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A NONLINEAR BEAM EQUATION WITH NONLINEARITY CROSSING AN EIGENVALUE

Q-HEUNG CHOI[1] AND HYEWON NAM

ABSTRACT. We investigate the existence of solutions of the nonlinear beam equation under the Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the varible t, $Lu + bu^+ - au^- = f(x,t)$, when the jumping nonlinearity crosses the first positive eigenvalue.

0. Introduction

In this paper, we investigate multiplicity of solutions $u(x,t)$ for a piecewise linear perturbation $-(bu^+ - au^-)$ of the beam operator L under the Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the variable t,

$$
\begin{aligned}
& Lu + bu^+ - au^- = f(x,t) \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R}, \\
(0.1) \qquad & u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\
& u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi),
\end{aligned}
$$

where L denote the beam operator $Lu = u_{tt} + u_{xxxx}$. The eigenvalues of L under the Dirichlet boundary condition and periodic condition on the variable t are given by $\lambda_{mn} = (2n+1)^4 - 4m^2 (m, n = 0, 1, 2, \cdots)$.

Let Q be the square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and H be the Hilbert space defined by

$$H = \{u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t\}.$$

Then equation (0.1) is represented by

$$(0.2) \qquad Lu + bu^+ - au^- = f \qquad \text{in } H.$$

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In [2,10], Choi and McKenna investigated multiplicity of solutions of a semilinear equation (0.2) when the nonlinearity $-bu^+$ crosses an eigenvalue λ_{10} and the forcing term f is supposed to be $1+\varepsilon h(\| h \|)$. In [5], Choi and Jung investigated multiplicity of solutions of a semilinear equation (0.2) when the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues $\lambda_{00}, \lambda_{10}$ and the source term f is generated by ϕ_{00} and ϕ_{10}, and when the nonlinearity $-(bu^+ - au^-)$ crosses an eigenvalue λ_{10} and the source term f is generated by ϕ_{00} and ϕ_{10}.

Our concern is to investigate multiplicity of solutions of (0.2) when $-17 < a < -1 < b < 3$ and the source term f is generated by two eigenfunctions ϕ_{00}, ϕ_{10}. In particular, we investigate multiplicity of solutions of (0.2).

In Section 1, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue λ_{00} and the source term f is generated by ϕ_{00} and ϕ_{10}. And we use the variational reduction method to reduce the problem from an infinite dimensional one to a finite dimensional one. Let $Q = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and V be the subspace of $L^2(Q)$ spanned by ϕ_{00} and ϕ_{10}. Let P be the orthogonal projection $L^2(Q)$ onto V. Then the beam equation (0.1) is reduced to a equation in V.

In Section 2, we define a map Φ by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \qquad v \in V$$

and we investigate the properties of the map Φ and we reveal a relation between multiplicity of solutions and source terms in equation (0.2) when f belongs to the two-dimensional space V. We also determine the region of source terms in which (0.2) has no solution.

1. A variational reduction method

We consider the beam equation under the Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the varible t

$$\begin{aligned} & u_{tt} + u_{xxxx} + bu^+ - au^- = f(x,t) \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R}, \\ (1.1) \qquad & u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ & u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi). \end{aligned}$$

Here we suppose that $-\lambda_{41} = -17 < a < -\lambda_{00} = -1 < b < -\lambda_{10} = 3$.

Let L be the differential operator $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem

$$
\begin{aligned}
&Lu = \lambda u \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\
&u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \qquad (1.2)\\
&u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi)
\end{aligned}
$$

has infinitely many eigenvalues λ_{mn} and corresponding eigenfunctions $\phi_{mn}(m,n \geq 0)$ given by

$$
\begin{aligned}
\lambda_{mn} &= (2n+1)^4 - 4m^2, \\
\phi_{mn} &= \cos 2mt \cos(2n+1)x \qquad (m,n = 0,1,2,\cdots).
\end{aligned}
$$

Let Q be the square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and H be the Hilbert space defined by

$$H = \{u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t\}.$$

Then the set $\{\phi_{mn} \mid m,n = 0,1,2,\cdots\}$ forms an orthogonal set in H.

Equation (1.1) is equivalent to

$$Lu + bu^+ - au^- = f \qquad \text{in } H, \tag{1.3}$$

where we assume that $f = s_1\phi_{00} + s_2\phi_{10}\ (s_1, s_2 \in R)$.

THEOREM 1.1. *If $s_1 < 0$, then (1.3) has no solution.*

Proof. We rewrite (1.3) as

$$(L - \lambda_{00})u + (b + \lambda_{00})u^+ - (a + \lambda_{00})u^- = s_1\phi_{00} + s_2\phi_{10} \qquad \text{in } H.$$

Multiply across by ϕ_{00} and integrate over Q. Since L is self-adjoint and $(L - \lambda_{00})\phi_{00} = 0$, $((L - \lambda_{00})u, \phi_{00}) = 0$. Thus we have

$$
\begin{aligned}
\int_Q \{(b + \lambda_{00})u^+ - (a + \lambda_{00})u^-\}\phi_{00} &= (s_1\phi_{00} + s_2\phi_{10}, \phi_{00}) \\
&= s_1 \int_Q {\phi_{00}}^2 \\
&= \frac{\pi}{2} s_1.
\end{aligned}
$$

We know that $(b+\lambda_{00})u^+ - (a+\lambda_{00})u^- \geq 0$ for all real valued function u. Also $\phi_{00} > 0$ in Q. Therefore $\int_Q \{(b+\lambda_{00})u^+ - (a+\lambda_{00})u^-\}\phi_{00} \geq 0$. Hence there is no solution of (1.3) if $s_1 < 0$. □

Let V be the subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W be the orthogonal complement of V in H. Let P be the orthogonal projection of H onto V. Then every $u \in H$ can be written as $u = v + w$, where $v = Pu$ and $w = (I-P)u$. Hence equation (1.3) is equivalent to a system

$$Lw + (I-P)(b(v+w)^+ - a(v+w)^-) = 0, \tag{1.4}$$

$$Lv + P(b(v+w)^+ - a(v+w)^-) = s_1\phi_{00} + s_2\phi_{10}. \tag{1.5}$$

LEMMA 1.2. *For a fixed $v \in V$, (1.4) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous in v.*

Proof. Let $\delta = \frac{a+b}{2}$. Rewrite (1.4) as

$$(-L-\delta)w = (I-P)(b(v+w)^+ - a(v+w)^- - \delta(v+w)), \tag{1.6}$$

or equivalently,

$$w = (-L-\delta)^{-1}(I-P)g_v(w),$$

where

$$g_v(w) = b(v+w)^+ - a(v+w)^- - \delta(v+w).$$

Since

$$|g_v(w_1) - g_v(w_2)| \leq \max\{|b-\delta|, |\delta-a|\}|w_1 - w_2|,$$

we have

$$\|g_v(w_1) - g_v(w_2)\| \leq \max\{|b-\delta|, |\delta-a|\}\|w_1 - w_2\|.$$

The operator $(-L-\delta)^{-1}(I-P)$ is a self-adjoint compact linear map from W into itself. Its eigenvalues in W are $(-\lambda_{mn}-\delta)^{-1}$, where $\lambda_{mn} \neq 1$. Therefore its L^2 norm is $\max\{\frac{1}{|-17-\delta|}, \frac{1}{|3-\delta|}\}$. Since $\max\{|b-\delta|, |\delta-a|\} < \min\{|-17-\delta|, |3-\delta|\}$, for fixed $v \in V$, the right hand side of (1.6) defines a Lipschitz mapping of W into itself with Lipschitz constant $\gamma < 1$.

By the contraction mapping principle, for each $v \in V$, there is a unique $w \in W$ which satisfies (1.4).

By the standard argument principle, $\theta(v)$ is Lipschitz continuous in v. □

By Lemma 1.2, the study of the multiplicity of solutions of (1.3) is reduced to that of an equivalent problem

$$(1.7) \qquad Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) = s_1\phi_{00} + s_2\phi_{10}$$

defined on V.

PROPOSITION. *If $v \geq 0$ or $v \leq 0$, then $\theta(v) = 0$.*

Proof. Let $v \geq 0$. Then $\theta(v) = 0$ and equation (1.4) is reduced to

$$L0 + (I-P)(bv^+ - av^-) = 0$$

because $v^+ = v, v^- = 0$ and $(I-P)v = 0$. Similarly if $v \leq 0$, then $\theta(v) = 0$. □

Since $V = span\{\phi_{00}, \phi_{10}\}$ and ϕ_{00} is a positive eigenfunction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq c_1\}$$

so that $v \geq 0$ for all $v \in C_1$, and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$. Thus $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

Now we set

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0, |c_1| \leq c_2\}$$
$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_2 \leq 0, |c_1| \leq |c_2|\}.$$

Then the union of C_1, C_2, C_3, and C_4 is the space V.

We define a map $\Phi : V \longrightarrow V$ by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \qquad v \in V.$$

Then Φ is continuous on V and we have the following lemma.

LEMMA 1.3. *$\Phi(cv) = c\Phi(v)$ for $c \geq 0$ and $v \in V$.*

Proof. Let $c \geq 0$. If v satisfies

$$L\theta(v) + (I - P)(b(v + \theta(v))^+ - a(v + \theta(v))^-) = 0,$$

then

$$L(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore

$$\begin{aligned}\Phi(cv) &= L(cv) + P(b(cv + \theta(cv))^+ - a(cv + \theta(cv))^-) \\ &= L(cv) + P(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) \\ &= c\Phi(v)\end{aligned}$$

□

We investigate the image of the cones C_1, C_3 under Φ.

First, we consider the image of C_1. If $v = c_1\phi_{00} + c_2\phi_{10} \geq 0$,

$$\begin{aligned}\Phi(v) &= Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= c_1\phi_{00} - 3c_2\phi_{10} + b(c_1\phi_{00} + c_2\phi_{10}) \\ &= (b+1)c_1\phi_{00} + (b-3)c_2\phi_{10}.\end{aligned}$$

Thus images of the rays $c_1\phi_{00} \pm c_1\phi_{10} (c_1 \geq 0)$ are

$$(b+1)c_1\phi_{00} + (b-3)c_1\phi_{10} \qquad (c_1 \geq 0).$$

Therefore Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \geq 0, |d_2| \leq \frac{-b+3}{b+1} d_1 \right\}.$$

Second, we consider the image of C_3. If $v = -c_1\phi_{00} + c_2\phi_{10} \leq 0\,(c_1 \geq 0, |c_2| \leq c_1)$,

$$\begin{aligned}\Phi(v) &= Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= -c_1\phi_{00} - 3c_2\phi_{10} + a(-c_1\phi_{00} + c_2\phi_{10}) \\ &= (-a-1)c_1\phi_{00} + (a-3)c_2\phi_{10}.\end{aligned}$$

Thus images of the rays $-c_1\phi_{00} \pm c_1\phi_{10}(c_1 \geq 0)$ are

$$(-a-1)c_1\phi_{00} \pm (a-3)c_1\phi_{10} \qquad (c_1 \geq 0).$$

Therefore Φ maps C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \geq 0, |d_2| \leq \frac{a-3}{a+1}d_1 \right\}.$$

We have three cases $R_1 \subsetneqq R_3$, $R_3 \subsetneqq R_1$, and $R_1 = R_3$. The relation $R_1 \subsetneqq R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{a+3}{a-1}$. The relation $R_3 \subsetneqq R_1$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b < \frac{a+3}{a-1}$. The relation $R_1 = R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{a+3}{a-1}$.

2. Multiplicity results

We consider the restrictions $\Phi|_{C_i} (1 \leq i \leq 4)$ of Φ to the cones C_i. Let $\Phi_i = \Phi|_{C_i}$, *i.e.*, $\Phi_i : C_i \longrightarrow V$.

First, we consider Φ_1. It maps C_1 onto R_1. Let l_1 be the segment defined by

$$l_1 = \left\{ \phi_{00} + d_2\phi_{10} \;\middle|\; |d_2| \leq \frac{-b-3}{b+1} \right\}.$$

Then the inverse image $\Phi_1^{-1}(l_1)$ is the segment

$$L_1 = \left\{ \frac{1}{b+1}(\phi_{00} + c_2\phi_{10}) \;\middle|\; |c_2| \leq 1 \right\}.$$

By Lemma 1.3, $\Phi_1 : C_1 \longrightarrow R_1$ is bijective.

Second, we consider Φ_3. It maps C_3 onto R_3. Let l_3 be the segment defined by

$$l_3 = \left\{ \phi_{00} + d_2\varphi_{10} \;\middle|\; |d_2| \leq \frac{a-3}{a+1} \right\}.$$

Then the inverse image $\Phi_3^{-1}(l_3)$ is the segment

$$L_3 = \left\{ \frac{1}{a+1}(\phi_{00} + c_2\phi_{10}) \;\middle|\; |c_2| \leq 1 \right\}.$$

By Lemma 1.3, $\Phi_3 : C_3 \longrightarrow R_3$ is bijective.

2.1 The nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{a+3}{a-1}$

The relation $R_1 \subsetneqq R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{a+3}{a-1}$. We investigate the images of the cones C_2, C_4 under Φ, where

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{10} \,|\, c_2 \geq 0, |c_1| \leq c_2\},$$
$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{10} \,|\, c_2 \leq 0, |c_1| \leq |c_2|\}.$$

By Theorem 1.1 and Lemma 1.2, the image of C_2 under Φ is a cone containing

$$R_2 = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \geq 0, \frac{-a+3}{a+1}d_1 \leq d_2 \leq \frac{b-3}{b+1}d_1 \right\}$$

and the image of C_4 under Φ is a cone containing

$$R_4 = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \geq 0, \frac{-b+3}{b+1}d_1 \leq d_2 \leq \frac{a-3}{a+1}d_1 \right\}.$$

We consider the restrictions Φ_2 and Φ_4. Define the segments l_2, l_4 as follows;

$$l_2 = \left\{ \phi_{00} + d_2\phi_{10} \;\middle|\; \frac{-a+3}{a+1} \leq d_2 \leq \frac{b-3}{b+1} \right\},$$
$$l_4 = \left\{ \phi_{00} + d_2\phi_{10} \;\middle|\; \frac{-b+3}{b+1} \leq d_2 \leq \frac{a-3}{a+1} \right\}.$$

We investigate the inverse images $\Phi_2^{-1}(l_2)$ and $\Phi_4^{-1}(l_4)$. We note that $\Phi_i(C_i)$ contains R_i, for $i = 2, 4$, respectively.

LEMMA 2.1.1. *For $i = 2, 4$, let γ be any simple path in R_i with end points on ∂R_i, where each ray in R_i (starting from the origin) intersects only one point of γ. Then the inverse image $\Phi_i^{-1}(\gamma)$ of γ is also a simple path in C_i with end points on ∂C_i, where any ray in C_i (starting from the origin) intersects only one point of this path.*

Proof. Since Φ is continuous and γ is closed in V, $\Phi_i^{-1}(\gamma)$ is closed. Suppose that there is a ray(starting from the origin) in C_i, which intersects two points of $\Phi_i^{-1}(\gamma)$, say p and $\alpha p (\alpha > 1)$. Then $\Phi(\alpha p) = \alpha\Phi(p)$, which implies $\Phi(p) \in \gamma$ and $\Phi(\alpha p) \in \gamma$. This contradicts the assumption that each ray(starting from the origin) in C_i intersects only one point of γ.

We regard a point $p \in V$ as a radius vector in the plan V. Define the argument $\arg p$ to be the angle from the positive ϕ_{oo}-axis to p.

We claim that $\Phi_i^{-1}(\gamma)$ meets all the rays(starting from the origin) in C_i. If not, $\Phi_i^{-1}(\gamma)$ is disconnected in C_i. Since $\Phi_i^{-1}(\gamma)$ is closed and meets at most one point of any ray in C_i, there are two points p_1 and p_2 in C_i such that $\Phi_i^{-1}(\gamma)$ does not contain a point $p \in C_i$ with $\arg p_1 < \arg p < \arg p_2$. Let l be the segment with end points p_1 and p_2 then $\Phi_i(l)$ is a path in R_i, where $\Phi_i(p_1)$ and $\Phi_i(p_2)$ belong to γ. Choose a point $q \in \Phi_i(l)$ such that $\arg q$ is between $\arg \Phi_i(p_1)$ and $\arg \Phi_i(p_2)$. Then there exist a point q' of γ such that $q' = \beta q$ for some $\beta > 0$. Hence $\Phi_i^{-1}(q)$ and $\Phi_i^{-1}(q')$ are on the same ray(starting from the origin) in C_i and $\arg p_1 < \arg \Phi_i^{-1}(q') < \arg p_2$, which is a contradiction. This completes the proof. □

Lemma 2.1.1 implies that $\Phi_i (i = 2, 4)$ is surjective. Hence we have the following theorem.

THEOREM 2.1.2. *For $1 \leq i \leq 4$, the restriction Φ_i maps C_i onto R_i. Therefore, Φ maps V onto R_3. In particular, Φ_1 and Φ_3 are bijective.*

The above theorem also implies the following result.

THEOREM 2.1.3. *Suppose $-17 < a < -1 < b < 3$ and $b > \frac{a+3}{a-1}$. Let $f = s_1\phi_{00} + s_2\phi_{10} \in V$. Then we have :*

(1) *If $f \in IntR$, then (1.3) has exactly two solutions, one of which is positive and the other the other is negative.*
(2) *If $f \in IntR_2 \bigcup IntR_4$, then (1.3) has a negative solution and at least one sign changing solution.*
(3) *If $f \in \partial R_3$, then (1.3) has a negative solution.*
(4) *If $f \in R_3^c$, then (1.3) has no solution.*

2.2 The nonlinearity $-(bu^+ - au^-)$ satisfies $b < \frac{a+3}{a-1}$

The relation $R_3 \subsetneq R_1$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b < \frac{a+3}{a-1}$. We investigate the images of the cones C_2, C_4 under Φ. By Theorem 1.1 and Lemma 1.2, the image of C_2 under Φ is a cone containing

$$R_2{}' = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \geq 0, \frac{b-3}{b+1}d_1 \leq d_2 \leq \frac{-a+3}{a+1}d_1 \right\}$$

and the image of C_4 under Φ is a cone containing

$$R_4{}' = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \geq 0, \frac{a-3}{a+1}d_1 \leq d_2 \leq \frac{-b+3}{b+1}d_1 \right\}$$

We consider the restrictions Φ_2 and Φ_4. Define the segments $l_2{}', l_4{}'$ as follows;

$$l_2{}' = \left\{ \phi_{00} + d_2\phi_{10} \;\middle|\; \frac{b-3}{b+1} \leq d_2 \leq \frac{-a+3}{a+1} \right\}$$
$$l_4{}' = \left\{ \phi_{00} + d_2\phi_{10} \;\middle|\; \frac{a-3}{a+1} \leq d_2 \leq \frac{-b+3}{b+1} \right\}$$

We investigate the inverse images $\Phi_2^{-1}(l_2{}')$ and $\Phi_4^{-1}(l_4{}')$. We note that $\Phi_i(C_i)(i = 2, 4)$ contains $R_i{}'(i = 2, 4)$.

LEMMA 2.2.1. *For $i = 2, 4$, let γ' be any simple path in $R_i{}'$ with end points on $\partial R_i{}'$, where each ray in $R_i{}'$ (starting from the origin) intersects only one point of γ'. Then the inverse image $\Phi_i^{-1}(\gamma')$ of γ' is also a simple path in C_i with end points on ∂C_i, where any ray in C_i (starting from the origin) intersects only one point of this path.*

Proof. The proof is similar to the proof given in Lemma 2.1.1. □

Lemma 2.2.1 implies that $\Phi_i(i = 2, 4)$ is surjective. Hence we have the following theorem.

THEOREM 2.2.2. *For $i = 2, 4$, the restriction Φ_i maps C_i onto $R_i{}'$. And Φ_1 and Φ_3 are bijective. Therefore, Φ maps V onto R_1.*

Above the theorem also implies the following results.

THEOREM 2.2.3. *Suppose* $-17 < a < -1 < b < 3$ *and* $b < \frac{a+3}{a-1}$. *Let* $f = s_1\phi_{00} + s_2\phi_{10} \in V$. *Then we have :*

(1) *If* $f \in IntR$, *then* (1.3) *has exactly two solutions, one of which is positive and the other the other is negative.*
(2) *If* $f \in IntR_2' \bigcup IntR_4'$, *then* (1.3) *has a positive solution and at least one sign changing solution.*
(3) *If* $f \in \partial R_1$, *then* (1.3) *has a positive solution.*
(4) *If* $f \in R_1^c$, *then* (1.3) *has no solution.*

2.3 The nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{a+3}{a-1}$

The relation $R_1 = R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{a+3}{a-1}$.

We considered the map $\Phi : V \longrightarrow V$ defined by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \qquad v \in V,$$

where $-17 < a < -1 < b < 3$ and $b = \frac{a+3}{a-1}$.

We investigate the images of C_2 and C_4 under Φ. For fixed v, we define a map $\Phi_v : (-1,3) \longrightarrow V$ as follows

$$\Phi_v(b) = Lv + P(b(v+w)^+ - a(v+w)^-), \qquad b \in (-1,3),$$

where $v \in V$ and a are fixed.

LEMMA 2.3.1. Φ_v *is continuous at* $b_0 = \frac{a+3}{a-1}$, *where* $-17 < a < -1 < b_0 < 3$ *and* a *is fixed.*

Proof. Let $\delta = \frac{a+b_0}{2}$ and $-1 < b < 3$. Rewrite (1.4) as

$$(2.3.1) \qquad (-L-\delta)w = (I-P)(b(v+w)^+ - a(v+w)^- - \delta(v+w)),$$

or equivalently,

$$(2.3.2) \qquad w = (-L-\delta)^{-1}(I-P)h(b,u),$$

where

$$h(b,w) = b(v+w)^+ - a(v+w)^- - \delta(v+w).$$

By Lemma 1.2, (2.3.2) has a unique solution $w = \theta_b(v)$ for fixed b with $-1 < b < 3$. Let $w_0 = \theta_{b_0}(v)$. Then we have

$$\begin{aligned} w - w_0 &= (-L-\delta)^{-1}(I-P)[h(b,w) - h(b_0,w_0)] \\ &= (-L-\delta)^{-1}(I-P)[h(b,w) - h(b,w_0)] \\ &\quad + (-L-\delta)^{-1}(I-P)[h(b,w_0) - h(b_0,w_0)]. \end{aligned}$$

Since

$$\|h(b,w) - h(b,w_0)\| \leq \max\{|b-\delta|, |\delta - a|\}\|w - w_0\|$$

and

$$\gamma = \max\{\frac{1}{|-17-\delta|}, \frac{1}{|3-\delta|}\}\max\{|b-\delta|, |\delta - a|\}| < 1,$$

we have

$$\|w - w_0\| \leq \gamma\|w - w_0\| + \max\{\frac{1}{|-17-\delta|}, \frac{1}{|3-\delta|}\}\|v + w_0\||b - b_0|.$$

Hence

$$\|w - w_0\| \leq \max\{\frac{1}{|-17-\delta|}, \frac{1}{|3-\delta|}\}\frac{\|v + w_0\|}{(1-\gamma)}|b - b_0|,$$

which shows that $\theta_b(v)$ is continuous at $b_0 = \frac{a+3}{a-1}$. Therefore $\Phi_v(b)$ is continuous at $b_0 = \frac{a+3}{a-1}$. □

First, we investigate the images of the cones C_2 under Φ. Let $p_1 = \phi_{00} + \frac{-a+3}{a+1}\phi_{10}$ and $p_2 = \phi_{00} + \frac{b-3}{b+1}\phi_{10}$. We fix a. Define

$$\theta = \begin{cases} \arg p_1 - \arg p_2, & \text{if } b > \frac{a+3}{a-1}; \\ \arg p_2 - \arg p_1, & \text{if } b < \frac{a+3}{a-1}. \end{cases}$$

Then $0 \leq \theta \leq \frac{\pi}{2}$ and

$$\tan\theta = \frac{|-ab + a + b + 3|}{-2a - 2b + 4}.$$

When b converges to $\frac{a+3}{a-1}$, $\tan\theta$ converges to 0. Then θ converges to 0 since $0 \le \theta \le \frac{\pi}{2}$. We note that Φ_2 maps C_2 onto R_2 when $b > \frac{a+3}{a-1}$ and that Φ_2 maps C_2 onto $R_2{}'$ when $b < \frac{a+3}{a-1}$. When b converges to $\frac{a+3}{a-1}$, the angle of two lines consisting ∂R_2 or $\partial R_2{}'$ converges to 0. Since Φ_2 is continuous at $\frac{a+3}{a-1}$, Φ_2 maps C_2 onto the ray

$$S_2 = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \ge 0, d_2 = \frac{b-3}{b+1}d_1 \right\}$$

when $b = \frac{a+3}{a-1}$.

Second, we investigate the images of the cones C_4 under Φ.

Let $q_1 = \phi_{00} + \frac{a-3}{a+1}\phi_{10}$ and $q_2 = \phi_{00} + \frac{-b+3}{b+1}\phi_{10}$. We fix a. Define

$$\theta' = \begin{cases} \arg q_1 - \arg q_2, & \text{if } b > \frac{a+3}{a-1}; \\ \arg q_2 - \arg q_1, & \text{if } b < \frac{a+3}{a-1}. \end{cases}$$

Then $0 \le \theta' \le \frac{\pi}{2}$ and

$$\tan\theta' = \frac{|-ab+a+b+3|}{-2a-2b+4}.$$

When b converges to $\frac{a+3}{a-1}$, $\tan\theta'$ converges to 0. Then θ' converges to 0, since $0 \le \theta' \le \frac{\pi}{2}$. We note that Φ_4 maps C_4 onto R_4 when $b > \frac{a+3}{a-1}$ and that Φ_4 maps C_4 onto $R_4{}'$ when $b < \frac{a+3}{a-1}$. When b converges to $\frac{a+3}{a-1}$, the angle of two lines consisting ∂R_4 or $\partial R_4{}'$ converges to 0. Since Φ_4 is continuous at $\frac{a+3}{a-1}$, Φ_4 maps C_4 onto the ray

$$S_4 = \left\{ d_1\phi_{00} + d_2\phi_{10} \;\middle|\; d_1 \ge 0, d_2 = \frac{-b+3}{b+1}d_1 \right\}$$

when $b = \frac{a+3}{a-1}$.

Hence we have the following theorem.

THEOREM 2.3.2. *For $i = 2, 4$, the restriction Φ_i maps C_i onto S_i. And Φ_1 and Φ_3 are bijective. Therefore, Φ maps V onto R, where $R = R_1 = R_3$.*

The above theorem also implis the following result.

THEOREM 2.3.3. *Suppose* $-17 < a < -1 < b < 3$ *and* $b = \frac{a+3}{a-1}$. *Let* $f = s_1\phi_{00} + s_2\phi_{10} \in V$. *Then we have :*

(1) *If* $f \in IntR$, *then* (1.3) *has exactly two solutions, one of which is positive and the other the other is negative.*

(2) *If* $f \in \partial R$, *then* (1.3) *has a positive solution, a negative solution, and infinitely many sign changing solutions.*

(3) *If* $f \in R^c$, *then* (1.3) *has no solution.*

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Department of Mathematics
Inha university
Incheon 402-751, Korea