \frac{k-1}{k} \int_{\partial D} \left| \frac{\partial u}{\partial \nu^{+}} \right|^2 |\omega| d\sigma. \tag{1.4}$$

If the Neumann data g is not zero, then $\frac{\partial u}{\partial \nu^{\pm}}$ is not zero and hence we have

COROLLARY 1.2. *There exists a constant C independent of h such that*

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \le C\sqrt{h}.$$

If ω_h converges to ω uniformly as $h \to 0$, then for small enough h

$$\frac{1}{C}\sqrt{h} \le \|\nabla(u-u_h)\|_{L^2(\Omega)}.$$

Corollary 1.2 says that $\sqrt{h}$ is the best possible.

The proof of Theorem 1.1 is based on our earlier result on the representation of the solution to $P[D,g]$ ([9]). So, we first review the representation formula in Section 2 and then prove Theorem 1.1 in Section 3.

One comment on a notation: the constants C in estimates may differ from one step to another. However, those constants do not depend on the quatities to be estimated.

2. Representation of solutions

For this work, we assume that Ω is a simply connected bounded C^2 domain in $\mathbb{R}^2$ and D be a simply connected subdomain with C^2 boundary which is compactly contained in Ω. The single and double layer potentials on D is defined by

$$\mathcal{S}_D f(X) = \frac{1}{2\pi}\int_{\partial D} \log|X-Q| f(Q)\,d\sigma_Q, \qquad X \in \mathbb{R}^2,$$
$$\mathcal{D}_D f(X) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle \nu_Q, X-Q\rangle}{|X-Q|^2} f(Q)\,d\sigma_Q, \qquad X \in \mathbb{R}^2.$$

The following trace formula is well known (see [F] or [8]):

$$\frac{\partial}{\partial \nu^{\pm}}\mathcal{S}_D f(P) = (\pm\frac{1}{2}I + \mathcal{K}_D^*)f(P) \qquad (P \in \partial D) \tag{2.1}$$

where

$$\mathcal{K}_D^* f(P) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle \nu_P, P-Q\rangle}{|P-Q|^n} f(Q)\,d\sigma_Q.$$

We denote by $\mathcal{K}_D$ the dual of $\mathcal{K}_D^*$.

Let $L^2_0(\partial\Omega) = \{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f d\sigma = 0\}$. Then the representation formula for the solution to the problem $P[D, g]$ is as follows.

Representation Formula [9].

If u is the weak solution to the Neumann problem $P[D, g]$, then there are unique harmonic function $H \in W^{1,2}(\Omega)$ and $\varphi_D \in L^2_0(\partial D)$ so that u can be expressed as

$$u(x) = H(x) + \mathcal{S}_D \varphi_D(x) \quad \text{for } x \in \Omega. \tag{2.2}$$

Moreover, if $f = u|_{\partial\Omega}$,

$$H(x) = -\mathcal{S}_\Omega g(x) + \mathcal{D}_\Omega f(x) \tag{2.3}$$

and

$$\left(\frac{k+1}{2(k-1)} I - \mathcal{K}^*_D\right)\varphi_D = \frac{\partial H}{\partial \nu}\Big|_{\partial D} \quad \text{on } \partial D. \tag{2.4}$$

See [9] for proof. We remark that the representation formula holds for Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$.

LEMMA 2.1 [9]. *If u is the weak solution to the Neumann problem $P[D, g]$, then*

$$\varphi_D = (k-1)\frac{\partial u}{\partial \nu^-} = \frac{k-1}{k}\frac{\partial u}{\partial \nu^+} \tag{2.6}$$

3. Proofs

Let D and D_h be as in Section 1. Write ∂D_h as

$$\partial D_h : \zeta + h\omega_h(\zeta)\nu(\zeta), \qquad \zeta \in \partial D$$

if slight abuse of notations is allowed, where $\omega_h(\zeta)$ is a C^1 function on ∂D whose C^1 norm is uniformly bounded and $\nu(\zeta)$ is the outward unit normal to ∂D at ζ. Let u and u_h be the weak solutions of $P[D, g]$ and $P[D_h, g]$, respectively. By the representation formula (2.2), the solutions u and u_h can be expressed uniquely as :

$$u = H + \mathcal{S}_D\varphi_D \quad \text{and} \quad u_h = H_h + \mathcal{S}_{D_h}\varphi_{D_h} \quad \text{in } \Omega \tag{3.1}$$

where H, φ_D, H_h, and φ_{D_h} satisfy the relations (2.3) and (2.4). To make the notations short, we put

$$\varphi = \varphi_D,\ \mathcal{S} = \mathcal{S}_D,\ \mathcal{K}^* = \mathcal{K}^*_D,\ \varphi_h = \varphi_{D_h},\ \mathcal{S}_h = \mathcal{S}_{D_h},\ \mathcal{K}^*_h = \mathcal{K}^*_{D_h}.$$

LEMMA 3.1. *There is a positive constant C such that*

$$\|\nabla(H - H_h)\|_{L^\infty(\Omega)} < Ch. \tag{3.2}$$

Lemma 3.1 is proved in [7].

By identifying the real 2-D vector (v_1, v_2) with the complex number $v_1 + iv_2$, we can see that

$$\nabla \mathcal{S}\varphi(z) = \frac{-1}{2\pi}\int_{\partial D} \frac{\varphi(\zeta)}{\bar{z} - \bar{\zeta}} d|\zeta| \tag{3.3}$$

and

$$\mathcal{K}\varphi(z) = \frac{1}{2\pi}\Im \int_{\partial D} \frac{\varphi(\zeta)}{z - \zeta} d\zeta. \tag{3.4}$$

Here $\Im$ is the imaginary part. Let Φ_h be the diffeomorphism from ∂D onto ∂D_h defined by $\Phi_h(\zeta) = \zeta + h\omega_h(\zeta)\nu(\zeta)$.

LEMMA 3.2. *There is a positive constant C such that*

$$\|\varphi_h \circ \Phi_h - \varphi\|_{L^2(\partial D)} \leq Ch$$

if h is small enough.

Proof. Let $\lambda = \frac{k+1}{2(k-1)}$. Since $(\lambda I - \mathcal{K}^*)$ is invertible on $L^2(\partial D)$ [5], we have from (2.4) that

$$\begin{aligned}
&\|\varphi_h \circ \Phi_h - \varphi\|_{L^2(\partial D)} \\
&\quad \leq C\|(\lambda I - \mathcal{K}^*)(\varphi_h \circ \Phi_h - \varphi)\|_{L^2(\partial D)} \\
&\quad \leq C\|(\lambda I - \mathcal{K}_h^*\varphi_h) \circ \Phi_h - (\lambda I - \mathcal{K}^*)\varphi\|_{L^2(\partial D)} \\
&\qquad\quad + C\|(\mathcal{K}_h^*\varphi_h) \circ \Phi_h - \mathcal{K}^*(\varphi_h \circ \Phi_h)\|_{L^2(\partial D)} \\
&\quad \leq C\|\frac{\partial H_h}{\partial \nu} \circ \Phi_h - \frac{\partial H}{\partial \nu}\|_{L^2(\partial D)} \\
&\qquad\quad + C\|(\mathcal{K}_h^*\varphi_h) \circ \Phi_h - \mathcal{K}^*(\varphi_h \circ \Phi_h)\|_{L^2(\partial D)}.
\end{aligned}$$

Since the first term in the most right hand side of the above inequalities is $O(h)$ by Lemma 3.1, Lemma 3.2 follows from the following lemma. □

SUBLEMMA. *For any function* $g \in L^2(\partial D_h)$

$$\|(\mathcal{K}_h^* g) \circ \Phi_h - \mathcal{K}^*(g \circ \Phi_h)\|_{L^2(\partial D)} \leq Ch\|g\|_{L^2(\partial D_h)}$$

if h *is small enough.*

Proof. By duality and boundedness of Φ_h', it suffices to show that

$$\|(\mathcal{K}_h g) \circ \Phi_h - \mathcal{K}(g \circ \Phi_h)\|_{L^2(\partial D)} \leq Ch\|g\|_{L^2(\partial D_h)}$$

for any $g \in L^2(\partial D_h)$. By (3.4),

$$\begin{aligned}
&(\mathcal{K}_h g) \circ \Phi_h(z) - \mathcal{K}(g \circ \Phi_h)(z) \\
&\quad = \frac{1}{2\pi} \Im \left[\int_{\partial D_h} \frac{g(\zeta)}{\Phi_h(z) - \zeta} d\zeta - \int_{\partial D} \frac{(g \circ \Phi_h)(\zeta)}{z - \zeta} d\zeta \right] \\
&\quad = \frac{1}{2\pi} \Im \int_{\partial D} \left[\frac{\Phi_h'(\zeta)}{\Phi_h(z) - \Phi_h(\zeta)} - \frac{1}{z - \zeta} \right] (g \circ \Phi_h)(\zeta) d\zeta \\
&\quad = \frac{1}{2\pi} \Im \int_{\partial D} \left[\frac{1}{\Phi_h(z) - \Phi_h(\zeta)} - \frac{1}{z - \zeta} \right] (g \circ \Phi_h)(\zeta) d\zeta \\
&\qquad + \frac{1}{2\pi} \Im \int_{\partial D} \frac{h(\omega_h \nu)'(\zeta)}{\Phi_h(z) - \Phi_h(\zeta)} (g \circ \Phi_h)(\zeta) d\zeta \\
&\quad := I(z) + II(z).
\end{aligned}$$

Since the Cauchy transform on C^2-curves (in fact, on Lipschitz curves) is bounded on L^2, we have

$$\int_{\partial D} |II(\zeta)|^2 d|\zeta| \leq Ch^2 \|g\|^2_{L^2(\partial D_h)}.$$

Note that

$$I(z) = \frac{1}{2\pi} \Im \int_{\partial D} \frac{1}{z - \zeta} \sum_{j=1}^{\infty} h^j \left(\frac{(\omega_h \nu)(z) - (\omega_h \nu)(\zeta)}{z - \zeta} \right)^j (g \circ \Phi_h)(\zeta) d\zeta.$$

It is proven in [3] that

$$\begin{aligned}
\int_{\partial D} |I(\zeta)|^2 d|\zeta| &\leq \sum_{j=1}^{\infty} (hC\|(\omega_h \nu)'\|_{L^\infty(\partial D)})^{2j} \|g\|^2_{L^2(\partial D_h)} \\
&\leq C' h^2 \|g\|^2_{L^2(\partial D_h)}
\end{aligned}$$

if h is small enough. This completes the proof. □

Finally the following lemma leads us to Theorem 1.1.

LEMMA 3.3. *There exists a constant C such that*

(3.5)

$$\lim_{h\to 0}\frac{1}{h}\int_{\Omega\setminus D\Delta D_h}\left|\nabla(\mathcal{S}\varphi-\mathcal{S}_h\varphi_h)(z)\right|^2 dV(z)=0$$

(3.6)

$$\limsup_{h\to 0}\frac{1}{h}\int_{D\Delta D_h}\left|\nabla(\mathcal{S}\varphi-\mathcal{S}_h\varphi_h)(z)\right|^2 dV(z)\le C\int_{\partial D}|\varphi(\zeta)|^2 d\sigma(\zeta).$$

Moreover, if $\omega_h\to\omega$ uniformly as $h\to 0$, then

(3.7)

$$\lim_{h\to 0}\frac{1}{h}\int_{D\Delta D_h}\left|\nabla(\mathcal{S}\varphi-\mathcal{S}_h\varphi_h)(z)\right|^2 dV(z)=\int_{\partial D}|\varphi(\zeta)|^2|\omega(\zeta)|d\sigma(\zeta).$$

Proof. It is easy to see that for each $\delta>0$

$$\lim_{h\to 0}\frac{1}{h}\int_{\substack{\mathrm{dist}(z,\partial D)>\delta\\ z\in\Omega}}\left|\nabla(\mathcal{S}\varphi-\mathcal{S}_h\varphi_h)(z)\right|^2 dV(z)=0.$$

So, we assume, from the beginning, that $\Omega=\{z=\zeta+t\nu(\zeta):|t|<\delta,\zeta\in\partial D\}$ for some δ. (δ is chosen so that the normal projection from Ω onto ∂D is well-defined.) Let $\epsilon>0$ be a fixed number to be determined later and let U be the tubular neighborhood of ∂D defined by $U=\{z=\zeta+t\nu(\zeta):|t|<\epsilon,\zeta\in\partial D\}$. If $z\in\Omega\setminus U$, then by (3.3)

$$\begin{aligned}
&\nabla(\mathcal{S}_h\varphi_h-\mathcal{S}\varphi)(z)\\
&=\frac{1}{2\pi}\left[\int_{\partial D}\frac{\varphi(\zeta)}{\bar z-\bar\zeta}d|\zeta|-\int_{\partial D_h}\frac{\varphi_h(\zeta)}{\bar z-\bar\zeta}d|\zeta|\right]\\
&=\frac{1}{2\pi}\left[\int_{\partial D}\frac{\varphi(\zeta)-\varphi_h\circ\Phi_h(\zeta)}{\bar z-\bar\zeta}d|\zeta|+h\int_{\partial D}\frac{(\omega_h\nu)(\zeta)}{(\bar z-\bar\zeta)(\bar z-\overline{\Phi_h(\zeta)})}\varphi_h\circ\Phi_h(\zeta)d|\zeta|\right.\\
&\qquad\left.+\int_{\partial D}\frac{\varphi_h\circ\Phi_h(\zeta)}{\bar z-\overline{\Phi_h(\zeta)}}[|\Phi_h'(\zeta)|-1]d|\zeta|\right]\\
&:=I_1(z)+I_2(z)+I_3(z).
\end{aligned}$$

Suppose $z=\xi+t\nu(\xi)\in\Omega\setminus D$, $\xi\in\partial D$. If N is the smallest integer such that $2^N t>\max\{|z-\zeta|:\zeta\in\partial D\}$, then $N\le C\log\frac{1}{t}$ and

$$\begin{aligned}
|I_1(z)|&\le\sum_{j=1}^{N}\int_{\substack{2^{j-1}t\le|\xi-\zeta|<2^j t\\ \zeta\in\partial D}}\frac{|\varphi(\zeta)-\varphi_h\circ\Phi_h(\zeta)|}{|z-\zeta|}d|\zeta|\\
&\le C|\log t|M(\varphi-\varphi_h\circ\Phi_h)(\xi)
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator on ∂D. Since M is bounded on $L^2(\partial D)$ ([10]), it follows from Lemma 3.2 that

$$\begin{aligned}\int_{\Omega\setminus U} |I_1(z)|^2 dV &= \int_{\epsilon<|t|<\delta}\int_{\partial D} |I_1(\xi+t\nu(\xi))|^2 d|\xi|\, dt \\ &\le C\int_{\epsilon<|t|<\delta} |\log t|^2 dt \int_{\partial D} |M(\varphi-\varphi_h\circ\Phi_h)(\xi)|^2 d|\xi| \\ &\le C\|\varphi-\varphi_h\circ\Phi_h\|^2_{L^2(\partial D)} \le Ch^2. \end{aligned} \tag{3.8}$$

For $I_2(z)$, we have

$$|I_2(z)| \le Ch\int_{\partial D} \frac{1}{|z-\zeta|^2} d|\zeta| \le Cht^{-1},$$

and hence

$$\int_{\Omega\setminus U} |I_2(z)|^2 dV \le Ch^2 \int_{\epsilon<|t|<\delta} t^{-2} dt \le Ch^2\epsilon^{-1}. \tag{3.9}$$

Since $|\Phi_h'(\zeta)|-1 = O(h)$, we have

$$|I_3(z)| \le Ch\int_{\partial D} \frac{1}{|z-\zeta|} d|\zeta| \le Ch\log\frac{1}{t}. \tag{3.10}$$

Thus, we have

$$\int_{\Omega\setminus U} |I_3(z)|^2 dV \le Ch^2. \tag{3.11}$$

Combining (3.8), (3.9), and (3.11), we have

$$\frac{1}{h}\int_{\Omega\setminus U} \left|\nabla(\mathcal{S}\varphi-\mathcal{S}_h\varphi_h)(z)\right|^2 dV \le Ch\epsilon^{-1}. \tag{3.12}$$

Now suppose that $z=\xi+t\nu(\xi)\in U$. Put $\xi_h=\xi+h\omega_h(\xi)\nu(\xi)$. Put $S^\epsilon=\{\zeta\in\partial D : |\zeta-\xi|<\epsilon\}$ and $S_h^\epsilon=\{\zeta+h\omega(\zeta)\nu(\zeta) : \zeta\in S^\epsilon\}$. Then,

$$\begin{aligned}\nabla(\mathcal{S}_h\varphi_h-\mathcal{S}\varphi)(z) &= \frac{1}{2\pi}\left[\int_{\partial D\setminus S^\epsilon} \frac{\varphi(\zeta)}{\bar z-\bar\zeta} d|\zeta| - \int_{\partial D_h\setminus S_h^\epsilon} \frac{\varphi_h(\zeta)}{\bar z-\bar\zeta} d|\zeta|\right] \\ &\quad + \frac{1}{2\pi}\left[\int_{S^\epsilon} \frac{\varphi(\zeta)-\varphi(\xi)}{\bar z-\bar\zeta} d|\zeta| - \int_{S_h^\epsilon} \frac{\varphi_h(\zeta)-\varphi_h(\xi_h)}{\bar z-\bar\zeta} d|\zeta|\right] \\ &\quad + \frac{\varphi_h(\xi_h)}{2\pi}\left[\int_{S^\epsilon} \frac{1}{\bar z-\bar\zeta} d|\zeta| - \int_{S_h^\epsilon} \frac{1}{\bar z-\bar\zeta} d|\zeta|\right] \\ &\quad + \frac{[\varphi(\xi)-\varphi_h(\xi_h)]}{2\pi}\int_{S^\epsilon} \frac{1}{\bar z-\bar\zeta} d|\zeta| \\ &:= II_1(z)+II_2(z)+II_3(z)+II_4(z). \end{aligned}$$

In the same way to derive (3.8), one can see that

$$\frac{1}{h}\int_U |II_1(z)|^2 dV \le Ch\epsilon^{-1}. \tag{3.13}$$

Since φ is C^α for every $\alpha < 1$ (see [4]), $|II_2(z)| \le C\epsilon^\alpha$ independently of h. Hence, we have

$$\frac{1}{h}\int_U |II_2(z)|^2 dV \le C_\alpha \epsilon^{2\alpha+1}h^{-1} \quad \text{for every } \alpha < 1. \tag{3.14}$$

By Lemma 3.2 and the estimate used in (3.10), we have

(3.15)
$$\begin{aligned}\frac{1}{h}\int_U |II_4(z)|^2 dV &= \frac{1}{h}\int_{-\epsilon}^{\epsilon}\int_{\partial D} |I_4(\xi + t\nu(\xi))|^2 d|\xi| dt \\ &\le \frac{C}{h}\int_{-\epsilon}^{\epsilon}\int_{\partial D} |\varphi(\xi) - \varphi_h(\xi_h)|^2 |\log t|^2 d|\xi| dt \\ &\le Ch.\end{aligned}$$

We now deal with $II_3(z)$. Put

$$\overline{W_h(z)} = \frac{1}{2\pi}\left[\int_{S^\epsilon} \frac{1}{z-\zeta} d|\zeta| - \int_{S_h^\epsilon} \frac{1}{z-\zeta} d|\zeta|\right].$$

For each $z \in U$, we may assume that S^ϵ is a graph by taking ϵ small enough if necessary, namely,

$$S^\epsilon : \zeta = x + ig(x), \qquad -\epsilon < x < \epsilon, \qquad g \in C^2,$$

$g(0) = g'(0) = 0$, and $z = it(|t| < h)$. Put

$$J(\zeta) = \frac{|1 + ig'(x)|}{1 + ig'(x)} \text{ and } J_h(\zeta) = \frac{|1 + ig'(x) + h(\omega_h\nu)'(x)|}{1 + ig'(x) + h(\omega_h\nu)'(x)}.$$

Let Γ_h (Γ_h', resp.) be the straight line connecting the left (right, resp.) endpoints of S^ϵ and S_h^ϵ and let C_h be the positively oriented closed curve composed of S^ϵ, S_h^ϵ, Γ_h, and Γ_h'. Then

$$\begin{aligned}2\pi\overline{W_h(z)} &= \int_{S^\epsilon} \frac{J(\zeta)}{z-\zeta}d\zeta - \int_{S_h^\epsilon} \frac{J_h(\zeta)}{z-\zeta}d\zeta \\ &= \int_{C_h} \frac{1}{z-\zeta}d\zeta - \int_{\Gamma_h+\Gamma_h'} \frac{1}{z-\zeta}d\zeta \\ &\qquad + \int_{S^\epsilon} \frac{J(\zeta)-1}{z-\zeta}d\zeta - \int_{S_h^\epsilon} \frac{J_h(\zeta)-1}{z-\zeta}d\zeta.\end{aligned}$$

By the Cauchy integral formula,

$$\left|\int_{C_h} \frac{1}{z-\zeta}d\zeta\right| = \begin{cases} 2\pi & \text{if } z \in D\Delta D_h \\ 0 & \text{if } z \in \Omega \setminus \overline{D\Delta D_h}. \end{cases}$$

It is easy to see that

$$\left|\int_{\Gamma_h+\Gamma_h'} \frac{1}{z-\zeta}d\zeta\right| \le Ch\epsilon^{-1}.$$

Note that

$$\begin{aligned}&|J(\zeta)-1| \le C|g'(x)| \le C|x| \le C|\zeta|, \quad \zeta \in \partial D \\ &|J_h(\zeta)-1| \le C|g'(x)| + Ch|\omega_h'(x)| \le C(|\zeta|+h), \quad \zeta \in \partial D_h.\end{aligned}$$

It then follows that for $z = \xi + t\nu(\xi)$,

$$\begin{aligned}&\left|\int_{S^\epsilon} \frac{J(\zeta)-1}{|z-\zeta|}d\zeta\right| \le C\int_{S^\epsilon} \frac{|\zeta|}{|z-\zeta|}d|\zeta| \le C\epsilon, \\ &\left|\int_{S_h^\epsilon} \frac{J_h(\zeta)-1}{z-\zeta}d\zeta\right| \le C\int_{S_h^\epsilon} \frac{|\zeta|+h}{|z-\zeta|}d|\zeta| \le C(\epsilon + h\log\frac{1}{|h\omega_h(\xi)-t|}).\end{aligned}$$

So far, we proved that

$$|W_h(z)| \le C(\epsilon + h\epsilon^{-1} + h\log\frac{1}{|h\omega_h(\xi)-t|}) \tag{3.16}$$

if $z = \xi + t\nu(\xi) \notin \overline{D\Delta D_h}$, and

$$||W_h(z)| - 1| \leq C(\epsilon + h\epsilon^{-1} + h \log \frac{1}{|(h-s)\omega_h(\xi)|}) \tag{3.17}$$

if $z = \xi + s\omega_h(\xi)\nu(\xi) \in D\Delta D_h$. It then follows from (3.16) that

$$\begin{aligned} &\frac{1}{h}\int_{U\setminus D\Delta D_h} |II_3(z)|^2 dV \\ &\quad = \frac{1}{h}\int_{h|\omega_h(\xi)|\leq t<\epsilon}\int_{\partial D} |\varphi_h(\xi_h)W_h(\xi + t\nu(\xi))|^2 d|\xi| dt \\ &\quad \leq C(\epsilon^3 h^{-1} + h\epsilon^{-1} + h). \end{aligned} \tag{3.18}$$

On the other hand, from (3.17), we have

$$\begin{aligned} &\left|\frac{1}{h}\int_{D\Delta D_h} |II_3(z)|^2 dV - \int_{\partial D} |\varphi(\xi)|^2 |\omega_h(\xi)| d|\xi|\right| \\ &\quad \leq \frac{1}{h}\int_{\partial D}\int_0^h \left||I_3(\xi + s\omega_h(\xi)\nu(\xi))|^2 - |\varphi(\xi)|^2\right| ds|\omega_h(\xi)| d|\xi| \\ &\quad \leq C\int_{\partial D} |\varphi_h(\xi_h) - \varphi(\xi)|^2 d|\xi| \\ &\qquad + \frac{C}{h}\int_{\partial D}\int_0^h \left||W_h(\xi + s\omega_h(\xi)\nu(\xi))|^2 - 1\right| ds|\varphi(\xi)|^2|\omega_h(\xi)| d|\xi| \\ &\quad \leq C(h^2 + \epsilon + h\epsilon^{-1} + \int_{\partial D} |h\omega_h(\xi)| \log \frac{1}{|h\omega_h(\xi)|} d|\xi|). \end{aligned} \tag{3.19}$$

Take $\epsilon = h^{2/3}$. Then (3.12)-(3.15), (3.18), and (3.19) prove Lemma 3.3. □

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