J. Korean Math. Soc. **34** (1997), No. 3, pp. 623–632

NONEXISTENCE OF GLOBAL SOLUTIONS OF SOME QUASILINEAR INITIAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, the nonexistence of the global solutions of semilinear wave equations with damping terms in the boundary conditions is investigated.

1. Introduction

We study the initial value problems

$$\Delta^2 u = f(u), \quad (t,x) \in (0,T) \times \Omega, \tag{1.1}$$

$$u = 0, \Delta u + \alpha(x)\frac{\partial u_t}{\partial \nu} = 0, \quad (t,x) \in (0,T) \times \Gamma, \tag{1.2}$$

$$u = 0, \Delta u + \alpha(x)\frac{\partial u_t}{\partial \nu} = 0, \quad (t,x) \in (0,T) \times \Gamma, \tag{1.3}$$

$$u(0,x) = u_0(x), \quad x \in \Omega \tag{1.4}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary Γ, $T > 0$ is an arbitrary real number, ν is the outward normal and $\alpha(x)$ is a smooth nonnegative function given on the boundary of the domain Ω. The nonexistence of the global solutions of quasilinear parabolic equations with no damping terms in the boundary conditions is investigated by H. A. Levine [4], O. A. Ladyzhenskaya and V. K. Kalantarov [3], J. L. Lions [8], R. T. Glassey and recently by H. A. Levine and J. Serrin [7]. In this paper we look at a new type of problem in which we have a semilinear wave equation with a damping term in *the boundary condition*

Received March 18, 1997.

1991 Mathematics Subject Classification: Primary 35L75; Secondary 35L35, 35L30.

Key words and phrases: global solution, damping.

[see 9,10] rather than in the equation as was considered in [2] in concrete cases and in [5] in an abstract setting.

Natures of the solutions to these equations have been investigated by several means. However the methods used for the investigation of the initial boundary value problems with no damping terms in the boundary conditions are quite insufficient to deal with the problems with the damping terms in the boundary conditions. The tool used in this work is a Lemma due to H.A. Levine [7]. From now on we will call it the Concavity Lemma. The most crucial point in the application of this tool is to find a functional that represents the dissipation on the boundary and satisfies the conditions of the Concavity Lemma.

The plan of the paper is as follows. The end of this section we will write down the Concavity Lemma and will prove it. In the next section we will prove the nonexistence of global solutions for (1.1),(1.2) and (1.4) for negative initial energy. In final section we will prove the nonexistence of global solutions for (1.1),(1.3) and (1.4) for negative initial energy if the structure conditions are satisfied.

LEMMA 1. *If a function*

$$F \in C^2, \quad F \geq 0$$

satisfies the inequality

$$F''(t)F(t) - (1+\gamma)[F'(t)]^2 \geq 0 \tag{1.5}$$

for some real number $\gamma > 0$, *as long as it is defined then if* $F(0) > 0, F'(0) > 0$ *then for the real number*

$$t_2 = \frac{F(0)}{\gamma F'(0)} \tag{1.6}$$

there exists a positive real number $t_1 < t_2$ *such that as* $t \to t_1$

$$F(t) \to +\infty \tag{1.7}$$

The proof of this lemma is quite easy. One observe that from (1.5) we have $(F^{-\gamma})'' \leq 0$ as long as $F > 0$. Since the differential inequality tells us that F is convex and $F'(0) > 0$, F must be increasing and hence cannot change sign. The rest of the lemma follows from the observations that $F^{-\gamma}$ must be below its tangent line at $(0, F^{-\gamma}(0))$ and that the slope of this line is negative. Therefore the line and hence $F^{-\gamma}$ must cross the

t axis. The line does so in time t_2 while the function $F^{-\gamma}$ does so in a possibly earlier time t_1. That is

$$[F(t)]^\gamma \geq \frac{[F(0)]^{1+\gamma}}{F(0) - \gamma F'(0)t}, \quad 0 < t < t_2.$$

2. The first problem

Let us assume that the initial-boundary value problem (1.1),(1.2) and (1.4) has a local classical solution.

Let $f(u)$ with the primitive

$$\mathcal{F}(u) = \int_0^u f(\xi)d\xi \tag{2.1}$$

satisfy the inequality

$$f(u) \cdot u \geq 2(1+\gamma)\mathcal{F}(u) \tag{2.2}$$

for some real number $\gamma > 0$, and for all $u \in R^1$. Let

$$E(t) = \frac{1}{2}\int_\Omega (\Delta u)^2 dx - \int_\Omega \mathcal{F}(u)dx \tag{2.3}$$

for function $u(t,x)$. Then we can prove the following theorem about the nonexistence of global solutions of the initial-boundary value problem (1.1), (1.2) and (1.4).

THEOREM 1. *Let u_0 be a smooth function such that $E(0) < 0$. Then there exists no global solution of the initial-boundary value problem* (1.1), (1.2) *and* (1.4).

Proof. We suppose that there exists a global solution $u(t,x)$ of (1.1), (1.2) and (1.4). To prove the theorem, it suffices to show that the functional

$$F(t) = \int_0^t \int_\Gamma \alpha(x)(\frac{\partial u}{\partial \nu})^2 dx d\eta + (T_0 - t)\int_\Gamma \alpha(x)(\frac{\partial u_0}{\partial \nu})^2 dx + \beta(t+t_0)^2 \tag{2.4}$$

satisfies the hypothesis of the Concavity Lemma. Here

$$0 < \beta \leq \frac{-2(\gamma+1)}{2\gamma+1}E(0), \; t_0 > \frac{\int_\Gamma \alpha(x)(\frac{\partial u_0}{\partial \nu})^2 dx}{2\gamma\beta}$$

and

$$T_0 \geq \frac{\beta t_0^2}{2\gamma\beta t_0 - \int_\Gamma \alpha(x)(\frac{\partial u_0}{\partial \nu})^2 dx}$$

are real constants. If we prove that the functional $F(t)$ satisfies the hypothesis of the Concavity Lemma, then there exists a real number $T < \infty$ such that

$$\lim_{t \to T} \int_0^t \int_\Gamma \alpha(x)(\frac{\partial u}{\partial \nu})^2 dx d\eta = \infty. \tag{2.5}$$

This means that the solution blows up in finite time.

To show the functional $F(t)$ satisfies the hypothesis of the Concavity Lemma let us make the observation

$$\begin{aligned} 0 &= -\int_\Omega u\Delta^2 u + \int_\Omega u f(u) dx \\ &= \int_\Omega \nabla u \cdot \nabla(\Delta u) dx - \int_\Gamma u \cdot \frac{\partial \Delta u}{\partial \nu} dx - \int_\Omega u f(u) dx \\ &= -\int_\Omega (\Delta u)^2 dx + \int_\Gamma \Delta u \frac{\partial u}{\partial \nu} dx + \int_\Omega u f(u) dx \\ &= -\int_\Omega (\Delta u)^2 dx - \int_\Gamma \alpha(x) \frac{\partial u}{\partial \nu} \frac{\partial u_t}{\partial \nu} dx + \int_\Omega u f(u) dx \end{aligned} \tag{2.6}$$

by using $\int_\Gamma u \cdot \frac{\partial \Delta u}{\partial \nu} dx = 0$ and (1.2).

So we have

$$\int_\Gamma \alpha(x) \frac{\partial u}{\partial \nu} \frac{\partial u_t}{\partial \nu} dx = -\int_\Omega (\Delta u)^2 dx + \int_\Omega u f(u) dx. \tag{2.7}$$

And we also observe

$$\begin{aligned} &\int_\Omega u_t \Delta^2 u dx \\ &= -\int_\Omega \nabla u_t \nabla(\Delta u) dx + \int_\Gamma u_t \frac{\partial \Delta u}{\partial \nu} dx \\ &= \int_\Omega \Delta u \Delta u_t dx - \int_\Gamma \Delta u \frac{\partial u_t}{\partial \nu} dx \\ &= \int_\Omega \Delta u \Delta u_t dx + \int_\Gamma \alpha(x) (\frac{\partial u_t}{\partial \nu})^2 dx \end{aligned}$$

by using (1.2) and $\int_\Gamma u_t \frac{\partial \Delta u}{\partial \nu} dx = 0$.

Hence we get

$$\begin{aligned} 0 &= -\int_\Omega u_t \Delta^2 u dx + \int_\Omega u_t f(u) dx \\ &= -\int_\Omega \Delta u \Delta u_t dx - \int_\Gamma \alpha(x)(\frac{\partial u_t}{\partial \nu})^2 dx + \int_\Omega u_t f(u) dx. \end{aligned} \tag{2.8}$$

Now we can get

$$\begin{aligned} &0 \\ &= \frac{1}{2}\frac{d}{dt} \int_\Omega (\Delta u)^2 dx + \int_\Gamma \alpha(x)(\frac{\partial u_t}{\partial \nu})^2 dx - \frac{d}{dt} \int_0^t \int_\Omega f(u) u_\eta dx d\eta \\ &= \frac{d}{dt}[\frac{1}{2} \int_\Omega (\Delta u)^2 dx - \int_\Omega \mathcal{F}(u) dx] + \int_\Gamma \alpha(x)(\frac{\partial u_t}{\partial \nu})^2 dx \end{aligned} \tag{2.9}$$

where $E(t)$ was defined in (2.3). Then integrating the above relation with respect to t, over the interval $[0, t]$ we get

$$E(t) = E(0) - \int_0^t \int_\Gamma \alpha(x)(\frac{\partial u_\eta}{\partial \nu})^2 dx d\eta.$$

Now we want to calculate F' and F''.

$$F'(t)=\int_\Gamma \alpha(x)(\frac{\partial u}{\partial\nu})^2dx-\int_\Gamma \alpha(x)(\frac{\partial u_0}{\partial\nu})^2dx+2\beta(t+t_0)$$

and

$$F''(t)=2\int_\Gamma \alpha(x)\frac{\partial u}{\partial\nu}\frac{\partial u_t}{\partial\nu}dx+2\beta.$$

Thus we have from (2.2) and (2.7)

$$\begin{array}{rcl} & & F''(t) \\ & \geq & -2\int_\Omega(\Delta u)^2dx+2\int_\Omega uf(u)dx+2\beta. \\ & \geq & 2[-\int_\Omega(\Delta u)^2dx+2(\gamma+1)\int_\Omega\mathcal{F}(u)dx+\beta] \\ & \geq & 2[-\int_\Omega(\Delta u)^2dx+(\gamma+1)\int_\Omega(\Delta u)^2dx \\ (2.10) & & +2(\gamma+1)\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u_\eta}{\partial\nu})^2dxd\eta-2(\gamma+1)E(0)+\beta] \\ & = & 2\gamma\int_\Omega(\Delta u)^2dx+4(\gamma+1)\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u_\eta}{\partial\nu})^2dxd\eta \\ & & +2\beta-4(\gamma+1)E(0) \\ & \geq & 4(\gamma+1)[\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u_\eta}{\partial\nu})^2dxd\eta+\beta]+D_0 \end{array}$$

where $D_0=2\beta-4(\gamma+1)\beta-4(\gamma+1)E(0)$. Since

$$\int_0^t\frac{d}{d\eta}\int_\Gamma\alpha(x)(\frac{\partial u}{\partial\nu})^2dxd\eta=2\int_0^t\int_\Gamma\alpha(x)\frac{\partial u}{\partial\nu}\frac{\partial u_\eta}{\partial\nu}dxd\eta,$$

$$\int_\Gamma\alpha(x)(\frac{\partial u}{\partial\nu})^2dx-\int_\Gamma\alpha(x)(\frac{\partial u_0}{\partial\nu})^2dx=2\int_0^t\int_\Gamma\alpha(x)\frac{\partial u}{\partial\nu}\frac{\partial u_\eta}{\partial\nu}dxd\eta.$$

By using Hölder's inequality and Young's inequality,

(2.18)

$$\begin{array}{rl} & [F'(t)]^2 \\ = & [\int_\Gamma\alpha(x)(\frac{\partial u}{\partial\nu})^2dx-\int_\Gamma\alpha(x)(\frac{\partial u_0}{\partial\nu})^2dx+2\beta(t+t_0)]^2 \\ = & [2\int_0^t\int_\Gamma\alpha(x)\frac{\partial u}{\partial\nu}\frac{\partial u_\eta}{\partial\nu}dxd\eta+2\beta(t+t_0)]^2 \\ \leq & 4[\{\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u}{\partial\nu})^2dxd\eta\}^{\frac12}\{\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u_\eta}{\partial\nu})^2dxd\eta\}^{\frac12}+\beta(t+t_0)]^2. \end{array}$$

Now we want to find a lower bound for the function

$$\Xi(t)=F''F-(1+\gamma)(F')^2. \tag{2.19}$$

Let

$$A=\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u_\eta}{\partial\nu})^2dxd\eta,\ B=\int_0^t\int_\Gamma\alpha(x)(\frac{\partial u}{\partial\nu})^2dxd\eta. \tag{2.20}$$

So we get
(2.21)

$$\begin{aligned}
&\Xi(t)\\
= \;& F''F-(1+\gamma)(F')^2\\
\geq \;& 4(1+\gamma)[(A+\beta)(B+\beta(t+t_0)^2)-\{A^{\frac{1}{2}}B^{\frac{1}{2}}+\beta(t+t_0)\}^2].
\end{aligned}$$

Now we note from the choice of β that

$$D_0 \geq 0 \tag{2.22}$$

if the initial energy is negative, that is $E(0)<0$. So, by the Schwarz inequality,

$$(A+\beta)(B+\beta(t+t_0)^2)-\{A^{\frac{1}{2}}B^{\frac{1}{2}}+\beta(t+t_0)\}^2 \geq 0.$$

Thus

$$\Xi(t) \geq 0$$

as long as it is defined. By the Concavity lemma, the theorem is proved. □

3. The second problem

We assume that the initial-boundary value problem (1.1),(1.3) and (1.4) has a local classical solution. Also we assume that the structure condition (2.2) for function f is satisfied.

Let

$$E(t)=\frac{1}{2}\int_\Omega(\Delta u)^2dx+\frac{1}{2}\int_\Gamma\alpha(x)(\frac{\partial u_t}{\partial\nu})^2dx-\int_\Omega\mathcal{F}(u)dx \tag{3.1}$$

for function $u(x,t)$. Then we have Theorem 2;

THEOREM 2. *Let u_0 be a smooth function such that $E(0)<0$. Then there exists no global solution of (1.1), (1.2) and (1.4).*

Proof. We suppose that there exists a global solution $u(t,x)$ of (1.1), (1.3) and (1.4). To prove the theorem, it suffices to show that the function

$$F(t)=\int_\Gamma\alpha(x)(\frac{\partial u}{\partial\nu})^2dx+\beta(t+t_0)^2 \tag{3.2}$$

satisfies the hypothesis of the Concavity Lemma. Here β, t_0 are real constants which will be specified later. If $F(t)$ satisfies the hypothesis of Concavity Lemma, then there exists a real number $T < \infty$ such that

$$\lim_{t \to T} \int_\Gamma \alpha(x)(\frac{\partial u}{\partial \nu})^2 dx = \infty. \tag{3.3}$$

To show the functional $F(t)$ satisfies the hypothesis of the Concavity Lemma let us make the observation

$$0 = -\int_\Omega (\Delta u)^2 dx - \int_\Gamma \alpha(x) \frac{\partial u}{\partial \nu}\frac{\partial u_{tt}}{\partial \nu} dx + \int_\Omega u f(u) dx \tag{3.4}$$

by using $\int_\Gamma u \cdot \frac{\partial \Delta u}{\partial \nu} dx = 0$ and (1.3).

So we have

$$\int_\Gamma \alpha(x) \frac{\partial u}{\partial \nu}\frac{\partial u_{tt}}{\partial \nu} dx = -\int_\Omega (\Delta u)^2 dx + \int_\Omega u f(u) dx. \tag{3.5}$$

And we also observe

$$\int_\Omega u_t \Delta^2 u dx = \int_\Omega \Delta u \Delta u_t dx + \int_\Gamma \alpha(x) \frac{\partial u_t}{\partial \nu}\frac{\partial u_{tt}}{\partial \nu} dx \tag{3.6}$$

by using (1.3) and $\int_\Gamma u_t \frac{\partial \Delta u}{\partial \nu} dx = 0$.

Hence we get

$$\begin{aligned} 0 &= -\int_\Omega u_t \Delta^2 u dx + \int_\Omega u_t f(u) dx \\ &= -\int_\Omega \Delta u \Delta u_t dx - \int_\Gamma \alpha(x) \tfrac{\partial u_t}{\partial \nu}\tfrac{\partial u_{tt}}{\partial \nu} + \int_\Omega u_t f(u) dx. \end{aligned} \tag{3.7}$$

Now we can get

(3.8)

$$\begin{aligned} 0 &= \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega (\Delta u)^2 dx + \tfrac{1}{2}\tfrac{d}{dt}\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial \nu})^2 dx - \tfrac{d}{dt}\int_0^t \int_\Omega f(u) u_\eta dx d\eta \\ &= \tfrac{d}{dt}[\tfrac{1}{2}\int_\Omega (\Delta u)^2 dx + \tfrac{1}{2}\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial \nu})^2 dx - \int_\Omega \mathcal{F}(u) dx] \\ &= \tfrac{d}{dt} E(t) \end{aligned}$$

where $E(t)$ was defined in (3.1). Thus we have

$$E(t) = E(0).$$

Now we want to calculate F' and F''.

$$F'(t) = 2\int_\Gamma \alpha(x) \frac{\partial u}{\partial \nu}\frac{\partial u_t}{\partial \nu} dx + 2\beta(t + t_0) \tag{3.9}$$

and

$$F''(t) = 2\int_\Gamma \alpha(x) \frac{\partial u}{\partial \nu}\frac{\partial u_{tt}}{\partial \nu} dx + 2\int_\Gamma \alpha(x)(\frac{\partial u_t}{\partial \nu})^2 dx + 2\beta. \tag{3.10}$$

Thus we have from (2.2) and (3.5)

(3.11)

$$\begin{aligned} & F''(t) \\ \geq\ & -2\int_\Omega(\Delta u)^2dx + 2\int_\Omega uf(u)dx + 2\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial\nu})^2dx + 2\beta. \\ \geq\ & 2[-\int_\Omega(\Delta u)^2dx + 2(\gamma+1)\int_\Omega \mathcal{F}(u)dx + \int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial\nu})^2dx + \beta] \\ \geq\ & 2[-\int_\Omega(\Delta u)^2dx + (\gamma+1)\int_\Omega(\Delta u)^2dx \\ & +(\gamma+2)\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial\nu})^2dx - 2(\gamma+1)E(0) + \beta] \\ =\ & 2\gamma\int_\Omega(\Delta u)^2dx + 2(\gamma+2)\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial\nu})^2dx \\ & +2\beta - 4(\gamma+1)E(0) \\ \geq\ & 4(\gamma_0+1)[\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial\nu})^2dx + \beta] + D_0 \end{aligned}$$

where $\gamma = 2\gamma_0$, $D_0 = -2(2\gamma_0+1)\beta - 4(\gamma+1)E(0)$. Now we choose β such that

$$D_0 > 0$$

and choose t_0 such that

$$F'(0) > 0.$$

By using Hölder's inequality and Young's inequality

$$\begin{aligned} & [F'(t)]^2 \\ (3.12)\qquad =\ & [2\int_\Gamma \alpha(x)\tfrac{\partial u}{\partial\nu}\tfrac{\partial u_t}{\partial\nu}dx + 2\beta(t+t_0)]^2 \\ \leq\ & 4[(\int_\Gamma \alpha(x)(\tfrac{\partial u}{\partial\nu})^2dx)^{\frac12}(\int_\Gamma \alpha(x)(\tfrac{\partial u_t}{\partial\nu})^2dx)^{\frac12} + \beta(t+t_0)]^2. \end{aligned}$$

Now we want to find a lower bound for the function

$$\Xi(t) = F''F - (1+\gamma_0)(F')^2. \tag{3.13}$$

Let

$$A = \int_0^t\int_\Gamma \alpha(x)(\frac{\partial u_\eta}{\partial\nu})^2dxd\eta,\ B = \int_0^t\int_\Gamma \alpha(x)(\frac{\partial u}{\partial\nu})^2dxd\eta. \tag{3.14}$$

So we get

(3.15)

$$\begin{aligned} & \Xi(t) \\ =\ & F''F - (1+\gamma_0)(F')^2 \\ \geq\ & 4(1+\gamma_0)[(A+\beta)(B+\beta(t+t_0)^2) - \{A^{\frac12}B^{\frac12} + \beta(t+t_0)\}^2]. \end{aligned}$$

Now we note from the choice of β that

$$D_0 \geq 0 \tag{3.16}$$

if the initial energy is negative, that is $E(0) < 0$. So, by the Schwarz inequality,

$$(A+\beta)(B+\beta(t+t_0)^2) - \{A^{\frac12}B^{\frac12} + \beta(t+t_0)\}^2 \geq 0.$$

Thus

$$\Xi(t) \geq 0$$

as long as it is defined. By the Concavity lemma, the theorem is proved. □

ACKNOWLEDGMENTS. I am grateful for many illuminating discussions with Howard A. Levine.

References

[1] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. of Diff. Equations **109** (1994), 295-308.

[2] R. T. Glassey, *Blow-up theorems for nonlinear wave equations*, Math. Z. **132** (1973), 183-203.

[3] O. A. Ladyzhenskaya and V. K. Kalantarov, *The occurrence of collapse for quasi-linear parabolic and hyperbolic types*. Zap. Nauckn. SLOMI Steklov **69, 77** (1977)

[4] H. A. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$*, Trans. Amer. Math. Soc. **192** (1974), 1-21.

[5] H. A. Levine, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*, SIAM J. Math. Anal. **5** (1974), 138-146.

[6] H. A. Levine, *Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: The method of unbounded Fourier coefficients*, Math. Ann. **214** (1975) 205-220.

[7] H. A. Levine and J. Serrin, *A global nonexistence theorem for quasilinear evolution equations with dissipation*, (manuscript).

[8] J. L. Lions, *Equations Differentielles Operationelles et Probleme aux Limites*, Springer 1961.

[9] S. R. Park, *Remarks on Quasilinear Hyperbolic Equations with Dynamic Boundary Conditions*, Mathematische Nachrichten (to appear).

[10] S. R. Park, M. Can and F. Aliyev, *Nonexistence of Global Solutions of Some Quasilinear Hyperbolic Equations*, J. Mathematical Analysis and Applications (to appear).

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