J. Korean Math. Soc. **34** (1997), No. 3, pp. 553-567

TIME-OPTIMAL BANG-BANG TRAJECTORIES USING BIFURCATION RESULT

CHANG EON SHIN

ABSTRACT. This paper is concerned with the control problem

$$\dot{x}(t) = F(x) + u(t)G(x), \quad t \in [0, T], \quad x(0) = 0,$$

where F and G are smooth vector fields on $\mathbb{R}^n$, and the admissible controls u satisfy the constraint $|u(t)| \leq 1$. We provide the sufficient condition that the bang-bang trajectories having different switching orders intersect.

1. Introduction

Consider the control system

$$\dot{x}(t) = F(x) + u(t)G(x), \quad t \in [0, T], \quad x(0) = 0, \tag{1.1}$$

where the vector fields F and G are smooth on $\mathbb{R}^n$ and admissible controls u are measurable functions taking values in $[-1, 1]$. The aim of this paper is to investigate the optimality of bang-bang trajectories steering the above system from the origin to a given point in $\mathbb{R}^n$ in minimum time.

We now review the main definitions and notations which will be used in this paper. If $u(t) = 1$ almost everywhere(a.e.) on an interval I, then the corresponding trajectory is called an $X-$arc on I, while if $u(t) = -1$ a.e., the corresponding trajectory is called a $Y-$arc on I, where $X = F + G$ and $Y = F - G$. A trajectory is called *bang-bang* if it is a concatenation of $X-$arcs and $Y-$arcs. We say that a control and its corresponding trajectory are extremal if they satisfy Pontryagin's Maximum Principle, which provides a necessary condition for a

Received October 2, 1996.
1991 Mathematics Subject Classification: 49J30.
Key words and phrases: bang-bang, optimal trajectory, bifurcation.
This work was supported by BSRI-94-1412.

trajectory to be optimal. For a small time optimal control problem, several authors have investigated the optimality of bang-bang trajectories in $\mathbb{R}^3$[2],[7],[11] and in $\mathbb{R}^4$[7] under the generic assumptions for the structures of X, Y and Lie brackets of X and Y showing that they lose optimality at third switching points in $\mathbb{R}^3$ and at fourth switching points in $\mathbb{R}^4$. If we can show that any two extremal bang-bang trajectories of (1.1) having different switching orders reach a point at a same total time and those points form a $(n-1)-$dimensional manifold , then they are not optimal from the point of intersection.[9] In this paper, we provide the sufficient condition for an existence of a surface consisting of those points. We call this surface by a cut-locus. We prove the main theorem by means of an application of general bifurcation theory from the simple eigenvalue.

We can develop the above program to the general bifurcation problem derived from a differential equation. Let $X_0, \cdots, X_n$ be smooth vector fields in $\mathbb{R}^n$. Denote lengths of time intervals by s_i, t_i and $\tau_0 = 0$, $\tau_i = s_1 + \cdots + s_i$, $T = \tau_n$. Consider the $(n+1)-$dimensional system

$$\Phi(s,t) = \begin{pmatrix} \Phi_0(s,t) \\ \Phi_1(s,t) \end{pmatrix} = 0 \in \mathbb{R}^{n+1}, \tag{1.2}$$

where

$$\Phi_0(s,t) = [s_1 + \cdots + s_n] - [t_0 + \cdots + t_{n-1}], \tag{1.3}$$

$$\Phi_1(s,t) = e^{s_n X_n} \cdots e^{s_1 X_1}(0) - e^{t_{n-1} X_{n-1}} \cdots e^{t_0 X_0}(0). \tag{1.4}$$

Here, $e^{\tau Z}(p)$ denotes the value at time τ of the solution to the Cauchy problem

$$\dot{y}(t) = Z(y(t)), \qquad y(0) = p.$$

System (1.2) have trivial solution branch:

$$t_0 = s_n = 0, \quad t_i = s_i, \text{ for } i = 1, \cdots, n-1.$$

If it occurs that there are solutions to (1.2) which branch off from the trivial solution, then by substituting X for $X_1, X_3, \cdots$, and Y

for $X_0, X_2, \cdots$, we can confirm optimality of bang-bang trajectories of (1.1) having n switchings.

We define the usual Lie bracket $[F, G](x)$ of smooth vector fields F, G in a given local coordinate by

$$(D_x G(x))F(x) - (D_x F(x))G(x).$$

The main result of this paper is:

THEOREM 1. *When $s_n = 0, t_0 = 0$ and $s_i = t_i$ for $i = 1, \cdots, n-1$, if there exists an adjoint vector $p(\cdot)$ (refer to §2) such that at some point $(s_1, \cdots, s_{n-1}) = (\bar{s}_1, \cdots, \bar{s}_{n-1})$,*

A1) *the equations*

$$p(\tau_i)\,[X_{i+1}(x(\tau_i)) - X_i(x(\tau_i))] = 0, \qquad i = 0, \cdots, n-2$$

determine the nonzero n-dimensional vector $p(T)$ uniquely up to a scalar multiplication and the vector $p(T)$ satisfies

$$p(T)[X_n(x(T)) - X_{n-1}(x(T))] = 0 \text{ and}$$

A2) $p(T)\,[X_{n-1}, X_n](x(T)) \neq 0$,
where $x(t)$ is the solution of the Cauchy problem

$$\dot{x}(t) = X_i(t), \quad \text{on } [\tau_{i-1}, \tau_i] \ (i = 1, \cdots, n), \quad x(0) = 0,$$

then the point $(s_1, \cdots, s_{n-1}) = (\bar{s}_1, \cdots, \bar{s}_{n-1})$ is the bifurcation point for Φ.

2. Bifurcation result

Let Λ, U and W be Banach spaces. Consider the equation

$$\Psi(\nu, \mu) = 0, \tag{2.1}$$

where $\Psi : \Lambda \times U \to W$. We assume that $\Psi \in C^2(\Lambda \times U, W)$ and that

$$\Psi(\nu, 0) = 0 \qquad \forall \nu \in \Lambda. \tag{2.2}$$

If there is a sequence $(\nu_n, \mu_n) \in \Lambda \times U$ with $\mu_n \neq 0$ such that $\Psi(\nu_n, \mu_n) = 0$ and $(\nu_n, \mu_n) \to (\bar{\nu}, 0)$, then the point $\bar{\nu}$ is called *the bifurcation point* for Ψ. Clearly, if $\nu = \bar{\nu}$ is a bifurcation point, by the implicit function theorem, the partial derivative $\frac{\partial \Psi}{\partial \mu}(\bar{\nu}, 0)$ is not invertible, where $\frac{\partial \Psi}{\partial \mu}(\bar{\nu}, 0) = \Psi_\mu(\bar{\nu}, 0)$ is the matrix of the first derivatives $\frac{\partial \Psi_i}{\partial \mu_j}(\bar{\nu}, 0)$. Let $\bar{\nu}$ be a point in Λ and we assume that

$$\nu = (\nu_1, \cdots, \nu_N) \in \Lambda = \mathbb{R}^N, \ \mu = (\mu_1, \cdots, \mu_m) \in U = \mathbb{R}^m, \ W = \mathbb{R}^m.$$

We set

$$B = \frac{\partial \Psi}{\partial \mu}(\bar{\nu}, 0), \qquad A_j = \frac{\partial}{\partial \nu_j}\frac{\partial \Psi}{\partial \mu}(\bar{\nu}, 0), \quad j = 1, \cdots, N,$$

which are $m \times m$ matrices. In a neighborhood of $(\bar{\nu}, 0)$, we expand Ψ in Taylor approximation, by writing

$$\Psi(\nu, \mu) = \Psi(\bar{\nu}, 0) + B\mu + \sum_{j=1}^{N}(\nu_j - \bar{\nu}_j)A_j\mu + \mathcal{N}(\nu, \mu), \tag{2.3}$$

where

$$\mathcal{N}(\nu, 0) \equiv 0, \qquad \text{and} \qquad \frac{\partial \mathcal{N}}{\partial \mu}(\bar{\nu}, 0) = 0$$

since $\Psi(\nu, 0) = 0$ for any $\nu \in \Lambda$.

THEOREM 2. *The point $\bar{\nu}$ is a bifurcation point for Ψ in (2.1) provided that*

B1) *dim(ker B) =1, and*

B2) *for some ℓ, Range(B) $\oplus$ [$A_\ell \cdot$ ker(B)] = W.*

Proof. By assumption B1), there exists $\mu_0 \in U$ such that $\ker(B)$ is spanned by the element μ_0. By B1) and B2), the spaces U and W can be decomposed as

$$U = U_0 \oplus U_1, \qquad W = W_0 \oplus W_1,$$

where $U_0 = \ker(B) = \text{span}\{\mu_0\}$, $W_1 = \text{Range}(B)$, $W_0 = A_\ell \cdot \ker(B)$ and U_1 is the topological complement of U_0 in U. Notice that for any

$\mu \in U$, there exist $\mu_1 \in U_1$ and $r \in \mathbb{R}$ such that $\mu = r\mu_0 + \mu_1$. Let's denote the projections from W onto W_0 and W_1, by π_0 and π_1, respectively. It is obvious that equation (2.1) is equivalent to

$$\pi_0(\Psi(\nu,\mu)) = 0, \tag{2.4}$$

and

$$\pi_1(\Psi(\nu,\mu)) = 0. \tag{2.5}$$

Rewriting (2.3) as $\Psi(\nu,\mu) = B\mu + \phi(\nu,\mu)$, equation (2.5) is equivalent to

$$B\mu_1 + \pi_1\phi(\nu, r\mu_0 + \mu_1) = 0. \tag{2.6}$$

We set

$$\psi(\nu, r, \mu_1) = B\mu_1 + \pi_1(\phi(\nu, r\mu_0 + \mu_1)).$$

Differentiating ψ with respect to μ_1 at $(\nu, r, \mu_1) = (\bar{\nu}, 0, 0)$, we get the map

$$\psi_{\mu_1}(\bar{\nu}, 0, 0) : \omega \mapsto B\omega + \pi_1(\phi_\mu(\bar{\nu}, 0))\omega.$$

It is clear that from (2.3), $\phi_\mu(\bar{\nu}, 0)$ is the zero map and $\psi_{\mu_1}(\bar{\nu}, 0, 0) = B$. Since the restriction of B to U_1 is bijective onto W_1, by the implicit function theorem, equation (2.5) can be solved uniquely for μ_1 in a neighborhood $\mathcal{U}$ of $(\nu, r) = (\bar{\nu}, 0)$, i.e.,

$$\mu_1 = \mu_1^*(\nu, r).$$

By the uniqueness of μ_1 on $\mathcal{U}$ satisfying (2.5), $\mu_1^*(\nu, 0) = 0$ for any $(\nu, 0) \in \mathcal{U}$. We can substitute $\mu_1^*(\nu, r)$ for μ_1 in (2.6) to get

$$B\mu_1^*(\nu, r) + \pi_1\phi(\nu, r\mu_0 + \mu_1^*(\nu, r)) = 0. \tag{2.7}$$

Differentiating (2.7) with respect to r at $(\nu, r) = (\bar{\nu}, 0)$, we have

$$B\frac{\partial\mu_1^*}{\partial r}(\bar{\nu}, 0)r + \pi_1\left(\phi_\mu(\bar{\nu}, \mu_1^*(\bar{\nu}, 0))[\mu_0 + \frac{\partial\mu_1^*}{\partial r}r]\right) = 0 \quad \text{for any } r \in \mathbb{R}.$$

Since $\mu_1^*(\bar{\nu},0)=0$ and $\phi_\mu(\bar{\nu},0)$ is the zero map,

$$B\frac{\partial \mu_1^*}{\partial r}(\bar{\nu},0)r=0 \text{ for any } r\in\mathbb{R}.$$

Hence, $\frac{\partial \mu_1^*}{\partial r}(\bar{\nu},0)r\in U_0\cap U_1=\{0\}$ for any $r\in\mathbb{R}$. In other words, $\frac{\partial \mu_1^*}{\partial r}(\bar{\nu},0)$ is the zero map from $\mathbb{R}$ to U_1.

Next, we consider equation (2.4) which is equivalent to

$$\begin{aligned}\pi_0[B(r\mu_0+\mu_1^*(\nu,r)]+\sum_{j=1}^{N}(\nu_j-\bar{\nu}_j)A_j(r\mu_0+\mu_1^*(\nu,r))+\\ \mathcal{N}(\nu,r\mu_0+\mu_1^*(\nu,r))]=0.\end{aligned} \tag{2.8}$$

Setting $\Psi_0(\nu,r)=\pi_0(\Psi(\nu,\mu))=\pi_0(\Psi(\nu,r\mu_0+\mu_1^*(\nu,r)))$, $\Psi_0\in C^2$ and $\Psi_0(\nu,0)=0$ since $\Psi(\nu,0)=0$ for any ν. Hence, we can define the map $G:\Lambda\times\mathbb{R}\to W_0$ by

$$G(\nu,r)=\begin{cases} r^{-1}\Psi_0(\nu,r) & \text{if } r\neq 0,\\ \dfrac{\partial\Psi_0}{\partial r}(\nu,0) & \text{if } r=0.\end{cases}$$

Then, $G(\nu,r)\in C^1$ and notice that

$$\frac{\partial\Psi_0}{\partial r}(\nu,0)=\pi_0\left(\frac{\partial\Psi}{\partial\mu}(\nu,0)[\mu_0+\frac{\partial\mu_1^*}{\partial r}(\nu,0)]\right)$$

and

$$\frac{\partial G}{\partial\nu_\ell}(\bar{\nu},0)=\pi_0\left(\frac{\partial}{\partial\nu_\ell}\frac{\partial\Psi}{\partial\mu}(\bar{\nu},0)\mu_0\right)$$

since $\frac{\partial\mu_1^*}{\partial r}(\bar{\nu},0)$ is the zero map. Assumption B2) implies that

$$\frac{\partial G}{\partial\nu_\ell}(\bar{\nu},0)=\pi_0(A_\ell\mu_0)\neq 0.$$

By the implicit function theorem, the equation $G(\nu,r)=0$ can be solved for ν_ℓ in a neighborhood $\mathcal{V}$ of $(\bar{\nu}_1,\cdots,\bar{\nu}_{\ell-1},\bar{\nu}_{\ell+1},\cdots,\bar{\nu}_N,0)$,

$$\text{i.e.}\quad \nu_\ell=\nu_\ell^*(\nu_1,\cdots,\nu_{\ell-1},\nu_{\ell+1},\cdots,\nu_N,r)\text{ on }\mathcal{V}.$$

Observe that $\bar{\nu}_\ell = \nu_\ell^*(\bar{\nu}_1, \cdots, \bar{\nu}_{\ell-1}, \bar{\nu}_{\ell+1}, \cdots, \bar{\nu}_N, 0)$ and by shrinking $\mathcal{V}$, we may assume that for $(\nu_1, \cdots, \nu_{\ell-1}, \nu_{\ell+1}, \cdots \nu_N, r) \in \mathcal{V}$,

$$(\nu_1, \cdots, \nu_\ell^*, \cdots, \nu_N, r) \in \mathcal{U}.$$

Taking into account that

$$\pi_0(\Psi(\nu, \mu)) = 0 \quad \text{if and only if} \quad G(\nu, r) = 0,$$

there exist nontrivial solutions of $\Psi(\nu, \mu) = 0$.

$$\nu_\ell = \nu_\ell^*(\nu_1, \cdots, \nu_{\ell-1}, \nu_{\ell+1}, \cdots, \nu_N, r),$$

$$\mu_1 = \mu_1^*(\nu_1, \cdots, \nu_\ell^*, \cdots, \nu_N, r).$$

where $(\nu_1, \cdots, \nu_{\ell-1}, \nu_{\ell+1}, \cdots, \nu_N, r) \in \mathcal{V}$, and for $(\nu_1, \cdots, \nu_{\ell-1}, \nu_{\ell+1}, \cdots, \nu_N, r) \in \mathcal{V}$,

$$(\nu_1, \cdots, \nu_\ell^*, \cdots, \nu_N, r) \in \mathcal{U}.$$

Therefore $\nu = \bar{\nu}$ is a bifurcation point. □

In (1.2), let $\mu_n = s_n, \mu_{n+1} = t_0, \mu_i = s_i - t_i$ and $\nu_i = s_i + t_i$ for $i = 1, \cdots, n-1$. The trivial branch of Φ in (1.2) is $\mu_i = 0$ for $i = 1, \cdots, n+1$. By replacing variables s_i, t_i in (1.2) by ν_i, μ_i, we can regard Φ as a map from $\mathbb{R}^{n-1} \times \mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ with variables ν_i and μ_i. We can explicitly compute matrix B to get

$$B = \begin{pmatrix} 1 & \cdots & 1 & -1 \\ \frac{\partial \Phi_1}{\partial \mu_1} & \cdots & \frac{\partial \Phi_1}{\partial \mu_n} & \frac{\partial \Phi_1}{\partial \mu_{n+1}} \end{pmatrix}.$$

Define the intervals I_j by $[\tau_{j-1}, \tau_j]$, $j = 1, \cdots, n$. The union of the time intervals I_j's is $[0, T]$. Call $p(\cdot)$ the adjoint vector satisfying the equation

$$\dot{p}(t) = -p(t) D_x X_j(x(t)) \quad \text{on } I_j,$$

where $x(t)$ is the trajectory in Theorem 1. Let $M(t, s)$ be the fundamental matrix solution of the variational equation

$$\dot{v}(t) = D_x X_j(x(t)) v(t) \quad \text{on } I_j$$

with $M(t, t)$ being the identity matrix.

3. Proof of theorem 1

When $\mu_i = 0$ for $i = 0, \cdots, n+1$, the matrix B is

$$\begin{pmatrix} 1 & \cdots & 1 & -1 \\ M(T,\tau_1)X_1(x(\tau_1)) & \cdots & X_n(x(\tau)) & -M(T,0)X_0(0) \end{pmatrix}.$$

We claim that the matrix B satisfies conditions B1) and B2). Let $\mu = (0, \cdots, 0)$. Observing that

$$p(T)M(T,\tau_i)X_i(x(\tau_i)) = p(\tau_i)X_i(x(\tau_i)) \quad \text{and}$$

$$M(T,\tau_i)X_i(x(\tau_i)) = M(T,\tau_{i-1})X_i(x(\tau_{i-1})) \qquad \text{for } i = 1, \cdots, n,$$

by assumption (A1), we obtain that

$$p(T) \cdot M(T,\tau_{i+1})X_{i+1}(x(\tau_{i+1})) = p(T) \cdot M(T,\tau_i)X_i(x(\tau_i))$$

$$\text{for } i = 0, \cdots, n-1.$$

Setting $p_0 = -p(T) \cdot M(T,\tau_1)X_1(x(\tau_1))$ and $\bar{\mathbf{p}} = (p_0, p(T))$, $\bar{\mathbf{p}} \cdot B = 0$ and by the uniqueness of $p(T)$, $\dim(\ker B) = 1$ and condition B1) is satisfied.

Next, we claim that $\text{Range}(B) \oplus [A_{n-1} \cdot \ker(B)] = W$. Observe that vector $\Delta\mathbf{x} = (\Delta x_1, \cdots, \Delta x_{n+1}) \in \ker(B)$ if and only if

$$\Delta x_1 + \cdots + \Delta x_n - \Delta x_{n+1} = 0, \tag{3.1}$$

and

$$\begin{aligned} &- M(T,0)X_0(x(0))\Delta x_{n+1} + M(T,\tau_1)X_1(x(\tau_1))\Delta x_1 + \cdots \\ &+ M(T,\tau_{n-1}))X_{n-1}(x(\tau_{n-1}))\Delta x_{n-1} + X_n(x(\tau_n))\Delta x_n = 0. \end{aligned} \tag{3.2}$$

To compute $\dfrac{\partial}{\partial \nu_{n-1}} \dfrac{\partial \Phi}{\partial \mu}$, extend the length of interval I_{n-1} by ε so that the terminal point becomes $T + \varepsilon$ instead of T, and τ_i are unchanged for $i = 0, \cdots, \tau_{n-2}$. If $\Delta\mathbf{x} \in \ker(B)$, then by (3.2)

$$\begin{aligned}\frac{\partial}{\partial\nu_{n-1}}\frac{\partial\Phi_1}{\partial\mu}\Delta\mathbf{x} &= \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}[M(T+\varepsilon,T)\big(-M(T,0)X_0(x(0))\Delta x_{n+1}+\\ &\qquad \cdots + M(T,\tau_{n-1})X_{n-1}(x(\tau_{n-1}))\Delta x_{n-1}\big)\\ &\qquad + X_n(x(\tau_n+\varepsilon))\Delta x_n]\\ &= \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}[X_n(x(\tau_n+\varepsilon))\\ &\qquad - M(T+\varepsilon,T)X_n(x(\tau_n))]\Delta x_n\\ &= [D_xX_n(x(T))X_{n-1}(x(T))\\ &\qquad - D_xX_{n-1}(x(T))X_n(x(T))]\Delta x_n\\ &= [X_{n-1},X_n](x(T))\Delta x_n.\end{aligned}$$

Since for any $v \in$Range(B), $\bar{\mathbf{p}}\cdot v = 0$, if

$$\bar{\mathbf{p}}\cdot\frac{\partial}{\partial\nu_{n-1}}\frac{\partial\Phi}{\partial\mu}\Delta\mathbf{x} = p(T)\cdot[X_{n-1},X_n](x(T))\Delta x_n \neq 0,$$

then $A_{n-1}\cdot\Delta\mathbf{x} \notin$ Range(B) and Range$(B) \oplus [A_{n-1}\cdot\ker B] = W$. By assumption A2), we only have to show that $\Delta x_n \neq 0$. Write

$$w_i = M(T,\tau_i)X_i(x(\tau_i)),$$

and

$$v_i = w_i - w_{i-1} \qquad \text{for } i = 1,\cdots,n.$$

By the definition of matrix $M(t,s)$, we obtain

$$M(T,\tau_i)X_i(x(\tau_i)) = M(T,\tau_{i-1})X_i(x(\tau_{i-1})),$$

and therefore

$$v_i = M(T,\tau_{i-1})[X_i(x(\tau_{i-1})) - X_{i-1}(x(\tau_{i-1}))].$$

If $(\Delta x_1,\cdots,\Delta x_{n+1}) \in \ker(B)$, $\Delta x_{n+1} = \Delta x_1 + \cdots + \Delta x_n$ and the last

n components of $B\Delta\mathbf{x}$ is

$$\begin{aligned} w_n\Delta x_n &+ w_{n-1}\Delta x_{n-1} + w_{n-2}\Delta x_{n-2} + \cdots + w_1\Delta x_1 \\ &\quad - w_0(\Delta x_1 + \cdots + \Delta x_n) \\ &= (w_n - w_{n-1})\Delta x_n + w_{n-1}(\Delta x_n + \Delta x_{n-1}) \\ &\quad + w_{n-2}\Delta x_{n-2} + \cdots \\ &\quad + w_1\Delta x_1 - w_0(\Delta x_1 + \cdots + \Delta x_n) \\ &= \\ &\qquad \vdots \\ &= (w_n - w_{n-1})\Delta x_n + (w_{n-1} - w_{n-2})(\Delta x_n + \Delta x_{n-1}) + \cdots \\ &\quad + (w_2 - w_1)(\Delta x_2 + \cdots + \Delta x_n) + w_1(\Delta x_1 + \cdots + \Delta x_n) \\ &\quad - w_0(\Delta x_1 + \cdots + \Delta x_n) \\ &= \sum_{i=1}^{n} \alpha_i v_i, \end{aligned}$$

where $\alpha_i = \Delta x_i + \cdots + \Delta x_n$.

Thus $\sum_{i=1}^{n} \alpha_i v_i = 0$ if and only if $(\Delta x_1, \cdots, \Delta x_n, \Delta x_{n+1}) \in \ker(B)$. By the uniqueness of $p(T)$ such that $p(T) \cdot v_i = 0$ for $i = 1, \cdots, n$,

$$\dim \operatorname{span}\{v_1, \cdots, v_n\} = n - 1.$$

Observing that $\alpha_n = 0$ if and only if $v_n \notin \operatorname{span}\{v_1, \cdots, v_{n-1}\}$, if $\alpha_n = 0$,

$$\dim \operatorname{span}\{v_1, \cdots, v_{n-1}\} = n - 2$$

and the equations

$$p(\tau_i)[X_{i+1}(x(\tau_i)) - X_i(x(\tau_i))] = 0 \qquad \text{for } i = 0, \cdots, n-2$$

do not determine $p(T)$ uniquely up to a scalar multiplication.

By assumption A1), $\Delta x_n = \alpha_n \neq 0$. Hence $\nu = \bar{\nu}$ is the bifurcation point for system (1.2).

4. Example.

Let F, G be smooth vector fields on a four-dimensional manifold $\mathcal{M}$ with $F(p_0) = 0$. Consider the control system

$$\dot{x}(t) = F(x(t)) + uG(x(t)), \quad x(0) = p_0, \tag{4.1}$$

where the control u is a measurable function taking values in $[-1, 1]$.

It is assumed that

the vectors $G, [G, F], [[G, F], F]$ and $[G, [G, F]]$ are linearly independent at p_0.

By performing a suitable rescaling of time and space coordinates [1], (4.1) takes the form

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) = (u, x_1, x_2, x_1^2/2) + h(x), \quad x(0) = 0, \tag{4.2}$$

where the vector field h plays the role of a small perturbation. In the special case $h \equiv 0$, we apply Theorem 1 to system (4.2).

Let $x(t)$ be the solution of (4.2) and $p(t)$ the adjoint vector with $p(s_1 + s_2 + s_3) = \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$ corresponding to the control $u(t)$ having the values 1 on $[0, s_1) \cup [s_1 + s_2, s_1 + s_2 + s_3]$ and -1 on $[s_1, s_1 + s_2)$. Hence $p(t) = (p_1(t), p_2(t), p_3(t), p_4(t))$ satisfies that

$$\begin{aligned} \dot{p}_1(t) &= -p_2 - p_4 x_1, \\ \dot{p}_2(t) &= -p_3, \\ \dot{p}_3(t) &= 0, \\ \dot{p}_4(t) &= 0. \end{aligned}$$

We can explicitly compute $x(t)$ and $[X, Y]$:

$$x_1(t) = \begin{cases} t & \text{on } [0, s_1), \\ -t + 2s_1 & \text{on } [s_1, s_1 + s_2), \\ t - 2s_2 & \text{on } [s_1 + s_2, s_1 + s_2 + s_3]. \end{cases}$$

$$[X, Y] = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2x_1 \end{pmatrix}.$$

Due to

$$Y - X = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

assumption A1) implies that $p_1(\tau_i) = 0$ for $i = 0, 1, 2$, where $\tau_i = s_0 + \cdots + s_i$. Since $p_1(\tau_3) = 0$, $\tilde{p}_1 = 0$. When $\tilde{p}_1 = 0$,

$$p_1(\tau_2) = \tilde{p}_2 s_3 + \tilde{p}_4 s_1 s_3 - \tilde{p}_4 s_2 s_3 + \frac{\tilde{p}_3 s_3^2}{2} + \frac{\tilde{p}_4 s_3^2}{2}, \tag{4.3}$$

$$\begin{aligned} p_1(\tau_1) =& \tilde{p}_2 s_2 + \tilde{p}_4 s_1 s_2 + \frac{\tilde{p}_3 s_2^2}{2} - \frac{\tilde{p}_4 s_2^2}{2} + \tilde{p}_2 s_3 + \tilde{p}_4 s_1 s_3 + \tilde{p}_3 s_2 s_3 \\ &- \tilde{p}_4 s_2 s_3 + \frac{\tilde{p}_3 s_3^2}{2} + \frac{\tilde{p}_4 s_3^2}{2}, \end{aligned} \tag{4.4}$$

(4.5)

$$\begin{aligned} p_1(0) =& [2\tilde{p}_2(s_1 + s_2 + s_3) + \tilde{p}_3(s_1^2 + 2s_1 s_2 + s_2^2 + 2s_1 s_3 + 2s_2 s_3 + s_3^2) \\ &+ \tilde{p}_4(s_1^2 + 2s_1 s_2 - s_2^2 + 2s_1 s_3 - 2s_2 s_3 + s_3^2)]/2. \end{aligned}$$

In (4.3), we can solve for $\tilde{p}_2$ to get

$$\tilde{p}_2 = -\tilde{p}_4(s_1 - s_2 + \frac{s_3}{2}) - \frac{\tilde{p}_3 s_3}{2} = p_2'.$$

Replacing $\tilde{p}_2$ by p_2', $p_1(\tau_1) = \dfrac{s_2[\tilde{p}_3(s_2 + s_3) + \tilde{p}_4(s_2 - s_3)]}{2} = 0$ and

$$\tilde{p}_3 = \frac{\tilde{p}_4(s_3 - s_2)}{s_2 + s_3} = p_3'.$$

When $\tilde{p}_3 = p_3'$, $p_2' = \dfrac{\tilde{p}_4(-s_1 s_2 + s_2^2 - s_1 s_3 + s_2 s_3 - s_3^2)}{s_2 + s_3} = p_2''$ and

$$p_1(0) = -\frac{\tilde{p}_4 s_1 s_2(s_1 - s_3)}{s_2 + s_3}.$$

If $s_1 = s_3$, then there exists nonzero vector

$$\tilde{p} = (0, \frac{s_2^2 - 2s_3^2}{s_2 + s_3}, \frac{s_3 - s_2}{s_2 + s_3}, 1)$$

which is unique up to a scalar multiplication.

In this case,

$$[X, Y](x(\tau_3)) = (0, 2, 0, -2s_2 + 4s_3),$$

$$\tilde{p} \cdot [X, Y](x(\tau_3)) = \frac{2s_2 s_3}{s_2 + s_3}$$

which does not vanish if $s_2 \neq 0$ or $s_3 \neq 0$. If $p_1(0) = p_1(\tau_1) = p_1(\tau_2) = 0$, ($p_1(\tau_3) = 0$ is excluded), and $s_1 = s_3$, then we have

$$p_1(\tau_2) = \tilde{p}_1 + s_3\tilde{p}_2 + \frac{s_3^2\tilde{p}_3 + 3s_3^2\tilde{p}_4}{2} - s_3 s_2 \tilde{p}_4, \tag{4.6}$$

$$\begin{aligned} p_1(\tau_1) =& \tilde{p}_1 + (s_3 + s_2)\tilde{p}_2 + \frac{(s_3 + s_2)^2}{2}\tilde{p}_3 \\ &+ \frac{2s_3^2 - s_2^2}{2}\tilde{p}_4, \end{aligned} \tag{4.7}$$

$$\begin{aligned} p_1(0) =& \tilde{p}_1 + (2s_3 + s_2)\tilde{p}_2 + (2s_3^2 + 2s_3 s_2 + \frac{s_2^2}{2})\tilde{p}_3 \\ &+ \frac{4s_3^2 - s_2^2}{2}\tilde{p}_4, \end{aligned} \tag{4.8}$$

and direct computation yields that the equations

$$p(\tau_i)[X_{i+1}(x(\tau_i)) - X_i(x(\tau_i))] = 0, \qquad i = 0, 1, 2,$$

where $X_1 = X_3 = X$ and $X_0 = X_2 = Y$, determine

$$\tilde{p} = (0, \frac{s_2^2 - 2s_3^2}{s_2 + s_3}, \frac{s_3 - s_2}{s_2 + s_3}, 1)$$

uniquely up to a scalar multiplication.

Hence, for small $s_4 > 0$, there exist $t_i (i = 0, 1, 2, 3)$ satisfying (1.3)-(1.4) for the system (4.2), but t_0 may be a negative number which is not acceptable. In the following computation, we show that $\frac{\partial s_4}{\partial t_0} > 0$, when $s_i = t_i, s_4 = 0, t_0 = 0$ for $i = 1, 2, 3$. Assume that

$$s_1 + s_2 + s_3 + s_4 = 1.$$

Let $u^+(t)$ and $u^-(t)$ be the controls such that

$$u^-(t) = \begin{cases} -1 & \text{on } [0, t_0) \cup [t_0 + t_1, 1 - t_2), \\ 1 & \text{on } [t_0, t_0 + t_1) \cup [1 - t_2, 1], \end{cases}$$

$$u^+(t) = \begin{cases} 1 & \text{on } [0, s_1) \cup [1 - s_3 - s_4, 1 - s_4) \\ -1 & \text{on } [s_1, 1 - s_3 - s_4) \cup [1 - s_4, 1], \end{cases}$$

and let $x^+(t) = (x_1^+(t), x_2^+(t), x_3^+(t), x_4^+(t))$ and $x^-(t) = (x_1^-(t), x_2^-(t), x_3^-(t), x_4^-(t))$ be the trajectories corresponding to the controls u^+ and u^-, respectively. By direct computation, at $t = 1$,

$$x_1^+ = -1 + 2s_1 + 2s_3,$$

$$x_2^+ = -\frac{1}{2} + 2s_1 - s_1^2 + s_3^2 + 2s_3 s_4,$$

$$x_3^+ = -\frac{1}{6} + s_1 - s_1^2 + \frac{s_1^3 + s_3^3}{3} + s_3^2 s_4 + s_3 s_4^2,$$

$$x_1^- = -1 + 2t_1 + 2t_2,$$

$$x_2^- = -\frac{1}{2} + 2t_1 - 2t_0 t_1 - t_1^2 + t_2^2,$$

$$x_3^- = -\frac{1}{6} + t_1 - 2t_0 t_1 + t_0^2 t_1 - t_1^2 + t_0 t_1^2 + \frac{t_1^3 + t_2^3}{3}.$$

Solving the equations

$$x_i^+ = x_i^- \qquad \text{for } i = 1, 2, 3$$

for s_i's and computing the derivative of s_4 with respect to t_0 at $t_0 = 0$, we obtain

$$\text{(4.9)} \qquad \frac{\partial s_4}{\partial t_0} = \frac{2t_1 t_2(1-t_2)(2-t_1-t_2)(1-t_1-t_2)}{1-t_1} > 0.$$

so t_0 is positive while $s_4 > 0$.

Hence at $s_1 = s_3$, bifurcation occurs and when $h \equiv 0$, any bang-bang trajectory of (4.2) loses optimality at or before fourth switching point if the cut-locus forms 3−dimensional manifold.

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Department of Mathematics,
Sogang University,
Seoul 121-742, Korea
E-mail: shinc@ccs.sogang.ac.kr