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GENERALIZED SOLUTIONS OF IMPULSIVE CONTROL SYSTEMS CORRESPONDING TO CONTROLS OF BOUNDED VARIATION

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ABSTRACT. This paper is concerned with the impulsive control problem

$$\dot{x}(t) = f(t, x) + g(t, x)\dot{u}(t), \quad t \in [0, T], \quad x(0) = \bar{x},$$

where u is a possibly discontinuous control function of bounded variation, $f : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a bounded and Lipschitz continuous function, and $g : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable w.r.t. the variable x and satisfies $|g(t, \cdot) - g(s, \cdot)| \leq \phi(t) - \phi(s)$, for some increasing function ϕ and every $s < t$. We show that the map $u \mapsto x_u$ is Lipschitz continuous when u ranges in the set of step functions whose total variations are uniformly bounded, where x_u is the solution of the impulsive control system corresponding to u. We also define the generalized solution of the impulsive control system corresponding to a measurable control function of bounded variation.

1. Introduction

Consider the control system

$$(1.1) \qquad \dot{x}(t) = f(t, x) + g(t, x)\dot{u}(t), \qquad t \in [0, T] \quad x(0) = \bar{x} \in \mathbb{R}^n,$$

where $\cdot = \frac{d}{dt}$. Let f, g be maps from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following conditions;

C1. there exists $M_1 > 0$ such that for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$(1.2) \qquad |f(t, x)| \leq M_1 \quad \text{and} \quad |g(t, x)| \leq M_1,$$

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C2. there exists $L > 0$ such that for any $(t, x), (t', x') \in \mathbb{R} \times \mathbb{R}^n$,

$$|f(t,x) - f(t',x')| \le L(|t-t'| + |x-x'|), \tag{1.3}$$

C3. for every $t \in \mathbb{R}$ the map $x \to g(t,x)$ is a C^1-map such that for any $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$,

$$|g(t,x) - g(t,y)| \le L|x-y|, \tag{1.4}$$

C4. there exists an increasing and right-continuous function ϕ such that for any $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$|g(t,x) - g(s,x)| \le |t-s| + |\phi(t) - \phi(s)|. \tag{1.5}$$

If u is a C^1-function, the solution of (1.1) is defined in the sense of classical theory of ODE. However, if u is just integrable, the solution of (1.1) is interpreted as a distribution which is not unique. The aim of this paper is to define the unique generalized solution of (1.1) corresponding to a measurable function of bounded variation under assumptions C1 $\sim$ C4.

In [3], when f and g are C^1 and C^2 maps respectively, the map $\Phi : u \to x_u$ from the set of bounded C^1-control functions assigning the corresponding solution of (1.1) is Lipschitz continuous in the suitable L^1-norms, so the map Φ can be continuously extended to bounded integrable control functions and the solution of (1.1) corresponding to a bounded integrable control function u is defined as the limit of the solutions corresponding to C^1-control functions u_n which converge to u in L^1. For C^1-maps f and g, Bressan [2] also defined the generalized solution of (1.1) corresponding to a measurable function u which describes the phenomenon occurring at the point of jump.

To define the generalized solution of (1.1), in §2 it is assumed that

C5. for every $t \in \mathbb{R}$ the map $x \to g(t,x)$ is a C^1-map such that g satisfies

$$|g(t,x) - g(t',x')| \le L(|t-t'| + |x-x'|), \quad \text{for any } (t,x), (t',x') \in \mathbb{R} \times \mathbb{R}^n \tag{1.6}$$

instead of C3 and C4, and the domain of Φ is restricted to the set of step functions whose total variations are uniformly bounded. Under the conditions C1, C2 and C5, the map Φ is Lipschitz continuous, so Φ can be continuously extended to a measurable function of bounded variation.

In §3, it is assumed that f, g satisfies the conditions C1 ~C4. By introducing an auxiliary function G satisfying C5 in the new time variable θ, the generalized solution of (1.1) corresponding to a measurable function of bounded variation is defined in terms of the solution of the new control system induced by G. We end up this paper with the example in which the generalized solution of (1.1) is uniquely determined while the solutions of the integral equation induced by (1.1) are not unique.

2. Basic estimate in the case that g is Lipschitz continuous

Let f, g be maps satisfying C1, C2, C5, and for $M > 0$, let $\mathcal{S}_M$ be the set of all step functions from $[0, T]$ to $\mathbb{R}$ whose total variations are less than or equal to M. For $u \in \mathcal{S}_M$, denote the solution of (1.1) corresponding to u by $x_u(\cdot)$ and define the map Φ on $\mathcal{S}_M$ by $\Phi(u) = x_u(\cdot)$. If $u \in \mathcal{S}_M$ has jumps at the points $0 \leq t_1 < \cdots < t_m \leq T$, then x_u is the solution of

$$\dot{x} = f(t, x), \qquad t \neq t_i \tag{2.1}$$

while at the points of jump t_i[2]

$$x(t_i+) = \left(e^{(u(t_i+)-u(t_i-))g(t_i,\cdot)}\right) x(t_i-). \tag{2.2}$$

where $\left(e^{\varepsilon g(t_i,\cdot)}\right)\bar{x}$ denotes the value at time $t = \varepsilon$ of the solution to the Cauchy problem

$$\frac{dx(t)}{dt} = g(t_i, x(t)), \qquad x(0) = \bar{x}. \tag{2.3}$$

For any $\varepsilon > 0$ and $\alpha > 0$, the control system (1.1) in the new time variable $\tau = \frac{t}{T}\varepsilon$ and the new space variable $y = \frac{x}{\alpha}$ changes to

$$\frac{dy}{d\tau} = \tilde{f}(\tau, y) + \tilde{g}(\tau, y)\frac{du_\alpha}{d\tau}, \quad x(0) = x, \quad 0 \leq \tau \leq \varepsilon, \tag{2.4}$$

where $u_\alpha(\tau) = \frac{u(\tau)}{\alpha}$ and $\tilde{f}$, $\tilde{g}$ are some bounded Lipschitz continuous maps in all the variables.

Thus we may assume that for sufficiently small $\varepsilon > 0$,

$$T = \varepsilon \quad \text{and} \quad \text{for any } u \in \mathcal{S}_M \text{ and } 0 \leq t \leq T, |u(t)| \leq \varepsilon/2.$$

To prove the Lipschitz continuity of the map Φ, it is necessary that some lemmas should be stated.

LEMMA 2.1. *Let $F(z)$ be a bounded Lipschitz continuous map from $\mathbb{R}^{n+1}$ to $\mathbb{R}$. Then there exists a sequence $\{F_m\}$ of smooth maps such that F_m converges uniformly to F and $\|D_z F_m\|$ are uniformly bounded, where $z = (z_0, \cdots, z_n)$, $D_z F_m = \left(\frac{\partial F_m}{\partial z_0}, \cdots, \frac{\partial F_m}{\partial z_n}\right)$ and $\|D_z F_m\| = \max\{\left|\frac{\partial F_m}{\partial z_i}(z)\right| \mid i = 0, \cdots, n, \quad z \in \mathbb{R}^{n+1}\}$.*

Proof. Let φ be a C^∞–function on $\mathbb{R}^{n+1}$ such that $\int_{\mathbb{R}^{n+1}} \varphi(z)dz = 1$ and $\varphi \equiv 0$ on $\{z \in \mathbb{R}^{n+1} \mid |z| \geq 1\}$. For $t > 0$, define the map φ_t on $\mathbb{R}^{n+1}$ by

$$\varphi_t(z) = t^{-n-1}\varphi(\frac{z}{t}) \tag{2.5}$$

and for any measurable function h from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, put

$$h * \varphi(z) = \int_{\mathbb{R}^{n+1}} h(z-y)\varphi(y)dy. \tag{2.6}$$

Then $F * \varphi_t(z)$ converges uniformly to $F(z)$ as $t \to 0+$ and the map $z \mapsto F * \varphi_t(z)$ is smooth.

For each $m \in I\!N$, define the map F_m by $F_m(z) = F * \varphi_{\frac{1}{m}}(z)$. Then F_m converges uniformly to F as $m \to \infty$ and for any $m \in I\!N$ the norm $\|D_z F_m\|$ of the Jacobian matrix is uniformly bounded since

$$\begin{aligned} |F_m(z) - F_m(\bar{z})| &= \left|\int_{\mathbb{R}^{n+1}} [F(z-y) - F(\bar{z}-y)]\varphi_{\frac{1}{m}}(y)dy\right| \\ &\leq \bar{L}|z - \bar{z}|, \end{aligned} \tag{2.7}$$

where $\bar{L}$ is the Lipschitz constant of F. □

Denote by $\tilde{e}^{sf}\tilde{x}$ the value at time $t = s$ of the solution to the Cauchy problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \qquad x(0) = \tilde{x}.$$

LEMMA 2.2. *There exists a constant $C > 0$ such that for any $|\varepsilon_1| \le \varepsilon, |\varepsilon_2| \le \varepsilon$, and $\tilde{x} \in \mathbb{R}^n$*

$$|\tilde{e}^{\varepsilon_1 f} e^{\varepsilon_2 \bar{g}^1} \tilde{x} - e^{\varepsilon_2 \bar{g}^2} \tilde{e}^{\varepsilon_1 f} \tilde{x}| \le C|\varepsilon_1 \varepsilon_2|, \tag{2.8}$$

where $\bar{g}^1(t,x) = g(0,x)$ and $\bar{g}^2(t,x) = g(\varepsilon_1, x)$.

Proof. Let $|\varepsilon_1| \le \varepsilon, |\varepsilon_2| \le \varepsilon$ and $\tilde{x} \in \mathbb{R}^n$. By lemma 2.1, there exist sequences $\{f_m\}$ and $\{g_m\}$ of smooth maps such that f_m and g_m converge uniformly to f and g respectively, and the gradients of f_m and g_m are uniformly bounded. We claim that there exists a constant $C_1 > 0$ such that for any $m \in I\!N$,

$$|\tilde{e}^{\varepsilon_1 f_m} e^{\varepsilon_2 \bar{g}^1_m} \tilde{x} - e^{\varepsilon_2 \bar{g}^2_m} \tilde{e}^{\varepsilon_1 f_m} \tilde{x}| \le C_1|\varepsilon_1 \varepsilon_2|, \tag{2.9}$$

where $\bar{g}^1_m(t,x) = g_m(0,x)$ and $\bar{g}^2_m(t,x) = g_m(\varepsilon_1, x)$. Adjoining the variable $x_0 = t$ to x, let $X = (x_0, x)$ with $x = (x_1, \cdots, x_n)$.

Since the gradients of f_m and g_m are uniformly bounded, there exists $M_2 > 0$ such that for any $m \in I\!N$,

$$\|D_X f_m\| \le M_2, \qquad \|D_X g_m\| \le M_2,$$

where $D_X f_m = (\frac{\partial f_m}{\partial x_0}, \cdots, \frac{\partial f_m}{\partial x_n})$, $f_m = (f_{m,1}, \cdots, f_{m,n})^T$, $g_m = (g_{m,1}, \cdots, g_{m,n})^T$ (v^T represents the transpose of the vector v) and

$$\|D_X f_m\| = \sup\{\left|\frac{\partial f_{m,i}}{\partial x_j}(x_0, x)\right| \mid i = 1, \cdots, n,\ j = 0, \cdots, n,$$
$$(x_0, x) \in [-\varepsilon, \varepsilon] \times \mathbb{R}^n\}.$$

For each $m \in I\!N$, define the maps $\tilde{F}_m, \tilde{G}^1_m, \tilde{G}^2_m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ by

$$\tilde{F}_m(x_0, x) = \begin{pmatrix} 1 \\ f_{m,1}(x_0, x) \\ \vdots \\ f_{m,n}(x_0, x) \end{pmatrix}, \tilde{G}^1_m(x_0, x) = \begin{pmatrix} 0 \\ g_{m,1}(0, x) \\ \vdots \\ g_{m,n}(0, x) \end{pmatrix},$$

$$\tilde{G}^2_m(x_0, x) = \begin{pmatrix} 0 \\ g_{m,1}(\varepsilon_1, x) \\ \vdots \\ g_{m,n}(\varepsilon_1, x) \end{pmatrix}.$$

Let $\hat{x} = (0, \tilde{x})$. Then the inside of absolute value of the left-hand side in (2.9) is the last n components of

$$h(\varepsilon_1, \varepsilon_2) = e^{\varepsilon_1 \tilde{F}_m} e^{\varepsilon_2 \tilde{G}^1_m} \hat{x} - e^{\varepsilon_2 \tilde{G}^2_m} e^{\varepsilon_1 \tilde{F}_m} \hat{x}.$$

We denote by $\Phi(t, \tilde{F}, z)$ the value at time t of the fundamental matrix solution to the linear equation

$$\dot{v}(t) = D_x \tilde{F}(e^{tF} z) \cdot v(t),$$

where $\Phi(0, \tilde{F}, z)$ is the identity matrix. To apply Taylor's theorem, compute the derivatives of h to get

$$\frac{\partial h}{\partial \varepsilon_1} = \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) - \Phi\left(\varepsilon_2, \tilde{G}^2_m, e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right) \cdot \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right),$$

$$\frac{\partial h}{\partial \varepsilon_2} = \Phi\left(\varepsilon_1, \tilde{F}_m, e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \cdot \tilde{G}^1_m\left(e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) - \tilde{G}^2_m\left(e^{\varepsilon_2 \tilde{G}^2_m} e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right),$$

$$\begin{aligned}\frac{\partial^2 h}{\partial \varepsilon_1^2} &= D_X \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \cdot \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \\ &\quad - \Phi\left(\varepsilon_2, \tilde{G}^2_m, e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right) \cdot D_X \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right) \cdot \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial \varepsilon_2^2} &= \Phi\left(\varepsilon_1, \tilde{F}_m, e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \cdot D_X \tilde{G}^1_m\left(e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \cdot \tilde{G}^1_m\left(e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \\ &\quad - D_X \tilde{G}^2_m\left(e^{\varepsilon_2 \tilde{G}^2_m} e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right) \cdot \tilde{G}^2_m\left(e^{\varepsilon_2 \tilde{G}^2_m} e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial \varepsilon_1 \partial \varepsilon_2} &= D_X \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} e^{\varepsilon_2 \tilde{G}^2_m} \hat{x}\right) \cdot \Phi\left(\varepsilon_1, \tilde{F}_m, e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \cdot \\ &\qquad \tilde{G}^1_m\left(e^{\varepsilon_2 \tilde{G}^1_m} \hat{x}\right) \\ &\quad - D_X \tilde{G}^2_m\left(e^{\varepsilon_2 \tilde{G}^2_m} e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right) \cdot \Phi\left(\varepsilon_2, \tilde{G}^2_m, e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right) \cdot \\ &\qquad \tilde{F}_m\left(e^{\varepsilon_1 \tilde{F}_m} \hat{x}\right).\end{aligned}$$

Thus at $(\varepsilon_1, \varepsilon_2) = (0, 0)$,

$$h = \frac{\partial h}{\partial \varepsilon_1} = \frac{\partial h}{\partial \varepsilon_2} = \frac{\partial^2 h}{\partial {\varepsilon_1}^2} = \frac{\partial^2 h}{\partial {\varepsilon_2}^2} = 0$$

and

$$\frac{\partial^2 h}{\partial\varepsilon_1\partial\varepsilon_2} = \begin{pmatrix} 0 \\ \sum_{i=1}^n \frac{\partial f_{m,1}}{\partial x_i} g_{m,i} \\ \vdots \\ \sum_{i=1}^n \frac{\partial f_{m,n}}{\partial x_i} g_{m,i} \end{pmatrix} - \begin{pmatrix} 0 \\ \sum_{i=1}^n \frac{\partial g_{m,1}}{\partial x_i} f_{m,i} \\ \vdots \\ \sum_{i=1}^n \frac{\partial g_{m,n}}{\partial x_i} f_{m,i} \end{pmatrix}.$$

At $(\varepsilon_1, \varepsilon_2) = (0,0)$, the last n components of $\frac{\partial^2 h}{\partial\varepsilon_1\partial\varepsilon_2}$ is

$$[g_m, f_m](0,\tilde{x}) = (D_x f_m(0,\tilde{x}))\, g_m(0,\tilde{x}) - (D_x g_m(0,\tilde{x}))\, f_m(0,\tilde{x}),$$

where $D_x f(x) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. Since $[f_m, g_m](t,x)$ are uniformly bounded for $m \in I\!N, |t| \leq \varepsilon$ and $x \in \mathbb{R}^n$, by Taylor's theorem there exists $C_1 > 0$ satisfying (2.9).

If necessary, by considering $|\varepsilon_1|$ and $|\varepsilon_2|$ instead of ε_1 and ε_2 we may assume that $\varepsilon_1, \varepsilon_2 > 0$. Let $x_m(t)$ be the solution of the Cauchy problem

$$(2.10) \qquad \dot{x}(t) = \begin{cases} f_m(t, x(t)), & 0 \leq t < \varepsilon_1 \\ \bar{g}_m^2(t, x(t)), & \varepsilon_1 \leq t \leq \varepsilon_1 + \varepsilon_2 \end{cases}, \qquad x(0) = \tilde{x}.$$

and let $y(t)$ be the solution of the Cauchy problem

$$(2.11) \qquad \dot{x}(t) = \begin{cases} f(t, x(t)), & 0 \leq t < \varepsilon_1 \\ \bar{g}^2(t, x(t)), & \varepsilon_1 \leq t \leq \varepsilon_1 + \varepsilon_2 \end{cases}, \qquad x(0) = \tilde{x}.$$

Choose $n_1 \in I\!N$ such that

$$|f(t,x) - f_{n_1}(t,x)| < \varepsilon_2 \quad \text{and} \quad |\bar{g}^2(t,x) - \bar{g}_{n_1}^2(t,x)| < \varepsilon_1$$

for any $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. For $0 \leq t \leq \varepsilon_1$,

$$\begin{aligned} |y(t) - x_{n_1}(t)| &\leq \int_0^t |f(s, y(s)) - f_{n_1}(s, x_{n_1}(s))| ds \\ &\leq \int_0^t |f(s, y(s)) - f(s, x_{n_1}(s))| ds \\ &\qquad + \int_0^t |f(s, x_{n_1}(s)) - f_{n_1}(s, x_{n_1}(s))| ds \\ &\leq L \int_0^t |y(s) - x_{n_1}(s))| ds + \varepsilon_1\varepsilon_2 \end{aligned}$$

and by Gronwall's inequality,

$$|y(t) - x_{n_1}(t)| \le \varepsilon_1\varepsilon_2 e^{tL}.$$

For $\varepsilon_1 \le t \le \varepsilon_1 + \varepsilon_2$,

$$\begin{aligned}
|y(t) - x_{n_1}(t)| = |y(\varepsilon_1) + \int_{\varepsilon_1}^{t} \bar{g}^2(s, y(s))ds \\
- x_{n_1}(\varepsilon_1) - \int_{\varepsilon_1}^{t} \bar{g}^2_{n_1}(s, x_{n_1}(s))ds| \\
&\le |y(\varepsilon_1) - x_{n_1}(\varepsilon_1)| + \int_{\varepsilon_1}^{t} |\bar{g}^2(s, y(s)) - \bar{g}^2(s, x_{n_1}(s))|ds \\
&\quad + \int_{\varepsilon_1}^{t} |\bar{g}^2(s, x_{n_1}(s)) - \bar{g}^2_{n_1}(s, x_{n_1}(s))|ds \\
&\le |y(\varepsilon_1) - x_{n_1}(\varepsilon_1)| + \int_{\varepsilon_1}^{t} L|y(s) - x_{n_1}(s)|ds + \int_{\varepsilon_1}^{t} \varepsilon_1 ds \\
&\le \varepsilon_1\varepsilon_2 e^{\varepsilon_1 L} + \varepsilon_1\varepsilon_2 + \int_{\varepsilon_1}^{t} L|y(s) - x_{n_1}(s)|ds,
\end{aligned}$$

so by Gronwall's inequality,

$$|y(\varepsilon_1 + \varepsilon_2) - x_{n_1}(\varepsilon_1 + \varepsilon_2)| \le \big(\varepsilon_1\varepsilon_2 e^{\varepsilon_1 L} + \varepsilon_1\varepsilon_2\big)e^{\varepsilon_2 L}.$$

We thus have

$$\begin{aligned}
(2.12) \qquad & |e^{\varepsilon_2 \bar{g}^2}\tilde{e}^{\varepsilon_1 f}\tilde{x} - e^{\varepsilon_2 \bar{g}^2_{n_1}}\tilde{e}^{\varepsilon_1 f_{n_1}}\tilde{x}| \\
&= |y(\varepsilon_1 + \varepsilon_2) - x_{n_1}(\varepsilon_1 + \varepsilon_2)| \le C_2\varepsilon_1\varepsilon_2
\end{aligned}$$

where $C_2 = e^{(\varepsilon_1+\varepsilon_2)L} + e^{\varepsilon_2 L}$. In the same way, we get

$$(2.13) \qquad |\tilde{e}^{\varepsilon_1 f}e^{\varepsilon_2 \bar{g}^1}\tilde{x} - \tilde{e}^{\varepsilon_1 f_{n_1}}e^{\varepsilon_2 \bar{g}^1_{n_1}}\tilde{x}| \le C_2\varepsilon_1\varepsilon_2.$$

Due to (2.9), (2.12) and (2.13), we have

$$\begin{aligned}
|\tilde{e}^{\varepsilon_1 f}e^{\varepsilon_2 \bar{g}^1}\tilde{x} - e^{\varepsilon_2 \bar{g}^2}\tilde{e}^{\varepsilon_1 f}\tilde{x}| &= |\tilde{e}^{\varepsilon_1 f}e^{\varepsilon_2 \bar{g}^1}\tilde{x} - \tilde{e}^{\varepsilon_1 f_{n_1}}e^{\varepsilon_2 \bar{g}^1_{n_1}}\tilde{x} \\
&\quad + \tilde{e}^{\varepsilon_1 f_{n_1}}e^{\varepsilon_2 \bar{g}^1_{n_1}}\tilde{x} - e^{\varepsilon_2 \bar{g}^2_{n_1}}\tilde{e}^{\varepsilon_1 f_{n_1}}\tilde{x} \\
&\quad + e^{\varepsilon_2 \bar{g}^2_{n_1}}\tilde{e}^{\varepsilon_1 f_{n_1}}\tilde{x} - e^{\varepsilon_2 \bar{g}^2}\tilde{e}^{\varepsilon_1 f}\tilde{x}| \\
&\le (2C_2 + C_1)\varepsilon_1\varepsilon_2. \qquad \square
\end{aligned}$$

Depending on the above lemma, we can prove Lipschitz continuity of the map Φ from $\mathcal{S}_M$ to the set of maps on $[0, T]$ defined by $\Phi(u) = x_u$.

THEOREM 2.3. *Under assumptions C1, C2 and C5, there exists a constant $\bar{M} > 0$ such that for any $u, v \in \mathcal{S}_M$ and $\tau \in [0, T]$,*

$$|x_u(\tau) - x_v(\tau)| \leq \bar{M}\Big[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^\tau |u(s) - v(s)|ds\Big]. \tag{2.14}$$

Proof. Let $u(t) = \sum_{i=1}^m \alpha_i \chi_{I_i}(t)$ and $v(t) = \sum_{i=1}^m \beta_i \chi_{I_i}(t)$ where χ_I is the characteristic function on the set I and I_i is an interval which is one of the forms $[a_{i-1}, a_i]$, $(a_{i-1}, a_i]$, $[a_{i-1}, a_i)$, (a_{i-1}, a_i), and $0 = a_0 \leq a_1 \leq \cdots \leq a_m = T$. If $\tau \in I_j$ for some j, by considering $u(t) = \sum_{i=1}^j \alpha_i \chi_{I_i^*}(t)$ and $v(t) = \sum_{i=1}^j \beta_i \chi_{I_i^*}(t)$ where $I_i^* = I_i$ for $i = 1, \cdots, j-1$ and $I_j^* = I_j \cap [a_{j-1}, \tau]$, we may assume that $\tau = T$.

Put

$$k_i = \alpha_{i+1} - \alpha_i, \quad s_i = \beta_{i+1} - \beta_i, \quad c_i = \alpha_{i+1} - \beta_i \quad \text{for } i = 1, \cdots, m-1$$

and

$$\ell_i = a_i - a_{i-1}, \quad d_i = \alpha_i - \beta_i \quad \text{for } i = 1, \cdots, m.$$

For each $i = 1, \cdots, m-1$, $|k_i| \leq \varepsilon, |d_i| \leq \varepsilon$ and $|d_m| \leq \varepsilon$. By (2.2), the solution x_u of (1.1) corresponding to u satisfies

$$x_u(a_i+) = \left(e^{(u(a_i+) - u(a_i-))g(a_i, \cdot)}\right) x_u(a_i-). \tag{2.15}$$

for $i = 1, \cdots, m-1$. Thus the points $x_u(T)$ and $x_v(T)$ can be written as

$$x_u(T) = \tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \cdots \tilde{e}^{\ell_2 f} e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}$$

and

$$x_v(T) = \tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \cdots \tilde{e}^{\ell_2 f} e^{s_1 g^1} \tilde{e}^{\ell_1 f} \bar{x},$$

where $g^i(t, x) = g(a_i, x)$, $i = 0, \cdots, m$. Let

$$s_m = k_0 = \ell_0 = 0.$$

For $i = 0, \cdots, m-1$, by adding and subtracting

$$\tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \cdots e^{s_{i+1} g^{i+1}} \tilde{e}^{\ell_{i+1} f} e^{-d_{i+1} g^i} e^{k_i g^i} \tilde{e}^{\ell_i f} \cdots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \tag{2.16}$$

accepting that when $i = m-1$, (2.16) is

$$\tilde{e}^{\ell_m f} e^{-d_m g^{m-1}} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}$$

and when $i = 0$, (2.16) is

$$\tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{s_1 g^1} \tilde{e}^{\ell_1 f} e^{-d_1 g^0} \bar{x},$$

we have

$$
\begin{aligned}
&|x_u(T) - x_v(T)| \\
&\le |\tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
&\quad - \tilde{e}^{\ell_m f} e^{-d_m g^{m-1}} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\quad + \sum_{i=1}^{m-1} |\tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{s_{i+1} g^{i+1}} \tilde{e}^{\ell_{i+1} f} \cdot \\
&\qquad e^{-d_{i+1} g^i} e^{k_i g^i} \tilde{e}^{\ell_i f} e^{k_{i-1} g^{i-1}} \tilde{e}^{\ell_{i-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
&\quad - \tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{s_{i+1} g^{i+1}} \tilde{e}^{\ell_{i+1} f} \cdot \\
&\qquad e^{s_i g^i} \tilde{e}^{\ell_i f} e^{-d_i g^{i-1}} e^{k_{i-1} g^{i-1}} \tilde{e}^{\ell_{i-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\quad + |\tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{s_1 g^1} \tilde{e}^{\ell_1 f} e^{-d_1 g^0} \bar{x} \\
&\quad - \tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{s_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}|.
\end{aligned}
\tag{2.17}
$$

The first two terms of the right-hand side of (2.17) are bounded since by lemma 2.2,

$$
\begin{aligned}
&|\tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
&\quad - \tilde{e}^{\ell_m f} e^{-d_m g^{m-1}} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&= |e^{d_m g^m} e^{-d_m g^m} \tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
&\quad - \tilde{e}^{\ell_m f} e^{-d_m g^{m-1}} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\le |e^{d_m g^m} e^{-d_m g^m} \tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
&\quad - e^{-d_m g^m} \tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\quad + |e^{-d_m g^m} \tilde{e}^{\ell_m f} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
&\quad - \tilde{e}^{\ell_m f} e^{-d_m g^{m-1}} e^{k_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \dots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\le d_m M_1 + C|\ell_m d_m|.
\end{aligned}
\tag{2.18}
$$

Observing that for $i = 1, \cdots, m-1$,

$$d_{i+1} = c_i - s_i \qquad \text{and} \qquad d_i = c_i - k_i,$$

and for any $x \in \mathbb{R}^n$,

$$e^{-d_{i+1}g^i} e^{k_i g^i} \tilde{e}^{\ell_i f} x = e^{s_i g^i} e^{-d_i g^i} \tilde{e}^{\ell_i f} x,$$

we have a bound for terms inside the summation of (2.17):

$$\begin{aligned}
&|\tilde{e}^{\ell_m f} e^{s_{m-1}g^{m-1}} \tilde{e}^{\ell_{m-1} f} \ldots e^{s_{i+1}g^{i+1}} \tilde{e}^{\ell_{i+1} f} \cdot \\
&\quad e^{-d_{i+1}g^i} e^{k_i g^i} \tilde{e}^{\ell_i f} e^{k_{i-1}g^{i-1}} \tilde{e}^{\ell_{i-1} f} \ldots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x} \\
(2.19) \qquad &- \tilde{e}^{\ell_m f} e^{s_{m-1}g^{m-1}} \tilde{e}^{\ell_{m-1} f} \ldots e^{s_{i+1}g^{i+1}} \tilde{e}^{\ell_{i+1} f} \cdot \\
&\quad e^{s_i g^i} \tilde{e}^{\ell_i f} e^{-d_i g^{i-1}} e^{k_{i-1}g^{i-1}} \tilde{e}^{\ell_{i-1} f} \ldots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\le C|d_i \ell_i| e^{L(M+T)}.
\end{aligned}$$

The last two terms of the right-hand side of (2.17) are also bounded:

$$\begin{aligned}
&|\tilde{e}^{\ell_m f} e^{s_{m-1}g^{m-1}} \tilde{e}^{\ell_{m-1} f} \ldots e^{s_1 g^1} \tilde{e}^{\ell_1 f} e^{-d_1 g^0} \bar{x} \\
(2.20) \qquad &\qquad - \tilde{e}^{\ell_m f} e^{s_{m-1}g^{m-1}} \tilde{e}^{\ell_{m-1} f} \ldots e^{s_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}| \\
&\le M_1 |d_1| e^{L(M+T)}
\end{aligned}$$

since $|e^{-d_1 g^0}\bar{x} - \bar{x}| \le M_1|d_1|$, $\sum_{i=1}^{m} \ell_i = T$, $\sum_{i=1}^{m-1} |s_i| \le M$ and g^i, f are Lipschitz continuous with constant L. By the definitions of d_i and ℓ_i, $|u(0) - v(0)| = d_1, |u(T) - v(T)| = d_m$ and $\int_0^T |u(t) - v(t)|dt = \sum_{i=1}^{m} |d_i \ell_i|$, so

$$\begin{aligned}
|x_u(T) - x_v(T)| \le \bar{M}\big[&|u(0) - v(0)| + |u(T) - v(T)| \\
&+ \int_0^T |u(t) - v(t)|dt\big],
\end{aligned}$$

where $\bar{M} = (M_1 + C + 1)e^{L(M+T)}$. □

REMARK 2.4. For $i = 1, \cdots, m-1$, the point

$$\tilde{e}^{\ell_m f} e^{s_{m-1} g^{m-1}} \tilde{e}^{\ell_{m-1} f} \cdots e^{s_{i+1} g^{i+1}} \tilde{e}^{\ell_{i+1} f} e^{-d_{i+1} g^i} e^{k_i g^i}$$
$$\tilde{e}^{\ell_i f} \cdots e^{k_1 g^1} \tilde{e}^{\ell_1 f} \bar{x}$$

can be reached by following x_u−trajectory on $[0, a_j]$ and then following x_v−trajectory on $(a_j, T]$.

Inequality (2.14) makes it possible to extend the map Φ continuously to measurable functions of bounded variation and to define the generalized solution of (1.1) corresponding to a measurable function u which is of bounded variation.

DEFINITION 2.5. Let f, g satisfy C1, C2 and C5. Given an equivalence class of a control function u of bounded variation with an initial value $u(0)$, a trajectory $t \mapsto x_u(t)$ is a generalized solution of (1.1) corresponding to u if there exists a sequence $\{v_k\}$ of control functions in $\mathcal{S}_M$ for some $M > 0$ such that $v_k(0) = u(0), v_k \to u$ in $L^1(m)$, and $x_{v_k}(t)$ tends to $x_u(t)$ for each $t \in [0, T]$, where m is the Lebesgue measure.

For any measurable control function u with a total variation M, there exists a sequence $\{v_k\}$ of control functions in $\mathcal{S}_M$ such that $v_k(0) = u(0)$ and $v_k \to u$ in $L^1(m)$. Thus there exists a set $\mathcal{N}$ of measure zero such that $v_k(t) \to u(t)$ for $t \in [0, T] \setminus \mathcal{N}$. By (2.14), there exists a map x_u on $[0, T]$ such that $x_{v_k}(t)$ converges to $x_u(t)$ for each $t \in [0, T] \setminus \mathcal{N}$. Hence the generalized solution of (1.1) corresponding to a measurable control function u of bounded variation is unique up to m−a.e. We can also define the trajectory $x_u(t)$ pointwise by the following procedure: for $\tau \in [0, T]$, choose a sequence $\{v_k\}$ in $\mathcal{S}_M$ such that $v_k(0) = u(0)$, $v_k(\tau) = u(\tau)$ and $v_k \to u$ in $L^1(m)$, and then $x_u(\tau)$ is defined as the limit of $x_{v_k}(\tau)$.

3. Estimate in the case that g is not Lipschitz continuous

Let f, g be maps satisfying C1∼ C4. To define an auxiliary time variable θ, define a function $t(\theta)$ from $[\bar{\phi}(0), \bar{\phi}(T)]$ into $[0, T]$ by

$$t(\theta) = \inf\{t \in [0, T] \mid \bar{\phi}(t) \geq \theta\} \qquad \text{for } \theta \in [\bar{\phi}(0), \bar{\phi}(T)], \tag{3.1}$$

where $\bar{\phi}(t) = t + \phi(t)$. Then the function $t(\theta)$ is nondecreasing, onto and continuous. By right-continuity of $\bar{\phi}$ and the definition of $t(\theta)$,

$$\bar{\phi}(t(\theta)) \geq \theta \quad \text{and} \quad \bar{\phi}(t(\theta)-) \leq \theta, \tag{3.2}$$

so for $\theta_1 > \theta_2$,

$$\theta_1 - \theta_2 \geq t(\theta_1) - t(\theta_2). \tag{3.3}$$

Define the functions θ^+, θ^- from $[0,T]$ to $[\bar{\phi}(0), \bar{\phi}(T)]$ by

$$\theta^+(\tau) = \sup\{\alpha \in [\bar{\phi}(0), \bar{\phi}(T)] \mid t(\alpha) = \tau\} \tag{3.4}$$

and

$$\theta^-(\tau) = \inf\{\alpha \in [\bar{\phi}(0), \bar{\phi}(T)] \mid t(\alpha) = \tau\}. \tag{3.5}$$

The functions $t(\cdot), \theta^+(\cdot), \theta^-(\cdot)$ satisfy the following properties.

LEMMA 3.1. *Let $\tilde{t} \in [0,T]$ and $\tilde{\theta} \in [\bar{\phi}(0), \bar{\phi}(T)]$. The following statements hold.*

(1) *If $\tilde{\theta} \leq \bar{\phi}(\tilde{t})$, then $t(\tilde{\theta}) \leq \tilde{t}$.*
(2) *If $\bar{\phi}(\tilde{t}) < \tilde{\theta}$, then $\tilde{t} < t(\tilde{\theta})$.*
(3) *$\theta^-(t(\tilde{\theta})) \leq \tilde{\theta}$ and $\theta^+(t(\tilde{\theta})) \geq \tilde{\theta}$*
(4) *$\theta^-(t(\tilde{\theta})) = \bar{\phi}(t(\tilde{\theta})-)$ and $\theta^+(t(\tilde{\theta})) = \bar{\phi}(t(\tilde{\theta}))$.*
(5) *$\bar{\phi}$ is continuous at $\tilde{t}$ if and only if $\theta^-(\tilde{t}) = \theta^+(\tilde{t})$.*

Proof. (1). If $\tilde{\theta} \leq \bar{\phi}(\tilde{t})$, then by the definition of $t(\tilde{\theta})$, $t(\tilde{\theta}) = \inf\{t \in [0,T] \mid \bar{\phi}(t) \geq \tilde{\theta}\} \leq \tilde{t}$.

(2). If $t(\tilde{\theta}) = \tilde{t}$, then $\bar{\phi}(\tilde{t}) = \bar{\phi}(t(\tilde{\theta})) \geq \tilde{\theta}$ by (3.2). Suppose that $t(\tilde{\theta}) < \tilde{t}$. There exists α with $t(\tilde{\theta}) \leq \alpha < \tilde{t}$ such that $\bar{\phi}(\alpha) \geq \tilde{\theta}$. Since $\bar{\phi}$ is increasing, $\bar{\phi}(\tilde{t}) \geq \tilde{\theta}$.

(3). By the definition of θ^- and θ^+, $\theta^-(t(\tilde{\theta})) = \inf\{\alpha \mid t(\alpha) = t(\tilde{\theta})\} \leq \tilde{\theta}$ and $\theta^+(t(\tilde{\theta})) = \sup\{\alpha \mid t(\alpha) = t(\tilde{\theta})\} \geq \tilde{\theta}$.

(4). Let $\tilde{t} < t(\tilde{\theta})$. We claim that $\bar{\phi}(\tilde{t}) \leq \theta^-(t(\tilde{\theta}))$. For any θ_1 with $t(\theta_1) = t(\tilde{\theta})$, if $\theta_1 \leq \bar{\phi}(\tilde{t})$, then by (1),$\tilde{t} \geq t(\theta_1) = t(\tilde{\theta})$ which contradicts the choice of $\tilde{t}$, so $\theta_1 > \bar{\phi}(\tilde{t})$ which implies that $\theta^-(t(\tilde{\theta})) \geq \bar{\phi}(\tilde{t})$. Hence $\theta^-(t(\tilde{\theta})) \geq \bar{\phi}(t(\tilde{\theta})-)$.

Next, we show that $\bar{\phi}(t(\tilde{\theta})-) \geq \theta^-(t(\tilde{\theta}))$. Let $\theta^-(t(\tilde{\theta})) > \theta_2$. By (3), $\tilde{\theta} > \theta_2$ and $t(\tilde{\theta}) \geq t(\theta_2)$. If $t(\tilde{\theta}) = t(\theta_2)$, then $\theta^-(t(\tilde{\theta})) = \inf\{\alpha \in [\bar{\phi}(0), \bar{\phi}(T)] \mid t(\alpha) = t(\tilde{\theta})\} \leq \theta_2$ which contradicts that $\theta^-(t(\tilde{\theta})) > \theta_2$. Hence

$$t(\tilde{\theta}) > t(\theta_2). \tag{3.6}$$

Let $\tilde{t} \in [0, t(\tilde{\theta}))$. Suppose that $\bar{\phi}(\tilde{t}) < \theta_2$. Then by (2), $\tilde{t} < t(\theta_2)$ and $t(\tilde{\theta}) \leq t(\theta_2)$ which contradicts (3.6). Hence there exists $\tilde{t} \in [0, t(\tilde{\theta}))$ such that $\bar{\phi}(\tilde{t}) \geq \theta_2$, so $\bar{\phi}(t(\tilde{\theta})-) \geq \theta_2$. Since θ_2 is an arbitrary number such that $\theta_2 < \theta^-(t(\tilde{\theta}))$, $\bar{\phi}(t(\tilde{\theta})-) \geq \theta^-(t(\tilde{\theta}))$.

By similar argument, we can show that $\theta^+(t(\tilde{\theta})) = \bar{\phi}(t(\tilde{\theta}+))$. On the other hand, $\bar{\phi}(t(\tilde{\theta}+)) = \bar{\phi}(t(\tilde{\theta}))$, so $\theta^+(t(\tilde{\theta})) = \bar{\phi}(t(\tilde{\theta}))$.

(5). Since the function $t(\theta)$ is onto, for $\tilde{t} \in [0, T]$ there exists $\theta_1 \in [\bar{\phi}(0), \bar{\phi}(T)]$ such that $t(\theta_1) = \tilde{t}$. By (4), $\bar{\phi}$ is continuous at $\tilde{t}$ if and only if $\theta^-(\tilde{t}) = \theta^+(\tilde{t})$. □

Since $\bar{\phi}$ is increasing, by (1.5) the limits

$$g(t+, x) = \lim_{s \to t+} g(s, x) \quad \text{and} \quad g(t-, x) = \lim_{s \to t-} g(s, x)$$

exist. Define the auxiliary function $G(\theta, x)$ on $[\bar{\phi}(0), \bar{\phi}(T)] \times \mathbb{R}^n$ by

$$G(\theta, x) = g(t(\theta), x) \tag{3.7}$$

if $\theta^+(t(\theta)) = \theta^-(t(\theta))$, while

$$G(\theta, x) = \lambda g(t(\theta)+, x) + (1-\lambda) g(t(\theta)-, x) \tag{3.8}$$

if $\theta = \lambda\theta^+(t(\theta)) + (1-\lambda)\theta^-(t(\theta))$ for some $\lambda \in [0, 1]$.

Next lemma shows that $G(\theta, x)$ is Lipschitz continuous.

LEMMA 3.2. *For any $x \in \mathbb{R}^n$ and $\theta_1, \theta_2 \in [\bar{\phi}(0), \bar{\phi}(T)]$,*

$$|G(\theta_1, x) - G(\theta_2, x)| \leq |\theta_1 - \theta_2|. \tag{3.9}$$

Proof. If $\theta^+(t(\theta)) = \theta^-(t(\theta))$, then by lemma 3.1 (5) $g(t(\theta)+, x) = g(t(\theta)-, x)$, so

$$g(t(\theta), x) = \lambda g(t(\theta)+, x) + (1-\lambda) g(t(\theta)-, x)$$

for any $\lambda \in [0, 1]$. Let $\theta_1, \theta_2 \in [\bar{\phi}(0), \bar{\phi}(T)]$. We can write $\theta_i = \lambda_i \theta^+(t(\theta_i)) + (1-\lambda_i)\theta^-(t(\theta_i))$ for $i = 1, 2$ and for some $\lambda_1, \lambda_2 \in [0, 1]$. Assume that $\theta_1 \geq \theta_2$ and $\lambda_1 \geq \lambda_2$. (The case that $\lambda_1 < \lambda_2$ can be covered by replacing λ_1, λ_2 with $1-\lambda_1$ and $1-\lambda_2$, respectively.) Then

(3.10)

$$\begin{aligned}
|\theta_1 - \theta_2| &= |\lambda_1 \theta^+(t(\theta_1)) + (1-\lambda_1)\theta^-(t(\theta_1)) \\
&\quad - \lambda_2 \theta^+(t(\theta_2)) + (1-\lambda_2)\theta^-(t(\theta_2))| \\
&= |\lambda_1(\theta^+(t(\theta_1)) - \theta^+(t(\theta_2))) \\
&\quad + (\lambda_1 - \lambda_2)(\theta^+(t(\theta_2)) - \theta^-(t(\theta_2))) \\
&\quad + (1-\lambda_1)(\theta^-(t(\theta_1)) - \theta^-(t(\theta_2)))| \\
&= \lambda_1(\bar{\phi}(t(\theta_1)) - \bar{\phi}(t(\theta_2))) + (\lambda_1 - \lambda_2)(\bar{\phi}(t(\theta_2)) - \bar{\phi}(t(\theta_2)-)) \\
&\quad + (1-\lambda_1)(\bar{\phi}(t(\theta_1)-) - \bar{\phi}(t(\theta_2)-)),
\end{aligned}$$

and

(3.11)

$$\begin{aligned}
|G(\theta_1, x) - G(\theta_2, x)| &= |\lambda_1 g(t(\theta_1)+, x) + (1-\lambda_1) g(t(\theta_1)-, x) \\
&\quad - \lambda_2 g(t(\theta_2)+, x) - (1-\lambda_2) g(t(\theta_2)-, x)| \\
&= |\lambda_1(g(t(\theta_1)+, x) - g(t(\theta_2)+, x)) \\
&\quad + (\lambda_1 - \lambda_2)(g(t(\theta_2)+, x) - g(t(\theta_2)-, x)) \\
&\quad + (1-\lambda_1)(g(t(\theta_1)-, x) - g(t(\theta_2)-, x)| \\
&\leq \lambda_1(\bar{\phi}(t(\theta_1)) - \bar{\phi}(t(\theta_2))) \\
&\quad + (\lambda_1 - \lambda_2)(\bar{\phi}(t(\theta_2)) - \bar{\phi}(t(\theta_2)-)) \\
&\quad + (1-\lambda_1)(\bar{\phi}(t(\theta_1)-) - \bar{\phi}(t(\theta_2)-)).
\end{aligned}$$

From (3.10) and (3.11), lemma is proved. □

For $u \in \mathcal{S}_M$, define a function U on $[\bar{\phi}(0), \bar{\phi}(T)]$ by

$$U(\theta) = u(t(\theta)).$$

Denote by $x(U,\cdot)$ the solution of the Cauchy problem

$$\frac{dx(\theta)}{d\theta} = f(t(\theta),x)\frac{dt(\theta)}{d\theta} + G(\theta,x)\frac{dU(\theta)}{d\theta}, \quad x(0)=\bar{x}. \tag{3.12}$$

Since G is Lipschitz continuous and $0 \le \frac{dt(\theta)}{d\theta} \le 1$ $m-$a.e., by definition 2.5 the solution of (3.12) for any Borel measurable function U of bounded variation is unique. Moreover, by (2.14), we have the estimate

$$\begin{aligned}|x(U,\bar{\theta}) - x(V,\bar{\theta})| \le C\big[&|U(0)-V(0)| + |U(\bar{\theta})-V(\bar{\theta})| \\ &+ \int_0^{\bar{\theta}} |U(s)-V(s)|ds\big]\end{aligned} \tag{3.13}$$

where $\theta \in [\bar{\phi}(0), \bar{\phi}(T)]$, $u, v \in \mathcal{S}_M$ and $V(\theta) = v(t(\theta))$.

For any Borel measurable function u which is of bounded variation, we define the trajectory $x(u,t) = x(U,\theta^+(t))$. In fact, when u is a smooth control function, $x(u,t)$ is a solution of (1.1) in the sense of classical theory of ODE.

The integral of (3.13) can be expressed as an integral in terms of original control functions u and v.

LEMMA 3.3. *Let u be a $\mu-$integrable and Borel measurable function on $[0,T]$. For any $\bar{t} \in [0,T]$ with $\bar{t} = t(\bar{\theta})$,*

$$\int_0^{\bar{t}} u(s)d\mu(s) = \int_0^{\bar{\theta}} U(s)ds, \tag{3.14}$$

where μ is the Radon measure defined by $\mu((a,b]) = \bar{\phi}(b) - \bar{\phi}(a)$ for any $0 \le a < b \le T$.

Proof. Let $\mathcal{F} = \{E \subset [0,T] : \int_0^{\bar{\theta}} \chi_E(t(\theta))d\theta = \int_0^{\bar{t}} \chi_E(t)dt\}$.
From the fact that for $0 \le a < b \le T$,

$$\chi_{(a,b]}(t(\theta)) = 1 \qquad \text{if and only if} \qquad \bar{\phi}(a) < \theta \le \bar{\phi}(b),$$

$\mathcal{F}$ contains all the half-open intervals. We can easily show that $\mathcal{F}$ is a $\sigma-$algebra, so $\mathcal{F}$ contains all the Borel sets of $[0,T]$. For any Borel measurable function u, approximating u by the simple functions which are all Borel measurable, (3.14) holds. □

Inequality (3.13) can be rewritten as in Theorem 3.4.

THEOREM 3.4. *Let f and g satisfy the assumptions C1~C4. For any $u, v \in \mathcal{S}_M$ and $\tau \in [0,T]$,*

$$|x(u,\tau) - x(v,\tau)| \leq \bar{M}\Big[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^\tau |u(s) - v(s)| d\mu(s)\Big]. \tag{3.15}$$

Now, we can define the generalized solution of (1.1) under assumptions C1~C4.

DEFINITION 3.5. Let f, g satisfy C1~C4. Given an equivalence class with respect to the Radon measure μ of a control function u of bounded variation with an initial value $u(0)$, a trajectory $t \mapsto x(u,t)$ is a generalized solution of (1.1) corresponding to u if there exists a sequence $\{v_k\}$ of controls in $\mathcal{S}_M$ for some $M > 0$ such that $v_k(0) = u(0), v_k \to u$ in $L^1(\mu)$, and $x(v_k, t)$ tends to $x(u,t)$ for each $t \in [0,T]$, where μ is the Radon measure defined by $\mu((a, b]) = \bar{\phi}(b) - \bar{\phi}(a)$ for any $0 \leq a < b \leq T$.

The above generalized solution $x(u,t)$ is unique up to μ-a.e. and the trajectory $x(u,t)$ can also be defined pointwise as in §2.

4. Example

Let

$$g(t,x) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ x\sqrt{t-1} & \text{if } 1 \leq t \leq 2 \text{ and } |x| \leq 2. \end{cases}$$

We extend $g(t,x)$ to $[0,2] \times \mathbb{R}$ so that g satisfies C1, C3, C4 with $\phi(t) = 0$ for $0 \leq t < 1$ and $\phi(t) = 3\sqrt{t-1}$ for $1 \leq t \leq 2$.

Consider the scalar impulsive control system

$$\dot{x}(t) = g(t,x)\dot{u}(t), \qquad t \in [0,2], \quad x(0) = 0$$

with $u(t) = t$ for $0 \leq t < 1$, $u(t) = t + 1$ for $1 \leq t \leq 2$.

For any $c \in \mathbb{R}$ with $0 \leq c < 2/e^{\frac{2}{3}}$,

$$x(t) = \begin{cases} 0 & 0 \leq t < 1, \\ c\,e^{2(t-1)^{\frac{3}{2}}/3} & 1 \leq t \leq 2 \end{cases}$$

is the solution of the integral equation [9, p.11]

$$x(t) = \int_0^t g(s, x(s))du(s), \qquad t \in [0, 2].$$

For $n \in \mathbb{N}$, let u_n be the step function on $[0, 2]$ such that

$$u_n(t) = \begin{cases} \frac{k}{n} & \text{if } \frac{k}{n} \le t < \frac{k+1}{n},\ k = 0, \cdots, n-1, \\ 2 + \frac{k}{n} & \text{if } 1 + \frac{k}{n} \le t < 1 + \frac{k+1}{n},\ k = 0, \cdots, n-2, \\ 2 + \frac{n-1}{n} & \text{if } 1 + \frac{n-1}{n} \le t \le 2. \end{cases}$$

Then $\int_0^2 |u(t) - u_n(t)| d\mu(t) \to 0$ as $n \to \infty$ and the solution $x_{u_n}(t)$ corresponding to the control function $u_n(t)$ is identically zero. Thus the only possible solution of the present paper is the trajectory for $c = 0$.

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