J. Korean Math. Soc. **34** (1997), No. 3, pp. 501–513

ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

JIN-GYO JEONG AND KI-YEON SHIN

ABSTRACT. The existence of a unique local generalized solution for the abstract functional evolution problem of the type

(FDE:ϕ) $\quad x'(t) + A(t, x_t)x(t) \ni G(t, x_t),\ t \in [0, T], \quad x_0 = \phi$

in a general Banach spaces is considered. It is shown that (FDE:ϕ) could be considered with well-known fixed point theory and recent results for the functional differential equations involving the operator $A(t)$.

1. Introduction and preliminaries

Let X be a real Banach space with norm $\|\cdot\|$. We let C denote the space of all continuous functions $f : [-r, 0] \to X$ for a fixed $r > 0$. For $f \in C$, $\|f\|_C = \sup_{-r\le s\le 0} \|f(s)\|$.

We consider the abstract nonlinear functional differential equation of the type

$$\text{(FDE : }\phi) \qquad \begin{aligned} &x'(t) + A(t, x_t)x(t) \ni G(t, x_t), \quad t \in [0, T], \\ &x_0 = \phi, \quad -r \le t \le 0 \end{aligned}$$

in a general Banach space, where for a function $f : [-r, T] \to X$, $f_t(s) = f(t + s)$, $t \in [0, T]$, $s \in [-r, 0]$ with a positive constant T.

An operator $A : D \subset X \to 2^X$ is called "accretive" if

$$\|x_1 - x_2\| \le \|x_1 - x_2 + \lambda(y_1 - y_2)\|$$

Received May 16, 1995.

1991 Mathematics Subject Classification: 34K30, 47H20, 34L20.

Key words and phrases: m-accretive operator, method of lines, generalized solution, contraction mapping.

for every $\lambda > 0$ and every $[x_1, y_1], [x_2, y_2] \in A$. It is called "$m$-accretive" if it is accretive and $R(I + \lambda A) = X$ for all $\lambda > 0$. If A is m-accretive, we set

$$|Ax| = \lim_{\lambda \downarrow 0} \|A_\lambda x\|, \quad x \in X,$$

where $A_\lambda = (I - J_\lambda)/\lambda$ with $J_\lambda = (I + \lambda A)^{-1}$. We also set

$$\hat{D} = \{x \in X \ : \ |Ax| < \infty\}.$$

It is known that $D(A) \subset \hat{D}(A) \subset \overline{D(A)}$. For other properities of these operators, the reader is referred to Barbu [1], Crandall [2], Crandall and Pazy [3] and Evans [4].

Tanaka [12] has recently obtained the existence of a unique limit solution of the abstract nonlinear functional evolution equation of the type

$$x'(t) + A(t)x(t) \ni G(t, x_t), \quad t \in [0, T], \quad x_0 = \phi$$

in a general Banach space by constructing the "lines" which satisfy certain approximate discrete scheme. The solution is obtained from the uniform limit of the "lines". Kartsatos and Parrott [10] also have the similar results with different method. For the operator $A(t, x_t)$, Kartsatos and Parrott [8], Kartsatos [7] have studied by use of fixed point theory and Crandall and Pazy's result [3].

The following conditions will be used in the sequel.

(A.1) For each $(t, \psi) \in [0, T] \times C$, $A(t, \psi) : D(A(t, \psi)) \subset X \to 2^X$ is m-accretive in X, where $D(A(t, \psi))$ is only dependent on t. We denote $D(A(t, \psi)) = D(t)$.

(A.2) For each $t, s \in [0, T]$, $\psi_1, \psi_2 \in C$, and $v \in X$,

$$\begin{aligned} &\|A_\lambda(t, \psi_1)v - A_\lambda(s, \psi_2)v\| \\ &\qquad \le L_0(\|v\|)[|t - s|(1 + \|A_\lambda(s, \psi_2)v\|) + \|\psi_1 - \psi_2\|_C] \end{aligned}$$

where $L_0 : \mathcal{R}^+ \to \mathcal{R}^+ = [0, \infty)$ is increasing, continuous function.

(A.3) For $t, s \in [0, T]$, and $\psi, \psi_1, \psi_2 \in C$,

$$\begin{aligned} &\|G(t, \psi_1) - G(t, \psi_2)\| \le k_1 \|\psi_1 - \psi_2\|_C, \\ &\|G(t, \psi) - G(s, \psi)\| \le L_1(\|\psi\|_C)|t - s|, \end{aligned}$$

where k_1 is a positive constant and $L_1 : \mathcal{R}^+ \to \mathcal{R}^+$ is nondecreasing, continuous function.

(A.4) ϕ is a given Lipschitz function with Lipschitz constant k_0 on $[-r, 0]$.

By virtue of (A.2), it is known that $\hat{D}(A(t, \psi))$ is independent of $(t, \psi) \in [0, T] \times C$. (See Evans [4].) We denote by $\hat{D} \equiv \hat{D}(A(t, \psi))$.

The main purpose of this paper is to obtain a "generalized solution" of (FDE:ϕ) with more simple method. When the functional term in A and $G$is fixed, (FDE:ϕ) is converted a very well known evolution problem. Then we employ the Banach contraction principle to get a local generalized solution.

We define a set E by

$$E = \{u : [-r, T] \to X \mid u(t) \text{ is continuous, } u(t) = \phi(t) \text{ for } t \in [-r, 0] \\ \text{and } \|u(t_1) - u(t_2)\| \le M|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T]\},$$

where $M \ge k_0$ is a constant. Clearly, $E \neq \phi$ since the function $u(t)$ defined by $u(t) = \phi(t)$ for $t \in [-r, 0]$, and $u(t) = \phi(0)$ for $t \in [0, T]$ belongs to E. Moreover, the set E is a Banach space with supremum norm. (cf. Ha, Shin and Jin [6]).

2. Main results

In the following discussion, we assume that the hypotheses (A.1)–(A.4) hold and $\phi(0) \in \hat{D}$. Let $u \in E$ be arbitrary but fixed. We shall first consider a more simple evolution problem which is converted from (FDE:ϕ) by employing the above $u \in E$.

By fixing the functional term with u, we consider (EE:ϕ, u) from (FDE:ϕ) by the type of

$$(\text{EE} : \phi, u) \qquad \begin{aligned} &x'(t) + A(t, u_t)x(t) \ni G(t, u_t), \quad t \in [0, T], \\ &x_0 = \phi(0). \end{aligned}$$

For the simplicity, we put $B(t) \equiv A(t, u_t)$ and $g(t) \equiv G(t, u_t)$ for $t \in [0, T]$. Then our hypotheses (A.1)–(A.4) are converted as follows.

(B.1) For each $t \in [0, T]$, $B(t) : D(t) \subset X \to 2^X$ is m-accretive.

(B.2) For each $t, s \in [0,T]$ and $v \in X$,

$$\begin{aligned}\|B_\lambda(t)v - B_\lambda(s)v\| &\le L_0(\|v\|)|t-s|(1+M)(1+\|B_\lambda(s)v\|)\\ &\equiv \tilde{L}_0(\|v\|)|t-s|(1+\|B_\lambda(s)v\|)\end{aligned}$$

where $\tilde{L}_0 : \mathcal{R}^+ \to \mathcal{R}^+$ is again nondecreasing continuous function with $\tilde{L}_0(p) = (1+M)L_0(p)$ and $B_\lambda(t)$ is the Yosida approximation of $B(t)$.
(B.3) For $t, s \in [0,T]$

$$\begin{aligned}\|g(t) - g(s)\| &\le \|G(t,u_t) - G(t,u_s)\| + \|G(t,u_s) - G(s,u_s)\|\\ &\le k_1\|u_t - u_s\|_C + L_1(\|u_s\|_C)|t-s|\\ &\le (k_1 M + L_1(\|u_s\|_C))|t-s|\\ &\le (k_1 M + L_1(\|\phi\|_C + MT))|t-s|\\ &\equiv \tilde{L}_1|t-s|\end{aligned}$$

where $\tilde{L}_1$ is a constant.

By (B.1)–(B.3), it is easy to show that there exist constants $C_1 = C_1(\phi)$ and $C_2 = C_2(\phi)$ such that for $t \in [0,T]$

$$|B(t)\phi(0)| = |A(t,u_t)\phi(0)| \le C_1, \quad \|g(t)\| = \|G(t,u_t)\| \le C_2.$$

Let $\{t_j^n\}_{j=0}^n$ be a partition of the interval $[0,T]$ for fixed n, where $t_j^n = jT/n$, $j = 0, 1, \cdots, n$. And we let $g_j^n = g(t_j^n)$. When we put $x_0^n = \phi(0)$, we construct a sequence $\{x_j^n\}_{j=0}^n$ of elements of X satisfying

$$\frac{x_j^n - x_{j-1}^n}{t_j^n - t_{j-1}^n} + B(t_j^n)x_j^n \ni g_j^n, \qquad j = 1, 2, \cdots, n$$

by m-accretiveness of B. The step function

$$x_n(t) = \begin{cases} x_0^n, & t = 0, \\ x_j^n, & t \in (t_{j-1}^n, t_j^n],\ j = 1, 2, \cdots, n \end{cases}$$

is called an approximate solution of (EE:ϕ,u). If the approximate solution converge to some continuous function uniformly on $[0,T]$, we call it the limit solution of (EE:ϕ,u) on $[0,T]$.

Since the conditions (B.1)–(B.3) satisfy the conditions (A) and (C.2) in Theorem 2 of Evans [4], we follow the steps to get constants $R_1 = R_1(\phi)$, $R_2 = R_2(\phi)$ satisfying

$$\sup_n\{\max_{0\leq j\leq n} \|x_j^n\|\} \leq R_1, \text{ and } \quad \sup_n\{\max_{0\leq j\leq n} \frac{\|x_j^n - x_{j-1}^n\|}{t_j^n - t_{j-1}^n}\} \leq R_2$$

for $\phi(0) \in \hat{D}$. Therefore, by similar method of Evans [4], we have constant $K = 2R_2$ to show that

$$\|x_n(t) - x_n(s)\| \leq K(|t-s| + T/n)$$

for sufficiently large n. Consequently, in considering (EE:ϕ,u) for fixed $u \in E$, we note that our hypotheses (B.1)–(B.3) implies there exist a limit solution $x_u(t)$ which is converged from approximate solutions $x_n(t)$ uniformly on $[0,T]$. More precisely, we have the following theorem.

THEOREM 1. (Evans [4]) *Let* (A.1)–(A.4) *hold and* $\phi(0) \in \hat{D}$. *Then there exist a limit solution* $x_u(t)$ *of* (EE:ϕ,u) *on* $[0,T]$ *for fixed* $u \in E$. *Moreover,* x_u *is Lipschitz continuous with Lipschitz constant* K *on* $[0,T]$.

Now we show the relation between the limit solutions of (EE:ϕ,u) and (EE:ϕ,v) for $u, v \in E$. Using this, we have the existence of a limit solution for (EE:ϕ) by the Banach contraction principle.

THEOREM 2. *Let* $x_u(t)$ *and* $y_v(t)$ *be the limit solutions of* (EE:ϕ,u) *and* (EE:ϕ,v) *in* Theorem 1, *respectively. Then there exists a constant* C_4 *such that*

$$\|x_u(t) - y_v(t)\| \leq \|x_u(\tau) - y_v(\tau)\| + C_4 T\|u - v\|_\infty + \int_\tau^t [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)]_+ \, d\eta$$

for $0 \leq \tau \leq t \leq T$.

Proof. Let x_u, y_v be the limit solutions of (EE:ϕ,u), (EE:ϕ,v), respectively. By the definition of the limit solution of (EE:ϕ,u), there exists an approximate solution $x_n(t)$ such that

$$\frac{x_j^n - x_{j-1}^n}{h_n} + A(t_j^n, u_{t_j^n})x_j^n \ni G(t_j^n, u_{t_j^n}), \tag{2.1}$$

$x_n(0) = x_0^n = \phi(0)$ and $x_n(t) = x_k^n$, $t \in (t_{j-1}^n, t_j^n]$, $j = 1, 2, \cdots, n$, where $h_n = t_j^n - t_{j-1}^n$. Also, there exists an approximate solution $y_m(t)$ such that

$$\frac{y_k^m - y_{k-1}^m}{\hat{h}_m} + A(s_k^m, v_{s_k^m})y_k^m \ni G(s_k^m, v_{s_k^m}), \tag{2.2}$$

$y_m(0) = y_0^m = \phi(0)$ and $y_m(t) = y_k^m$, $t \in (s_{k-1}^m, s_k^m]$, $k = 1, 2, \cdots, m$, where $\hat{h}_m = s_k^m - s_{k-1}^m$. Let $\delta \in (0, T/2)$ and assume that n and m are sufficiently large such that $\max(h_n, \hat{h}_m) < \delta$. Then there is a positive constant C_4 such that for $p \in \{0, 1, \ldots, n\}$ and $q \in \{0, 1, \ldots, m\}$

$$\begin{aligned} \|x_j^n - y_k^m\| \leq \|x_p^n - y_q^m\| &+ C_4 D_{j,k} + \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m \\ &+ jh_n\{(\delta^{-1}\rho(T) + C_4)(D_{j,k} + |t_p^n - \hat{t}_q^m|) \\ &+ \rho(2\delta) + C_4(h_n + \|u - v\|_\infty)\} \end{aligned} \tag{2.3}$$

for $j = p, \cdots, n$ and $k = q, \cdots, m$. Here the symbols used above are defined by

$$\delta_j^n = \left[x_j^n - y_v(t_j^n), G(t_j^n, u_{t_j^n}) - G(t_j^n, (y_v)_{t_j^n})\right]_\lambda,$$

where $[x, y]_\lambda = \lambda^{-1}(\|x + \lambda y\| - \|x\|)$ for $\lambda > 0$,

$$\begin{aligned} \hat{\delta}_k^m &= \|G(s_k^m, u_{s_k^m}) - G(s_k^m, (y_v)_{s_k^m})\| + \frac{2}{\lambda}\|y_k^m - y_v(s_k^m)\|, \\ \rho(\hat{t}) &= \sup\{\frac{2}{\lambda}\|y_v(t) - y_v(r)\| \\ &\quad + \|G(r, (y_v)_r) - G(t, (y_v)_t)\| : |t - r| \leq \hat{t}\}. \end{aligned}$$

and

$$\begin{aligned} D_{j,k} &= \{((t_j^n - t_p^n) - (s_k^m - s_q^m))^2 + (t_j^n - t_p^n)h_n + (s_k^m - s_q^m)\hat{h}_m\}^{\frac{1}{2}} \\ &\quad + \{((t_j^n - t_p^n) - (s_k^m - s_q^m))^2 + (t_j^n - t_p^n)h_n + (s_k^m - s_q^m)\hat{h}_m\}. \end{aligned}$$

First, we prove that (2.3) holds. we let $\sigma = h_n \hat{h}_m / (h_n + \hat{h}_m)$. From (2.1) and (2.2), we have

$$\begin{aligned} A(t_j^n, u_{t_j^n})x_j^n &\ni G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n}, \\ A(s_k^m, v_{s_k^m})y_k^m &\ni G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}. \end{aligned}$$

Choose $0 < \lambda < 1$. Then, with the similar steps in the paper of Ha, Shin and Jin [6],

$$J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(x_j^n + \sigma\lambda\big(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n}\big)\big) = x_j^n,$$

$$J_{\sigma\lambda}(s_k^m, v_{s_k^m})\big(y_k^m + \sigma\lambda\big(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}\big)\big) = y_k^m.$$

From (A.2)–(A.4),

$$\begin{aligned} &\|x_j^n - y_k^m\| \\ &\le \|J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(x_j^n + \sigma\lambda(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n})\big) \\ &\qquad - J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big)\| \\ &\quad + \|J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big) \\ &\qquad - J_{\sigma\lambda}(s_k^m, v_{s_k^m})\big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big)\| \\ &\le \|\big(x_j^n + \sigma\lambda(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n})\big) \\ &\qquad - \big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big)\| \\ &\quad + \sigma\lambda L_0(\|y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|)[|t_j^n - s_k^m| \end{aligned}$$

$$\cdot (1 + \|A_{\sigma\lambda}(s_k^m, v_{\hat{t}_k^m})(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|)$$
$$+ \|u_{t_j^n} - v_{s_k^m}\|_C].$$

Let $A_{j,k} = \|x_j^n - y_k^m\|$. Then

$$\begin{aligned}
A_{j,k} \le & \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\
& + \frac{h_n \hat{h}_m}{h_n + \hat{h}_m} \{L_0(R_1)(1 + C_2 + R_2)|t_j^n - s_k^m| \\
& + L_0(R_1)\|u_{t_j^n} - v_{s_k^m}\|_C + [x_j^n - y_k^m, G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m})]_+\} \\
\le & \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\
& + \frac{h_n \hat{h}_m}{h_n + \hat{h}_m} \{L_0(R_1)(1 + C_2 + R_2)|t_j^n - s_k^m| \\
& + L_0(R_1)\|u_{t_j^n} - v_{s_k^m}\|_C + \delta_j^n + \hat{\delta}_k^m + \rho(|t_j^n - s_k^m|)\}
\end{aligned}$$

by the fact that

$$\begin{aligned}
& [x_j^n - y_k^m, G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m})]_+ \\
& \quad \le [x_j^n - y_v(t_j^n), G(t_j^n, u_{t_j^n}) - G(t_j^n, (y_v)_{t_j^n})]_\lambda \\
& \quad + \|G(s_k^m, u_{s_k^m}) - G(s_k^m, (y_v)_{s_k^m})\| \\
& \quad + \|G(s_k^m, (y_v)_{s_k^m}) - G(t_j^n, (y_v)_{t_j^n})\| \\
& \quad + \frac{2}{\lambda}\|y_k^m - y_v(s_k^m)\| + \frac{2}{\lambda}\|y_v(s_k^m) - y_v(t_j^n)\|.
\end{aligned}$$

Since

$$\begin{aligned}
|t_j^n - s_k^m| & \le |(t_j^n - s_k^m) - h_n| + h_n \\
& \le |(t_j^n - t_p^n) - (s_k^m - s_q^m) - h_n| + |t_p^n - s_q^m| + h_n \\
& \le D_{j-1,k} + |t_p^n - s_q^m| + h_n,
\end{aligned}$$

$$\begin{aligned}
\rho(|t_j^n - s_k^m|) & \le \delta^{-1}\rho(T)(|t_j^n - s_k^m|) - h_n) + \rho(2\delta) \\
& \le \delta^{-1}\rho(T)(D_{j-1,k} + |t_p^n - \hat{t}_q^m|) + \rho(2\delta),
\end{aligned}$$

and

$$\begin{aligned}\|u_{t_j^n} - v_{s_k^m}\|_C &\le \|u_{t_j^n} - u_{s_k^m}\|_C + \|u_{s_k^m} - v_{s_k^m}\|_C \\ &\le M|t_j^n - s_k^m| + \|u_{s_k^m} - v_{s_k^m}\|_C \\ &\le MD_{j-1,k} + M|t_p^n - s_q^m| + Mh_n + \|u_{s_k^m} - v_{s_k^m}\|_C,\end{aligned}$$

we have

$$\begin{aligned}A_{j,k} \le{}& \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\ &+ \frac{h_n \hat{h}_m}{h_n + \hat{h}_m}\{(R_4 + \delta^{-1}\rho(T))(D_{j-1,k} + |t_p^n - \hat{t}_q^m|) \\ &+ R_4 h_n + \delta_j^n + \hat{\delta}_k^m + \rho(2\delta) + R_4\|u - v\|_\infty\}.\end{aligned}$$

For $i = p+1, \cdots, n$,

$$\begin{aligned}&\|x_i^n - x_p^n\| \\ &\le \|J_{h_n}(t_i^n, u_{t_i^n})(x_{i-1}^n + h_n G(t_i^n, u_{t_i^n})) - J_{h_n}(t_i^n, u_{t_i^n})x_p^n\| \\ &\quad + \|J_{h_n}(t_i^n, u_{t_i^n})x_p^n - x_p^n\| \\ &\le \|x_{i-1}^n - x_p^n\| + h_n\|G(t_i^n, u_{t_i^n})\| + h_n|A(t_i^n, u_{t_i^n})x_p^n| \\ &\le \|x_{i-1}^n - x_p^n\| + 2h_nC_3 + h_nL_0(\|x_p^n\|)|t_i^n - t_p^n|(1 + C_3 + M) \\ &\le \|x_{i-1}^n - x_p^n\| + h_nC_4|t_i^n - t_p^n| + C_4h_n,\end{aligned}$$

where $C_4 = \max\{2C_3, R_4\}$ with the bound $C_3 \ge |A(t_j^n, u_{t_j^n})x_p^n|$. If we add this inequality for $i = p+1, \cdots, j$, we have

$$\begin{aligned}\|x_j^n - x_p^n\| &\le C_4h_n(j-p) + C_4h_n\sum_{i=p+1}^{j}|t_i^n - t_p^n| \\ &\le C_4h_n(j-p) + C_4(j-p)^2{h_n}^2 \\ &= C_4|t_j^n - t_p^n| + C_4|t_j^n - t_p^n|^2 \\ &\le C_4D_{j,q}.\end{aligned}$$

For $p \le j \le n$ and $k = q$,

$$\begin{aligned}\|x_j^n - x_p^n\| &\le C_4(|t_j^n - t_p^n| + |t_j^n - t_j^n|^2) \\ &\le C_4D_{j,q},\end{aligned}$$

which yields

$$\begin{aligned}\|x_j^n - y_k^m\| &\le \|x_j^n - x_p^n\| + \|x_p^n - y_q^m\| \\ &\le \|x_p^n - y_q^m\| + C_4 D_{j,q}.\end{aligned}$$

Similarly, the above inequality also holds for $j = p$ and $q \le k \le m$. Next, let $p+1 \le j \le n$ and $q+1 \le k \le m$, and suppose that (2.3) holds for the pair $(j-1,k)$ and $(j,k-1)$. Since $R_4 \le C_4$, from (2.3),

$$\begin{aligned}
A_{j,k} &\le \frac{h_n}{h_n+\hat h_m}\Big\{\|x_p^n - y_q^m\| + C_4 D_{j,k-1} + \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k-1}\hat\delta_i^m \hat h_m \\
&\quad + j h_n\big[(\delta^{-1}\rho(T) + C_4)(D_{j,k-1} + |t_p^n - \hat t_q^m|) + C_4 h_n + \rho(2\delta) \\
&\quad + C_4\|u-v\|_\infty\big]\Big\} \\
&\quad + \frac{\hat h_m}{h_n+\hat h_m}\Big\{\|x_p^n - y_q^m\| + C_4 D_{j-1,k} + \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k-1}\hat\delta_i^m \hat h_m \\
&\quad + (j-1)h_n\big[(\delta^{-1}\rho(T) + C_4)(D_{j-1,k} + |t_p^n - \hat t_q^m|) + C_4 h_n + \rho(2\delta) \\
&\quad + C_4\|u-v\|_\infty\big]\Big\} \\
&\quad + \frac{h_n\hat h_m}{h_n+\hat h_m}\Big\{(\delta^{-1}\rho(T) + C_4)(D_{j-1,k} + |t_p^n - \hat t_q^m|) + C_4 h_n + \rho(2\delta) \\
&\quad + \delta_j^n + \hat\delta_k^m + C_4\|u-v\|_\infty\Big\} \\
&= \|x_p^n - y_q^m\| + C_4\Big(\frac{h_n}{h_n+\hat h_m} D_{j,k-1} + \frac{\hat h_m}{h_n+\hat h_m} D_{j-1,k}\Big) \\
&\quad + \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k}\hat\delta_i^m \hat h_m + j h_n\Big\{(\delta^{-1}\rho(T) + C_4)(D_{j,k} + |t_p^n - \hat t_q^m|) \\
&\quad + C_4 h_n + \rho(2\delta) + C_4\|u-v\|_\infty\Big\} \\
&\le \|x_p^n - y_q^m\| + C_4 D_{j,k} + \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k}\hat\delta_i^m \hat h_m \\
&\quad + j h_n\Big\{(\delta^{-1}\rho(T) + C_4)(D_{j,k} + |t_p^n - \hat t_q^m|) \\
&\quad + C_4 h_n + \rho(2\delta) + C_4\|u-v\|_\infty\Big\}
\end{aligned}$$

Here we have used

$$\frac{h_n}{h_n+\hat{h}_m}D_{j,k-1}+\frac{\hat{h}_m}{h_n+\hat{h}_m}D_{j-1,k}\le D_{j,k}.$$

Thus it turns out that (2.3) holds for the pair (j,k). Hence, we conclude that (2.3) holds for all $p\le j\le n$ and $q\le k\le m$.
Let $\tau\in(t_{p-1}^n,t_p^n]\cap(s_{q-1}^m,s_q^m]$ and $t\in(t_{j-1}^n,t_j^n]\cap(s_{k-1}^m,s_k^m]$. Letting $n,m\to\infty$ in (2.3),

$$\begin{aligned}(2.4)\qquad &\|x_u(t)-y_v(t)\|\\ &\le\|x_u(\tau)-y_v(\tau)\|+\limsup_{n\to\infty}\sum_{i=p}^{j}\delta_i^n h_n\\ &+\limsup_{m\to\infty}\sum_{i=q}^{k}\hat{\delta}_i^m\hat{h}_m+T\rho(2\delta)+C_4T\|u-v\|_\infty.\end{aligned}$$

Since

$$\lim_{n\to\infty}\sum_{i=p}^{j}\delta_i^n h_n=\int_\tau^t[x_u(\eta)-y_v(\eta),G(\eta,(x_u)_\eta)-G(\eta,(y_v)_\eta)]_\lambda d\eta$$

and $\lim_{m\to\infty}\sum_{i=q}^{k}\hat{\delta}_i^m\hat{h}_m=0$, letting $\delta\downarrow 0$ in (2.4)

$$\begin{aligned}\|x_u(t)-y_v(t)\|&\le\|x_u(\tau)-y_v(\tau)\|+C_4T\|u-v\|_\infty\\ &+\int_\tau^t[x_u(\eta)-y_v(\eta),G(\eta,(x_u)_\eta)-G(\eta,(y_v)_\eta)]_\lambda d\eta\end{aligned}$$

By letting $\lambda\downarrow 0$ for the above inequality, we finally have desired result.

THEOREM 3. *Let $\phi(0)\in\hat{D}$ and* (A.1)–(A.4) *hold. Then there exists $T_1\in(0,T]$ such that* (FDE:ϕ) *has a limit solution on* $[0,T_1]$.

Proof. Let $T_1\in(0,T]$ be sufficiently small so that $(C_4+k_1)T_1<1$ and $K\le M$, $\max\{k_0,2(C_1+C_2)\}<M$. Then by Theorem 2, for $t\in[0,T_1]$

$$
\begin{aligned}
\|x_u(t) - y_v(t)\| &\le C_4 T_1 \|u - v\|_\infty + \int_0^t \|G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)\| \, d\eta \\
&\le C_4 T_1 \|u - v\|_\infty + \int_0^t k_1 \|x_u - y_v\|_\infty \, d\eta \\
&\le C_4 T_1 \|u - v\|_\infty + k_1 T_1 \|x_u - y_v\|_\infty .
\end{aligned}
$$

Therefore, $\|x_u - y_v\|_\infty < \|u - v\|_\infty$. Moreover, the limit solution satisfies $\|x_u(t) - x_u(s)\| \le K|t - s| \le M|t - s|$ for $t, s \in [-r, T_1]$ and for $u \in E$. Therefore, $x_u \in E$ for all $u \in E$. If we define an operator $F : E \to E$ by $u \mapsto x_u$, where $x_u(t)$ is the limit solution of (EE:ϕ,u), then F is a strict contraction on a complete metric space E. By the Banach fixed point theorem, there is a unique fixed point of F in E, say $x(t)$ for $t \in [-r, T_1]$. Then, $x(t)$ is the limit solution of (EE:ϕ) which is Lipschitz continuous on $[-r, T_1]$.

Remark 1. It is obvious from the proof of the above theorems that the interval $[0, T]$ can be replaced by $[T_1, T]$. Then the solution $x(t)$ of (EE:ϕ) exists beyond T_1. With this processing, we may conclude that there exists a maximal interval of existence of solutions of (FDE:ϕ) on $[0, T]$.

Remark 2. Using the result of Theorem 2, we may have similar result of Ha, Shin and Jin [6] with the concept of integral solution defined by Benilan. It is quite interested in investigating the relation between two evolution operators generated by operators in (EE:ϕ) with different second terms. Also, for a just continuous perturbation $G(t, \cdot)$, we may apply the method in the paper of Kartsatos and Shin [11].

References

[1] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noorhoff Int. Publ., Leyden (The Netherlands), 1976.

[2] M. G. Crandall, *A generalized domain for semigroup generators*, Proc. Amer. Math. Soc. **37** (1973), 434–440.

[3] M. G. Crandall and A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel J. Math. **11** (1972), 57–94.

[4] L. C. Evans, *Nonlinear evolution equations in an arbitrary Banach space*, Israel J. Math. **26** (1977), 1–42.

[5] K. S. Ha and K. Shin, *Solutions of nonlinear functional differential equations in L^p spaces*, Comm. Kor. Math. Soc. **9** (1994), 303–315.

[6] K. S. Ha, K. Shin and B. J. Jin, *Existence of solutions of nonlinear functional integro-differential equations in Banach spaces*, Differential and Integral Eqs. **8**, 553–566.

[7] A. G. Kartsatos, *The existence of bounded solutions on the real line of perturbed nonlinear evolution equations in general Banach spaces*, Nonl. Anal. TMA **17** (1991), 1085–1092.

[8] A. G. Kartsatos and M. E. Parrott, *Existence of solutions and Galerkin approximations for nonlinear functional evolution equations*, Tohoku Math. J. **34** (1982), 509–523.

[9] A. G. Kartsatos and M. E. Parrott, *A method of lines for a nonlinear abstract functional evolution equation*, Trans. Amer. Math. Soc. **286** (1984), 73–89.

[10] A. G. Kartsatos and M. E. Parrott, *The weak solution of functional differential equation in a general Banach space*, J. Diff. Eqs. **75** (1988), 290–302.

[11] A. G. Kartsatos and K. Shin, *Solvability of functional evolutions via compactness methods in general Banach spaces*, Nonl. Anal. TMA **21** (1993), 517–536.

[12] N. Tanaka, *On the existence of solutions for functional evolution equations*, Nonl. Anal. TMA **12** (1988), 1087–1104.

[13] K. Yosida, *Functional analysis* 2nd Ed., Springer-Verlag, Berlin and New York, 1968.

Department of Mathematics
Pusan National University
Pusan 609–735, Korea
E-mail: kyshin@hyowon.cc.pusan.ac.kr