J. Korean Math. Soc. **34** (1997), No. 2, pp. 483–500

ON THE SPECTRAL GEOMETRY FOR THE JACOBI OPERATORS OF HARMONIC MAPS INTO PRODUCT MANIFOLDS

Tae Ho Kang, U-Hang Ki and Jin Suk Pak

Abstract. We investigate the geometric properties reflected by the spectra of the Jacobi operator of a harmonic map when the target manifold is a Riemannian product manifold or a Kaehlerian product manifold. And also we study the spectral characterization of Riemannian sumersions when the target manifold is $S^n \times S^n$ or $CP^n \times CP^n$.

1. Introduction

The inverse eigenvalue problem of the second order operators arising in Riemannian geometry has been studied by many authors. Among them, the Jacobi operator for a harmonic map was studied in [7,8,9], and for the functional area was studied in [1,4,6]. The Jacobi operator of a harmonic map f arises in the second variation formula of the energy of the harmonic map f. This formula can be expressed in terms of an elliptic differential operator J_f(called the *Jacobi operator*) defined on the space of cross sections of the induced bundle of the target manifold.

The examples of harmonic maps include harmonic functions, geodesics, isometric minimal immersions and holomorphic or anti-holomorphic maps of Kaehler manifolds etc..

In this paper we characterize both F-invariant and F-anti-invariant (resp. Kaehlerian and totally real) immersions into the Riemannian product manifolds(resp. the Kaehlerian product manifolds) by the

Received February 25, 1997.

1991 Mathematics Subject Classification: 58G25.

Key words and phrases: Spectral geometry, Jacobi operators, harmonic maps, Kaehlerian product manifolds, harmonic morphisms.

The Present Studies were Supported (in part) by the Basic Science Research Institute Program, Ministry of Education, (1996) Project No. BSRI-96-1404 and TGRC-KOSEF.

spectra of Jacobi operators. And also we study the spectral characterization of Riemannian submersions when the target manifold is $S^n \times S^n$ or $CP^n \times CP^n$.

2. Preliminaries

Let (M, g) be an m-dimensional connected, closed(i.e., compact without boundary) Riemannian manifold with the metric g and (N, h) an n-dimensional Riemannian manifold with the metric h. A smooth map $f : (M, g) \longrightarrow (N, h)$ is said to be *harmonic* if it is a critical point of the energy

$$E(f) := \int_M e(f)\, dv_g,$$

where the energy density $e(f)$ of f is defined to be $e(f) := \frac{1}{2}\sum_{i=1}^m h(f_* e_i, f_* e_i)$, f_* is the differential of f, $\{e_1, \cdots, e_m\}$ a local orthonormal frame field on M, and dv_g is the volume element with respect to g. Let us consider the Jacobi operator J_f for a harmonic map f defined by

$$J_f V := \tilde{\Delta}_f V - \mathcal{R}_f V$$

for $V \in \Gamma(E)$(the space of smooth sections of E), where $\tilde{\Delta}$ is the rough Laplacian associated to the induced connection $\tilde{\nabla}$ of the induced bundle $E := f^*TN$ defined by $\tilde{\nabla}_X V := \nabla^h_{f_*X} V$ (for X a tangent vector of M, ∇^h the Levi-Civita connection of the metric h), and $\mathcal{R}_f V := \sum_{i=1}^m R_h(V, f_*e_i) f_*e_i$($R_h$ is the Riemannian curvature tensor of (N, h)). In this paper, we take the convention

$$R_h(\tilde{X}, \tilde{Y}) := [\nabla^h_{\tilde{X}}, \nabla^h_{\tilde{Y}}] - \nabla^h_{[\tilde{X}, \tilde{Y}]},$$

where $\tilde{X}$ and $\tilde{Y}$ are tangent vector fields on N. Then J_f is self-adjoint, elliptic of second order and has a discrete spectrum as a consequence of the compactness of M.

Consider the semigroup e^{-tJ_f} given by

$$e^{-tJ_f} V(x) = \int_M K(t, x, y, J_f) V(y)\, dv_g(y),$$

where $K(t,x,y,J_f) \in \mathrm{Hom}(E_y, E_x)$ is the kernel function ($x, y \in M$, E_x is the fibre of E over x). Then we have asymptotic expansions for the L^2-trace

$$(2.1) \qquad Tr(e^{-tJ_f}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n(J_f) \quad (t \downarrow 0^+),$$

where each $a_n(J_f)$ is the spectral invariant of J_f, which depends only on the discrete spectrum ;

$$Spec(J_f) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots \uparrow +\infty\}.$$

Applying the Jacobi operator J_f of a harmonic map f to the Gilkey's results in [3,p.327], we obtain

THEOREM 2.1 [cf.9]. *For a harmonic map* $f : (M, g) \longrightarrow (N, h)$

$$(2.2) \qquad a_0(J_f) = n\,Vol(M, g),$$

$$(2.3) \qquad a_1(J_f) = \frac{n}{6}\int_M \tau_g\, dv_g + \int_M Tr(\mathcal{R}_f)\, dv_g,$$

(2.4)

$$a_2(J_f) = \frac{n}{360}\int_M \left[5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2\right] dv_g + \frac{1}{360}\int_M \left[-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_g Tr(\mathcal{R}_f) + 180 Tr({\mathcal{R}_f}^2)\right] dv_g,$$

where $R^{\tilde{\nabla}}$ *is the curvature tensor of the connection* $\tilde{\nabla}$ *on* E*,which is defined by* $R^{\tilde{\nabla}} := f^* R_h$*, and* R_g, ρ_g, τ_g *are the curvature tensor, Ricci tensor, scalar curvature on* M*, respectively.*

REMARK. $\dim(M) = m$ is determined by $Spec(J_f)$ (i.e., dim (M) is a spectral invariant of J_f) because of the asymptotic expansion (2.1).

3. The spectral geometry for J_f of a harmonic map f into a Riemannian product manifold

To begin with we define an almost product manifold. Let N be an n-dimensional manifold with a tensor F of type (1,1) such that

$$F^2 = I,$$

where I denotes the identity transformation. Then we say that N is an *almost product manifold* with *almost product structure* F. If an almost product manifold N admits a Riemannian metric h such that

$$h(F\tilde{X}, F\tilde{Y}) = h(\tilde{X}, \tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on N, then N is called to be an *almost product Riemannian manifold.*

Let N^{n_1} and N^{n_2} be n_1 and n_2-dimensional Riemannian manifolds with Riemannian metrics h_1 and h_2 respectively. Now we suppose that N^{n_1} and N^{n_2} are real space forms with constant sectional curvatures c_1 and c_2 and denote them by $N^{n_1}(c_1)$ and $N^{n_2}(c_2)$ respectively. Then the Riemannian curvature tensor R_1 of $N^{n_1}(c_1)$ is given by

$$R_1(X,Y)Z = c_1\{h_1(Y,Z)X - h_1(X,Z)Y\}$$

and the Riemannian curvature tensor R_2 of $N^{n_2}(c_2)$ is given by

$$R_2(X,Y)Z = c_2\{h_2(Y,Z)X - h_2(X,Z)Y\},$$

where X, Y and Z are vector fields tangent to N^{n_1} or N^{n_2}. We consider the Riemannian product manifold $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$. We denote by P and Q the projection operators of tangent space of N to the tangent space of N^{n_1} and N^{n_2} respectively. Then we have

$$P^2 = P, Q^2 = Q, PQ = 0 = QP.$$

Putting $F = P - Q$, we get $F^2 = I$. Thus F is an almost product structure on N. Moreover for any tangent vector fields $\tilde{X}, \tilde{Y}$ on N, we define a Riemannian metric h on N by

$$h(\tilde{X}, \tilde{Y}) = h_1(P\tilde{X}, P\tilde{Y}) + h_2(Q\tilde{X}, Q\tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ of N. It is clear that

$$h(F\tilde{X},\tilde{Y}) = h(F\tilde{Y},\tilde{X}).$$

The Riemannian curvature tensor R_h of the Riemannian product manifold $N{=}N^{n_1}(c_1) \times N^{n_2}(c_2)$ $(n_1, n_2 \geq 2)$ is given by

$$\begin{aligned}R_h(\tilde{X},\tilde{Y})\tilde{Z} = &\alpha\{h(\tilde{Y},\tilde{Z})\tilde{X} - h(\tilde{X},\tilde{Z})\tilde{Y} + h(F\tilde{Y},\tilde{Z})F\tilde{X} - h(F\tilde{X},\tilde{Z})F\tilde{Y}\} \\ &+ \beta\{h(\tilde{Y},\tilde{Z})F\tilde{X} - h(\tilde{X},\tilde{Z})F\tilde{Y} + h(F\tilde{Y},\tilde{Z})\tilde{X} - h(F\tilde{X},\tilde{Z})\tilde{Y}\}\end{aligned} \tag{3.1}$$

for any vector fields $\tilde{X},\tilde{Y}$ and $\tilde{Z}$ on N, where $\alpha = \frac{1}{4}(c_1 + c_2)$ and $\beta = \frac{1}{4}(c_1 - c_2)$(cf.[10]).

Define a symmetric 2-form Ω on N by $\Omega(\tilde{X},\tilde{Y}) = h(\tilde{X},F\tilde{Y})$. Then for a harmonic map $f : (M,g) \longrightarrow (N,h)$ we obtain from (3.1)

$$\begin{aligned}Tr(\mathcal{R}_f) = &2(n-2)\alpha e(f) + \alpha(Tr_h F)(Tr_g f^*\Omega) \\ &+ (n-2)\beta Tr_g f^*\Omega + 2\beta e(f)(Tr_h F),\end{aligned} \tag{3.2}$$

(3.3)

$$\begin{aligned}Tr(\mathcal{R}_f^2) = &\sum_{i,j=1}^{m}\sum_{a=1}^{4n} h(R_h(f_*e_i,v_a)f_*e_i, R_h(f_*e_j,v_a)f_*e_j) \\ &= \alpha^2\big[2\|f^*h\|^2 + 4(n-4)e(f)^2 + 2\|f^*\Omega\|^2 + (n-4)(Tr_g f^*\Omega)^2 \\ &+ 4e(f)(Tr_g f^*\Omega)(Tr_h F)\big] \\ &+ 2\alpha\beta\big[4\sum_{i,j=1}^{m} f^*(\Omega \boxtimes h)(e_i,e_j) + (4n-16)e(f)(Tr_g f^*\Omega) \\ &+ 4e(f)^2(Tr_h F) + (Tr_h F)(Tr_g f^*\Omega)^2\big] \\ &+ \beta^2\big[2\|f^*h\|^2 + 4(n-4)e(f)^2 + 2\|f^*\Omega\|^2 + (n-4)(Tr_g f^*\Omega)^2 \\ &+ 2e(f)(Tr_g f^*\Omega)(Tr_h F)\big]\end{aligned}$$

(3.4)

$$\begin{aligned}\|R^{\tilde{\nabla}}\|^2 = &\sum_{i,j=1}^{m}\sum_{a,b=1}^{n} h(R_h(f_*e_i,f_*e_j)v_a,v_b)h(R_h(f_*e_i,f_*e_j)v_a,v_b) \\ &= \alpha^2\big[16e(f)^2 - 4\|f^*h\|^2 + 4(Tr_g f^*\Omega)^2 - 4\|f^*\Omega\|^2\big]\end{aligned}$$

$$+ 2\alpha\beta\big[16e(f)(Tr_g f^*\Omega) - 8\sum_{i,j=1}^{m} f^*(\Omega \boxtimes h)(e_i, e_j)\big]$$
$$+ \beta^2\big[16e(f)^2 - 4\|f^*h\|^2 + 4(Tr_g f^*\Omega)^2 - 4\|f^*\Omega\|^2\big],$$

where $\|f^*h\|^2 = \sum_{i,j=1}^m h(f_*e_i, f_*e_j)^2$, $\|f^*\Omega\|^2 = \sum_{i,j=1}^m h(f_*e_i, Ff_*e_j)^2$, $\sum_{i,j=1}^m f^*(\Omega \boxtimes h)(e_i, e_j) = \sum_{i,j=1}^m \Omega(f_*e_i, f_*e_j)h(f_*e_i, f_*e_j)$, $\{v_a : a = 1, \cdots, n\}$ is a local orthonormal frame field on N, and $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M.

Thus substituting (3.2) $\sim$ (3.4) into Theorem 2.1, we get

THEOREM 3.1. *Let $f : (M, g) \longrightarrow N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ be a harmonic map of an m-dimensional compact Riemannian manifold (M, g) into an $n(= n_1 + n_2)$-dimensional Riemannian product manifold N. Then the coefficients $a_0(J_f)$, $a_1(J_f)$ and $a_2(J_f)$ of the asymptotic expansion for the Jacobi operator J_f are respectively given by*

$$a_0(J_f) = nVol(M, g), \tag{3.5}$$

$$\begin{aligned} a_1(J_f) = \frac{n}{6}\int_M \tau_g dv_g &+ \int_M \big[2(n-2)\alpha e(f) + \alpha(Tr_h F)(Tr_g f^*\Omega) \\ &+ (n-2)\beta(Tr_g f^*\Omega) + 2\beta e(f)(Tr_h F)\big]\, dv_g \end{aligned} \tag{3.6}$$

(3.7)
$$\begin{aligned} a_2(J_f) = {} & \frac{n}{360}\int_M [5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2] dv_g \\ &+ \frac{1}{12}\int_M \big[(24n - 112)(\alpha^2 + \beta^2)e(f)^2 + 16(\alpha^2 + \beta^2)\|f^*h\|^2 \\ &+ (6n - 28)(\alpha^2 + \beta^2)(Tr_g f^*\Omega)^2 + 16(\alpha^2 + \beta^2)\|f^*\Omega\|^2 \\ &+ 16(3n - 14)\alpha\beta e(f)(Tr_g f^*\Omega) + 12\alpha\beta(Tr_h F)(Tr_g f^*\Omega)^2 \\ &+ 12(2\alpha^2 + \beta^2)e(f)(Tr_g f^*\Omega)(Tr_h F) + 48\alpha\beta e(f)^2(Tr_h F) \\ &+ 64\alpha\beta\sum_{i,j} f^*(\Omega \boxtimes h)(e_i, e_j)\big]\, dv_g \end{aligned}$$

$$+\frac{1}{6}\int_M \big[2(n-2)\alpha e(f) + \alpha(Tr_h F)(Tr_g f^*\Omega) + (n-2)\beta(Tr_g f^*\Omega) + 2\beta e(f)(Tr_h F)\big]\tau_g\, dv_g$$

COROLLARY 3.2. *Let f, f' be harmonic maps of (M, g) with constant scalar curvature into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 = n_2$ and $c_1 = c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then we have*
(i) $E(f) = E(f')$.
(ii) $\int_M \big[(24n-112)e(f)^2 + 16\|f^*\|^2$
$(6n-28)(Tr_g f^*\Omega)^2 + 16\|f^*\Omega\|^2\big]\, dv_g$
$= \int_M \big[(24n-112)e(f')^2 + 16\|f'^*\|^2$
$(6n-28)(Tr_g f'^*\Omega)^2 + 16\|f'^*\Omega\|^2\big]\, dv_g.$

Proof. It is clear from our assumptions that $Tr_h F = 0$ and $\beta = 0$. Then (i) and (ii) follow from (3.6) and (3.7) respectively. □

COROLLARY 3.3. *Let (M, g) be a compact Riemannian manifold whose scalar curvature is constant. Let f, f' be isometric minimal immersions of (M, g) into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 \neq n_2$ or $c_1 \neq c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then we have*

$$\int_M Tr_g f^*\Omega\, dv_g = \int_M Tr_g f'^*\Omega\, dv_g, \tag{3.8}$$

$$\begin{aligned}
&\int_M \big[\{(6n-28)(\alpha^2+\beta^2) + 12\alpha\beta\}(Tr_g f^*\Omega)^2 \\
&\quad + 16(\alpha^2+\beta^2)\|f^*\Omega\|^2\big]\, dv_g \\
&= \int_M \big[\{(6n-28)(\alpha^2+\beta^2) + 12\alpha\beta\}(Tr_g f'^*\Omega)^2 \\
&\quad + 16(\alpha^2+\beta^2)\|f'^*\Omega\|^2\big]\, dv_g.
\end{aligned} \tag{3.9}$$

Proof. Note that $e(f) = \frac{m}{2}$, $Tr_h F =$ constant and $\sum_{i,j} f^*(\Omega \boxtimes h)(e_i, e_j) = Tr_g f^*\Omega$. Then (3.8) and (3.9) follow from (3.6) and (3.7) respectively. □

Let N be an almost product Riemannian manifold with almost product structure F. Let $f : M \longrightarrow N$ be an isometric immersion of a Riemannian manifold M into N. If $Ff_*(T_xM) \subset f_*(T_xM)$ ($Ff_*(T_xM) \subset f_*(T_xM)^\perp$ resp.) for each $x \in M$, then f is said to be an *F-invariant* (*F-anti-invariant* resp.) immersion.

LEMMA 3.4. *Let N be an almost product Riemannian manifold with almost product structure F. Let f be an isometric immersion of a compact Riemannian manifold (M, g) into (N, h). Then we have the inequality*

$$0 \leq \int_M \|f^*\Omega\|^2 \, dv_g \leq dim(M)\, Vol(M, g).$$

Moreover,

(i) *the equality $\int_M \|f^*\Omega\|^2 \, dv_g = 0$ holds if and only if the immersion f is F-anti-invariant,*

(ii) *the equality $\int_M \|f^*\Omega\|^2 \, dv_g = dim(M)\, Vol(M, g)$ holds if and only if the immersion f is F-invariant.*

Proof. The proof is similar to that of Lemma 6.5([9]). □

PROPOSITION 3.5. *Let (M, g) be a compact Riemannian manifold whose scalar curvature is constant. Let f, f' be isometric minimal immersions of (M, g) into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 > n_2, n = n_1 + n_2 \geq 5$ and $c_1 > c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is an F-anti-invariant immersion, then so is f'.*

Proof. If f is an F-anti-invariant immersion, then we have from (3.9)

$$\begin{aligned} 0 = \int_M \big[&\{(6n - 28)(\alpha^2 + \beta^2) + 12\alpha\beta\}(Tr_g f'^*\Omega)^2 \\ &+ 16(\alpha^2 + \beta^2)\|f'^*\Omega\|^2 \big] \, dv_g. \end{aligned}$$

Since the coefficients of $(Tr_g f'^*\Omega)^2$ and $\|f'^*\Omega\|^2$ are positive respectively, we get $\|f'^*\Omega\|^2 = 0$, which and Lemma 3.4 complete the proof. □

PROPOSITION 3.6. *Let (M,g) be a compact Riemannian manifold whose scalar curvature is constant. Let f, f' be isometric minimal immersions of (M,g) into $N = N^2(c_1) \times N^2(c_2)$ with $c_1 < c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is an F-invariant immersion, then so is f'.*

Proof. Assume that f is an F-invariant immersion. Then we have from Lemma 3.4 and (3.9)

(3.10)

$$\begin{aligned} 0 \leq & 16(\alpha^2+\beta^2)\{\dim(M) - \|f'^*\Omega\|^2\}\, dv_g \\ = & \{-4(\alpha^2+\beta^2)\} + 12\alpha\beta\} \int_M \{(Tr_g f'^*\Omega)^2 - (Tr_g f^*\Omega)^2\}\, dv_g. \end{aligned}$$

On the other hand, the restricted structure tensor $F|_{f_*TM}$ to $f(M)$ is also an almost product structure on $f(M)$ whose eigenvalues are $+1$ or -1. Hence $Tr_g f^*\Omega$ is constant on M, because it is continuous function on M. This fact and (3.10) yield the inequality

$$\int_M (Tr_g f'^*\Omega)^2\, dv_g \geq \int_M (Tr_g f^*\Omega)^2\, dv_g$$

because of (3.8) and the Cauchy-Schwarz inequality. Hence (3.10) imply that $\dim(M)\mathrm{Vol}(M,g) = \int_M \|f'^*\Omega\|^2\, dv_g$. Therefore Lemma 3.4 completes the proof. □

4. The spectral geometry for J_f of a harmonic map f into a Kaehlerian product manifold

Let N^{n_1} and N^{n_2} be n_1 and n_2-dimensional Kaeherian manifolds with almost complex structures J_1 and J_2 respectively. Now we suppose that N^{n_1} and N^{n_2} are complex space forms with constant holomorphic sectional curvatures c_1 and c_2 and denote them by $N^{n_1}(c_1)$ and $N^{n_2}(c_2)$ respectively. Then the Riemannian curvature tensor R_1 of $N^{n_1}(c_1)$ is given by

$$\begin{aligned} R_1(X,Y)Z = \frac{c_1}{4}\big[& h_1(Y,Z)X - h_1(X,Z)Y + h_1(J_1Y,Z)J_1X \\ & - h_1(J_1X,Z)J_1Y + 2h_1(X,J_1Y)J_1Z\big] \end{aligned}$$

and the Riemannian curvature tensor R_2 of $N^{n_2}(c_2)$ is given by

$$R_2(X,Y)Z = \frac{c_2}{4}\big[h_2(Y,Z)X - h_2(X,Z)Y + h_2(J_2Y,Z)J_2X - h_2(J_2X,Z)J_2Y + 2h_2(X,J_2Y)J_2Z\big]$$

where X, Y and Z are vector fields tangent to N^{n_1} or N^{n_2}. We consider the Kaehlerian product manifold $N{=}N^{n_1}(c_1) \times N^{n_2}(c_2)$. In the same way as in the case of Riemannian product, we will denote by P and Q the projection operators of tangent space of the Kaehlerian product manifold N to the tangent space of N^{n_1} and N^{n_2} respectively, and $F = P - Q$ an almost product structure on N. Let us put $J\tilde{X} = J_1P\tilde{X} + J_2Q\tilde{X}$ for any vector field $\tilde{X}$ on N. Then we see that

$$J_1P = PJ, \;\; J_2Q = QJ, \;\; FJ = JF,$$

$$J^2 = -I, h(J\tilde{X}, J\tilde{Y}) = h(\tilde{X}, \tilde{Y}), \;\; \nabla^h_{\tilde{X}} J = 0,$$

where h is the Riemannian metric on N defined by h_1 and h_2 as in the case of Riemannian product. Thus J is a Kaehlerian structure on N. The Riemannian curvature tensor R_h of a Kaehlerian product manifold $N{=}N^{n_1}(c_1) \times N^{n_2}(c_2)$ $(n_1, n_2 \geq 2)$ is given by

(4.1)

$$\begin{aligned}R_h(\tilde{X},\tilde{Y})\tilde{Z} = \alpha\big[&h(\tilde{Y},\tilde{Z})\tilde{X} - h(\tilde{X},\tilde{Z})\tilde{Y} + h(J\tilde{Y},\tilde{Z})J\tilde{X}\\ &- h(J\tilde{X},\tilde{Z})J\tilde{Y} + 2h(\tilde{X},J\tilde{Y})J\tilde{Z} + h(F\tilde{Y},\tilde{Z})F\tilde{X}\\ &- h(F\tilde{X},\tilde{Z})F\tilde{Y} + h(FJ\tilde{Y},\tilde{Z})FJ\tilde{X} - h(FJ\tilde{X},\tilde{Z})FJ\tilde{Y}\\ &+ 2h(F\tilde{X},J\tilde{Y})FJ\tilde{Z}\big]\\ + \beta\big[&h(F\tilde{Y},\tilde{Z})\tilde{X} - h(F\tilde{X},\tilde{Z})\tilde{Y} + h(\tilde{Y},\tilde{Z})F\tilde{X}\\ &- h(\tilde{X},\tilde{Z})F\tilde{Y} + h(FJ\tilde{Y},\tilde{Z})J\tilde{X} - h(FJ\tilde{X},\tilde{Z})J\tilde{Y}\\ &+ h(J\tilde{Y},\tilde{Z})FJ\tilde{X} - h(J\tilde{X},\tilde{Z})FJ\tilde{Y} + 2h(F\tilde{X},J\tilde{Y})J\tilde{Z}\\ &+ 2h(\tilde{X},J\tilde{Y})FJ\tilde{Z}\big]\end{aligned}$$

for any vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ on N, where $\alpha = \frac{1}{16}(c_1 + c_2)$ and $\beta = \frac{1}{16}(c_1 - c_2)$ (cf.[10]).

Put $\Omega(\tilde{X},\tilde{Y}) = h(\tilde{X},F\tilde{Y}), \theta(\tilde{X},\tilde{Y}) = h(\tilde{X},FJ\tilde{Y}), \omega(\tilde{X},\tilde{Y}) = h(\tilde{X},J\tilde{Y})$ and $\sum_{i,j} f^*(\theta \boxtimes \omega)(e_i,e_j) = \sum_{i,j} \theta(f_*e_i, f_*e_j)\omega(f_*e_i,f_*e_j)$. Then for a harmonic map $f:(M,g)\longrightarrow(N,h)$ we obtain from (4.1)

(4.2)
$$\begin{aligned}Tr(\mathcal{R}_f) &= (2n+8)\alpha e(f) + \alpha(Tr_hF)(Tr_g f^*\Omega)\\ &\quad + (n+4)\beta Tr_g f^*\Omega + 2\beta e(f)(Tr_hF),\end{aligned}$$

(4.3)
$$\begin{aligned}Tr(\mathcal{R}_f^2) &= \sum_{i,j=1}^{m}\sum_{a=1}^{4n} h(R_h(f_*e_i,v_a)f_*e_i, R_h(f_*e_j,v_a)f_*e_j)\\
&= \alpha^2\big[20\|f^*h\|^2 + 4(n+8)e(f)^2 - 12\|f^*\omega\|^2 + 20\|f^*\Omega\|^2\\
&+ (n+8)(Tr_gf^*\Omega)^2 - 12\|f^*\theta\|^2 + 4e(f)(Tr_gf^*\Omega)(Tr_hF)\big]\\
&+ 2\alpha\beta\big[40\sum_{i,j=1}^{m} f^*(\Omega\boxtimes h)(e_i,e_j) - 24\sum_{i,j=1}^{m} f^*(\theta\boxtimes\omega)(e_i,e_j)\\
&+ 4(n+8)e(f)(Tr_gf^*\Omega) + 4e(f)^2(Tr_hF) + (Tr_hF)(Tr_gf^*\Omega)^2\big]\\
&+ \beta^2\big[20\|f^*h\|^2 + 4(n+8)e(f)^2 + 20\|f^*\Omega\|^2 - 12\|f^*\theta\|^2\\
&+ (n+8)(Tr_gf^*\Omega)^2 - 12\|f^*\omega\|^2 + 4e(f)(Tr_gf^*\Omega)(Tr_hF)\big],\end{aligned}$$

(4.4)
$$\begin{aligned}\|R^{\tilde{\nabla}}\|^2 &= \sum_{i,j=1}^{m}\sum_{a,b=1}^{n} h(R_h(f_*e_i,f_*e_j)v_a,v_b)h(R_h(f_*e_i,f_*e_j)v_a,v_b)\\
&= \alpha^2\big[32e(f)^2 - 8\|f^*h\|^2 + 4(n+10)\|f^*\omega\|^2\\
&+ 8(Tr_gf^*\Omega)^2 - 8\|f^*\Omega\|^2 + 4(n+10)\|f^*\theta\|^2\\
&+ 8\sum_{i,j=1}^{m} f^*(\theta\boxtimes\omega)(e_i,e_j)(Tr_hF)\big]\\
&+ 2\alpha\beta\big[32e(f)(Tr_gf^*\Omega) + 4(Tr_hF)\|f^*\omega\|^2 + 4(Tr_hF)\|f^*\theta\|^2\\
&+ 8(n+10)\sum_{i,j=1}^{m} f^*(\theta\boxtimes\omega)(e_i,e_j) - 16\sum_{i,j=1}^{m} f^*(\Omega\boxtimes h)(e_i,e_j)\big]\\
&+ \beta^2\big[32e(f)^2 - 8\|f^*h\|^2 + 8(Tr_gf^*\Omega)^2\end{aligned}$$

$$- 8\|f^*\Omega\|^2 + 4(n+10)\|f^*\omega\|^2 + 4(n+10)\|f^*\theta\|^2 + 8(Tr_h F)\sum_{i,j=1}^{m} f^*(\theta \boxtimes \omega)(e_i, e_j)\Big],$$

where $\{v_a : a = 1, \cdots, n\}$ is a local orthonormal frame field on N and $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M.

Substituting (4.2) $\sim$ (4.4) into Theorem 2.1, we get

THEOREM 4.1. *Let $f : (M, g) \longrightarrow N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ be a harmonic map of an m-dimensional compact Riemannian manifold (M, g) into an $n(= n_1 + n_2)$-dimensional Kaehlerian product manifold N. Then the coefficients $a_0(J_f)$, $a_1(J_f)$ and $a_2(J_f)$ of the asymptotic expansion for the Jacobi operator J_f are respectively given by*

$$a_0(J_f) = nVol(M, g), \tag{4.5}$$

$$\begin{aligned} a_1(J_f) = {} & \frac{n}{6}\int_M \tau_g dv_g \\ & + \int_M \big[2(n+4)\alpha e(f) + \alpha(Tr_h F)(Tr_g f^*\Omega) \\ & + (n+4)\beta(Tr_g f^*\Omega) + 2\beta e(f)(Tr_h F)\big]\, dv_g, \end{aligned} \tag{4.6}$$

$$\begin{aligned} a_2(J_f) = {} & \frac{n}{360}\int_M [5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2] dv_g \\ & + \frac{1}{12}\int_M \big[8(3n+20)(\alpha^2+\beta^2)e(f)^2 \\ & + 128(\alpha^2+\beta^2)\|f^*h\|^2 + 128(\alpha^2+\beta^2)\|f^*\Omega\|^2 \\ & - 4(n+28)(\alpha^2+\beta^2)\|f^*\omega\|^2 \\ & + 2(3n+20)(\alpha^2+\beta^2)(Tr_g f^*\Omega)^2 \\ & - 4(n+28)(\alpha^2+\beta^2)\|f^*\theta\|^2 \\ & - 8(\alpha^2+\beta^2)\sum_{i,j} f^*(\theta \boxtimes \omega)(e_i, e_j)(Tr_h F) \end{aligned} \tag{4.7}$$

$$+ 16(3n+20)\alpha\beta e(f)(Tr_g f^*\Omega) + 512\alpha\beta \sum_{i,j} f^*(\Omega \boxtimes h)(e_i, e_j)$$
$$- 16(n+28)\alpha\beta \sum_{i,j} f^*(\theta \boxtimes \omega)(e_i, e_j) - 8\alpha\beta(Tr_h F)\|f^*\omega\|^2$$
$$- 8\alpha\beta(Tr_h F)\|f^*\theta\|^2 + 24(\alpha^2+\beta^2)e(f)(Tr_g f^*\Omega)(Tr_h F)$$
$$+ 48\alpha\beta e(f)^2(Tr_h F) + 12\alpha\beta(Tr_h F)(Tr_g f^*\Omega)^2\big]\, dv_g$$
$$+ \frac{1}{6}\int_M \big[2(n+4)\alpha e(f) + \alpha(Tr_h F)(Tr_g f^*\Omega)$$
$$+ (n+4)\beta(Tr_g f^*\Omega) + 2\beta e(f)(Tr_h F)\big]\tau_g\, dv_g.$$

COROLLARY 4.2. *Let f, f' be harmonic maps of (M,g) with constant scalar curvature into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 = n_2$ and $c_1 = c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then we obtain*
(i) $E(f) = E(f')$.
(ii) $\int_M \big[8(3n+20)e(f)^2 + 128\|f^*h\|^2 + 128\|f^*\Omega\|^2$
$-4(n+28)\|f^*\omega\|^2 + 2(3n+20)(Tr_g f^*\Omega)^2 - 4(n+28)\|f^*\theta\|^2\big]\, dv_g$
$= \int_M \big[8(3n+20)e(f')^2 + 128\|f'^*h\|^2 + 128\|f'^*\Omega\|^2$
$-4(n+28)\|f'^*\omega\|^2 + 2(3n+20)(Tr_g f'^*\Omega)^2 - 4(n+28)\|f'^*\theta\|^2\big]\, dv_g$

Proof. (i) and (ii) follow from (4.6) and (4.7) respectively. □

COROLLARY 4.3. *Let f, f' be isometric minimal immersions of (M,g) into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 = n_2$ and $c_1 = c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then we have*

(4.8)
$$\int_M \big[128\alpha^2\|f^*\Omega\|^2 - 4(n+28)\alpha^2\|f^*\omega\|^2$$
$$+ 2(3n+20)\alpha^2(Tr_g f^*\Omega)^2 - 4(n+28)\alpha^2\|f^*\theta\|^2\big]\, dv_g$$
$$= \int_M \big[128\alpha^2\|f'^*\Omega\|^2 - 4(n+28)\alpha^2\|f'^*\omega\|^2$$
$$+ 2(3n+20)\alpha^2(Tr_g f'^*\Omega)^2 - 4(n+28)\alpha^2\|f'^*\theta\|^2\big]\, dv_g.$$

Proof. Note that $\beta = 0 = Tr_h F$ and $e(f) = \frac{m}{2}$. Then (4.8) follows from (4.7). □

LEMMA 4.4 [7]. *Let (N, h, J) be a Hermitian manifold with the Kaehler form ω. Let f be an isometric immersion of a compact Riemannian manifold (M, g) into (N, h). Then we have the equality*

$$0 \leq \int_M \|f^*\omega\|^2 \, dv_g \leq dim(M) Vol(M, g).$$

Moreover,

(i) *the equality $\int_M \|f^*\omega\|^2 \, dv_g = 0$ holds if and only if the immersion f is totally real,*

(ii) *the equality $\int_M \|f^*\omega\|^2 \, dv_g = dim(M) Vol(M, g)$ holds if and only if the immersion f is Kaehlerian.*

PROPOSITION 4.5. *Let f, f' be F-invariant minimal immersions of (M, g) into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 = n_2$ and $c_1 = c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then*

(i) *if f is a totally real immersion, then so is f',*

(ii) *if f is a Kaehlerian immersion, then so is f'.*

Proof. It follows from (4.8) that

$$\int_M (\|f^*\omega\|^2 + \|f^*\theta\|^2) = \int_M (\|f'^*\omega\|^2 + \|f'^*\theta\|^2) \tag{4.9}$$

because that f and f' are F-invariant immersions. Assume that f is totally real immersion. Then we have $\|f^*\omega\| = 0 = \|f^*\theta\|$. It follows from (4.9) that $\|f'^*\omega\| = 0$. Then Lemma 4.4 implies that f' is a totally real immersion. Next, assume that f is a Kaehlerian immersion. Then we have $\|f^*\omega\|^2 = \dim(M) = \|f^*\theta\|^2$. From (4.9) and Lemma 4.4 we obtain $0 \leq \int_M [(m - \|f'^*\omega\|^2) + (m - \|f'^*\theta\|^2)] \, dv_g$. This means that $\|f'^*\omega\|^2 = m$ because that the two terms of the inequality are positive respectively. Hence Lemma 4.4 shows that f' is also a Kaehlerian immersion. □

PROPOSITION 4.6. *Let f, f' be F-anti-invariant minimal immersions of (M, g) into $N = N^{n_1}(c_1) \times N^{n_2}(c_2)$ with $n_1 = n_2$ and $c_1 = c_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then*

(i) *if f is a totally real immersion, then so is f',*

(ii) *if f is a Kaehlerian immersion, then so is f'.*

Proof. Since f and f' are F-anti-invariant immersions, $\|f^*\Omega\| = 0 = Tr_g f^*\Omega$ because of Lemma 3.4. From this and (4.8), we get the same equation (4.9). The other part of the proof is similar to that of Proposition 4.5. □

5. Harmonic morphisms and harmonic Riemannian submersions

First of all we introduce the notion of harmonic morphisms (for details, see [3,8]).

A smooth map $f : (M, g) \longrightarrow (N, h)$ is a *harmonic morphism* if $\nu \circ f$ is a harmonic function in $f^{-1}(V)$ for every function ν which is harmonic in an open set $V \subset N$ such that $f^{-1}(V) \neq \phi$.

A smooth map $f : (M, g) \longrightarrow (N, h)$ is *horizontally weakly conformal* if (i) $f_{*x} : T_x M \longrightarrow T_{f(x)} N$ is surjective at each point x with $e(f)(x) \neq 0$, (ii) there exists a smooth function λ on M such that for each $x \in M$ with $e(f)(x) \neq 0$, $f^*h(X, Y) = \lambda^2(x) g(X, Y)$ for $X, Y \in H_x$, where H_x is the orthogonal complement of $\mathrm{Ker} f_*$ with respect to $g_x, x \in M$.

The spectral charaterization of harmonic Riemannian submersions among the set of all harmonic morphisms when the target manifolds are the standard sphere, complex projective space ([9]) and the quaternionic projective space ([7]) has been studied, by using the following theorem.

THEOREM 5.1 [3,8]. (i) *if $dim(M) < dim(N)$, every harmonic morphism is constant.*

(ii) *If $dim(M) \geq dim(N)$, a smooth map $f : (M, g) \longrightarrow (N, h)$ is a harmonic morphism if and only if f is horizontally weakly conformal and harmonic.*

It is known (cf.[3]) that the set $M^* := \{x \in M : e(f)(x) \neq 0\}$ is open and dense in M and the function λ^2 is given by $\lambda^2 = 2e(f)\mathrm{dim}(N)^{-1}$, and $\|f^*h\|^2 = \dim(N)\lambda^4$. A smooth map $f : (M,g) \longrightarrow (N,h)$ is a *Riemannian submersion* if it is semi-conformal with $\lambda = 1$ on M.

THEOREM 5.2. *Let f, f' be harmonic morphisms of a compact Riemannian manifold (M,g) with constant scalar curvature into the $S^n \times S^n$ or $CP^n \times CP^n$, where S^n and CP^n denote the standard n-sphere with the canonical metric and the complex projective space with the Fubini Study metric, respectively. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is a Riemannian submersion, then so is f'.*

Proof. It is sufficient to show that the function λ^2 for f' satisfies $\lambda^2 = 1$ everywhere on M.

Case 1. $(N = S^n \times S^n, h)$. In this case, $e(f') = n\lambda^2$, $\|f'^*h\|^2 = 2n\lambda^4$, $e(f) = n$ and $\|f^*h\|^2 = 2n$.

Now we show that if f is a harmonic morphism of (M,g) into $(N = S^n \times S^n, h)$, then

$$\|Tr_g f^*\Omega\|^2 = \lambda^4 (Tr\tilde{F})^2,$$

$$\|f^*\Omega\|^2 = \|f^*h\|^2 \text{ on } M^*.$$

In fact, at each point $x \in M^*$, we can define a linear transformation $\tilde{F}$ of H_x into itself such that $F \circ f_* = f_* \circ \tilde{F}$. Then

$$\tilde{F}^2 = I, g(\tilde{F}X, \tilde{F}Y) = g(X,Y), \ \ X, Y \in H_x.$$

Taking an orthonormal basis $\{e_a; a = 1, ..., 2n\}$ of (H_x, g_x), we obtain

$$\begin{aligned}(Tr_g f^*\Omega)^2 &= \Big[\sum_{a=1}^{2n} h(f_*e_a, Ff_*e_a)\Big]^2 = \Big[\sum_{a=1}^{2n} h(f_*e_a, f_*\tilde{F}e_a)\Big]^2 \\ &= \Big[\sum_{a=1}^{2n} \lambda^2 g(e_a, \tilde{F}e_a)\Big]^2 =: \lambda^4 (Tr\tilde{F})^2\end{aligned}$$

and

$$\begin{aligned}\|f^*\Omega\|^2 &= \sum_{a,b=1}^{2n} h(f_*e_a, Ff_*e_b)^2 = \sum_{a,b=1}^{2n} h(f_*e_a, f_*\tilde{F}e_b)^2 \\ &= 2n\lambda^4 = \|f^*h\|^2,\end{aligned}$$

where $Tr\tilde{F}$ is constant on M^*.

If f' is a harmonic morphism, then we can also define a linear transformation $\tilde{F}'$ of $H_x{}'$ into itself such that $F \circ f'_* = f'_* \circ \tilde{F}'$, where $H_x{}'$ is the orthogonal complement of $\mathrm{Ker} f'_*$ with respect to $g_x, x \in M$. Note that $Tr\tilde{F}' = Tr\tilde{F} = $ const. on M^*.

Now let f, f' be harmonic morphisms of (M, g) into $(S^n \times S^n, h)$ with $Spec(J_f) = Spec(J_{f'})$. Then, by Corollary 3.2, we have

(1) $E(f) = E(f')$

and

(2) $\int_M \{(24n - 112)e(f)^2 + 32\|f^*h\|^2 + (6n - 28)(Tr_g f^*\Omega)^2\}\, dv_g$
$= \int_M \{(24n - 112)e(f')^2 + 32\|f'^*h\|^2 + (6n - 28)(Tr_g f'^*\Omega)^2\}\, dv_g.$

If f is a Riemannian submersion, then (1) is equivalent to $\int_M \lambda^2\, dv_g = \int_M dv_g$, and (2) is equivalent to $\int_M \lambda^4\, dv_g = \int_M dv_g$. Therefore we get $\lambda^2 = 1$ everywhere on M by the Cauchy-Schwarz inequality.

Case 2. $N = (CP^n \times CP^n, h)$. At each point $x \in M^*$, we define a linear transformation $\tilde{J}$ of H_x into itself such that $J \circ f_* = f_* \circ \tilde{J}$ and $\tilde{J}^2 = -I$, where J is the complex structure of $N = (CP^n \times CP^n, h)$. Then $g(\tilde{J}X, \tilde{J}Y) = g(X, Y)$ and $g(\tilde{J}X, X) = 0$, $X, Y \in H_x$. Taking $\{e_a, \tilde{J}e_a; a = 1, ..., n\}$ as an orthonormal basis of (H_x, g_x), then we can obtain

$$\|f^*\theta\|^2 = \|f^*\omega\|^2 = \|f^*h\|^2 = \|f^*\Omega\|^2 = 2n\lambda^4,$$

where n is of complex dimension. Now let f and f' be harmonic morphisms (M, g) into $N = (CP^n \times CP^n, h)$ with $Spec(J_f) = Spec(J_{f'})$. Then Corollary 4.2 implies that

(3) $E(f) = E(f')$

and

(4) $\int_M \{8(3n + 20)e(f)^2 + 8(4 - n)\|f^*h\|^2 + 2(3n + 20)(Tr_g f^*\Omega)^2\}\, dv_g$
$= \int_M \{8(3n + 20)e(f')^2 + 8(4 - n)\|f'^*h\|^2 + 2(3n + 20)(Tr_g f'^*\Omega)^2\}\, dv_g.$

Then by a similar argument to that in Case 1, we complete the proof. □

References

[1] H. Donnelly, *Spectral invariants of the second variation operator*, Illinois J. Math. **21** (1977), 185–189.

[2] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier(Grenoble) **28** (1978), 107–144.

[3] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Publish or Perish, 1984.

[4] T. Hasegawa, *Spectral Geometry of closed minimal submanifolds in a space form, real and complex*, Kodai Math. J. **3** (1980), 224–252.

[5] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. **19** (1979), 215–229.

[6] T. H. Kang and H. S. Kim, *On the spectral geometry of closed minimal submanifolds in a Sasakian or cosympletic manifold with constant ϕ-sectional curvature*, Nihonkai Math. J. **6(1)** (1995), 43–61.

[7] T. H. Kang and J. S. Pak, *On the spectral geometry for the Jacobi operators of harmonic maps into a quaternionic projective space*, Geometriae Dedicata **60** (1996), 153–161.

[8] S. Nishikawa, P. Tondeur and L. Vanhecke, *Spectral Geometry for Riemannian Foliations*, Annals of Global Analysis and Geometry **10** (1992), 291–304.

[9] H. Urakawa, *Spectral Geometry of the second variation operator of harmonic maps*, Illinois J. Math. **33(2)** (1989), 250-267.

[10] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Math. Vol 3, World Scientific, Singapore, 1984.

Tae Ho Kang
Department of Mathematics
University of Ulsan
Ulsan, 680–749, KOREA

U-Hang Ki and Jin Suk Pak
Department of Mathematics
Kyungpook National University
Taegu, 702-701, KOREA