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LINEAR PROGRAMMING APPROACH IN COOPERATIVE GAMES

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ABSTRACT. In this paper we consider TU-cooperative games in the form of characteristic function. We notice that if one uses the necessary and sufficient condition for the core to be not empty in a dual form, it may be used for selecting the final outcome in the core. Using the linear programming approach for constructing the subcore, which is a subset of the core, we represent it in a simple form. We consider reduced games due to Davis–Mashler, Moulin and Funaki and formulate the sufficient conditions for the subcore to be $\mathcal{S}$-consistent.

1. Introduction

In the framework of classical cooperative game theory with transferable utilities(TU-cooperative game), many optimality concepts were introduced and treated, concluding the famous concept of the *core*.

The core proposed by Gillies [6] is a generalization of the *contract curve* of Edgeworth [4]. Edgeworth described a market in which there are two consumers and two commodities. The core in this game is defined as the part of Pareto optimal curve which lies in the area of those allocation for which the utility of each consumer is at least as great as he would receive if he consumed his initial endowment.

The well-known von Neumann-Morgenstern(VNM) solution is related to the core but is less satisfactory concept. It is described by von Neumann and Morgenstern [13] and Luce and Raifa [7]. One can say that the core is a generalized VNM. The unsatisfactory element of

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VNM solution is that some payoff vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ may be in a solution, yet be dominated by something not in the solution (but attainable).

Consider any coalition $\mathbf{S}$ which is a subset of the complete set of players $\mathbf{N} = \{1, 2, \cdots, n\}$. We regard the characteristic function $v(s)$ as a function defined on various coalitions $\mathbf{S}$ which gives a *security* level for these coalitions including those consisting of single players and the grand coalition $\mathbf{N}$. The system $(\mathbf{N}, v)$ is called a TU-cooperative game in characteristic function form.

Let $\mathbf{R}$ and $\mathbf{S}$ be any coalitions that are disjoint and assume the following inequality holds:

$$v(\mathbf{R} \cup \mathbf{S}) \geq v(\mathbf{R}) + v(\mathbf{S}).$$

This says that the function v is *superadditive.* Scarf [10] showed that the core is not empty for a class of convex superadditive games in characteristic function form. Generalization of some of Scarf's results may be found in Billera [1] and Shapley [12]. The necessary and sufficient conditions for the core to be not empty were formulated by Bondareva [2] and Shapley [11]. A notion which plays a key role in the proofs is that of a *balanced collection of coalitions* and the related *balanced game.* Unfortunately, these conditions do not yield the method to choose selectors of the core.

In this paper we treated the concept of *subcore* introduced by Zakharov [14]. The concept of subcore uses an optimal solution of linear programming problem which we shall consider in the next section as a some kind *status quo* point in the game. One of the most important advantage of the linear programming approach is that the *subcore* may be represented in a simple analytical form. In addition, we can easily check the necessary and sufficient conditions for all one point solutions like *Shapley value*, *Banzaf value*, *egalitarian nonseparable contribution value(ENSC value)*, etc., to be selectors of the subcore.

We also discuss in this paper the problem of consistency of the subcore in reduced games due to Davis-Mashler [3], Moulin [8] and Funaki [5], and stand conditions for the subcore to be *consistent.* A general consistency property of a solution is described as follows. Choose any payoff vector in a solution set for some game. Suppose that some coalitions $\mathbf{S}$ want to renegotiate payoff distribution among their members coalitions $\mathbf{S}$ want to renegotiate payoff distribution among their members and may cooperate with other members outside $\mathbf{S}$ paying for them agreed payoff. Such a process is described by a *reduced game*. The solution is called *consistent* if it recommends the same payoff distribution for the reduced game as initially. A consistent solution provides a stability of an agreement against wishes of any coalitions to renegotiate agreed payoff distribution.

2. Existence of the core and subcore

Consider a normal form game

$$\Gamma = \langle \mathbf{N},\ \{h_i\}_{i\in\mathbf{N}},\ \{U_i\}_{i\in\mathbf{N}}\rangle$$

where $\mathbf{N} = \{1, 2, \cdots, n\}$ is the set of players, $h_i(u_1, u_2, \cdots, u_n)$ a nonnegative payoff of the i-th player and U_i a set of admissible strategies of player i.

Assume that utility of any player is transferable. For any coalition $\mathbf{S} \subset \mathbf{N}$ one may be define the characteristic function

$$v(\mathbf{S}) = \max_{u_\mathbf{S}} \min_{u_{\mathbf{N}\setminus\mathbf{S}}} \sum_{i\in\mathbf{S}} h_i(u_\mathbf{S},\ u_{\mathbf{N}\setminus\mathbf{S}}),$$

where $u_\mathbf{S}$ and $u_{\mathbf{N}\setminus\mathbf{S}}$ are vectors of admissible strategies of coalitions $\mathbf{S}$ and $\mathbf{N}\setminus\mathbf{S}$, respectively.

It is well known [9] that the characteristic function $v(\mathbf{S})$ constructed in this way is superadditive: that is,

$$v(\mathbf{R}\cup\mathbf{S}) \geq v(\mathbf{R}) + v(\mathbf{S})$$

for any disjoint subsets $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{N}$.

DEFINITION 1. An n-dimensional vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ which satisfies the conditions

(1) $$\xi_i \geq v(\{i\}),\ \forall i \in \mathbf{N}$$

(2) $$\sum_{i\in\mathbf{N}} \xi_i = v(\mathbf{N}).$$

is called an *imputation*.

The analytical description of the core is provided by the following theorem.

THEOREM 1. *For the imputation $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ to belong to the core, it is necessary and sufficient that the inequality*

$$\xi(\mathbf{S}) = \sum_{i \in \mathbf{S}} \xi_i \geq v(\mathbf{S})$$

holds for all $\mathbf{S} \subset \mathbf{N}$.

Thus the *core* is the following set:

$$\begin{aligned} \mathbf{C}(v) =\{ &\xi = (\xi_1, \cdots, \xi_n) : \sum_{i \in \mathbf{S}} \xi_i \geq v(\mathbf{S}), \\ &\sum_{i \in \mathbf{N}} \xi_i = v(\mathbf{N}), \mathbf{S} \subset \mathbf{N}, \mathbf{S} \neq \mathbf{N}\}. \end{aligned}$$

Consider the linear programming problem

$$\text{(3)} \qquad \text{Minimize} \quad \sum_{i \in \mathbf{N}} \xi_i$$

$$\text{(4)} \qquad \text{subject to} \quad \sum_{i \in \mathbf{S}} \xi_i \geq v(\mathbf{S}), \ \mathbf{S} \subset \mathbf{N}, \ \mathbf{S} \neq \mathbf{N}.$$

Let $\xi^0 = (\xi_1^0, \xi_2^0, \cdots, \xi_n^0)$ be an optimal solution and $X^0(v)$ be a set of all optimal solutions of the problem (3) and (4). As shown in [14], a necessary and sufficient condition for the existence of the core may be considered in the form provided by the following theorem.

THEOREM 2. *The core of TU-cooperative game $(\mathbf{N}, v)$ is not empty if and only if the following inequality holds:*

$$\text{(5)} \qquad \sum_{i \in \mathbf{N}} \xi_i^0 \leq v(\mathbf{N}),$$

where ξ^0 is an arbitrary solution from the set $X^0(v)$.

We now introduce the subcore of TU-cooperative game $(\mathbf{N}, v)$.

DEFINITION 2. We call a set

$$\mathbf{SC}(v,\,\xi^0) = \{\xi = (\xi_1,\,\cdots,\,\xi_n) : \xi_i \geq \xi_i^0,\ \sum_{i\in\mathbf{N}} \xi_i = v(\mathbf{N})\}$$

$$= \{\xi : \xi = \xi^0 + \frac{v(\mathbf{N}) - \sum_{i=1}^{n} \xi_i^0}{n} \cdot I,\ I = (1,\,\cdots,\,1) \in \mathbf{R}^n,$$

$$\sum_{i=1}^{n} \xi_i^0 \leq v(\mathbf{N})\}$$

the *subcore* of cooperative game $(\mathbf{N},\,v)$ with respect to ξ^0.

The importance of the notion of the subcore consists of the fact provided by the following theorem [14].

THEOREM 3. *The core* $\mathbf{C}(v)$ *of the cooperative game* $(\mathbf{N},\,v)$ *is empty if and only if the subcore* $\mathbf{SC}(v,\,\xi^0)$ *is empty for arbitrary* $\xi^0 \in X^0(v)$.

One can also notice that $\mathbf{SC}(v,\,\xi^0) \subset \mathbf{C}(v)$ for all $\xi^0 \in X^0(v)$ if $\mathbf{C}(v) \neq \emptyset$.

3. Balanced games

Next, we study the *balancedness* of a game, which had introduced by Bondareva(1963) and Shapley(1967).

DEFINITION 3. Let $\mathcal{W}$ be an arbitrary collection of coalitions for n-person game. $\mathcal{W}$ is called a *balanced collection* of coalitions if it is possible to find a set of weights $\delta_\mathbf{S} > 0$ for some $\mathbf{S} \in \mathcal{W}$ such that

$$\sum_{\substack{\mathbf{S}\in\mathcal{W}\\ \mathbf{S}\ni i}} \delta_\mathbf{S} = 1,\ i = 1,\,2,\,\cdots,\,n.$$

A TU-cooperative game $(\mathbf{N},\,v)$ is called *balanced* if, for every balanced collection $\mathcal{W}$ of coalitions, the following inequality holds:

$$\sum_{\mathbf{S}\in\mathcal{W}} \delta_\mathbf{S} v(\mathbf{S}) \leq v(\mathbf{N}).$$

THEOREM 4. *For the TU-cooperative game* $(\mathbf{N}, v)$ *to be balanced it is necessary and sufficient that the inequality* (5) *holds for one of the optimal solution* $\xi^0 \in X^0(v)$.

Proof. Consider the linear programming problem which is dual to (3) – (4):

$$\text{Maximize} \sum_{\substack{\mathbf{S} \subset \mathbf{N} \\ \mathbf{S} \neq \mathbf{N}}} \mathbf{y_S}\, v(\mathbf{S}) \tag{6}$$

$$\text{subject to} \sum_{\substack{\mathbf{S} \subset \mathbf{N} \\ \mathbf{S} \neq \mathbf{N}}} \mathbf{e}^i_\mathbf{S}\, \mathbf{y_S} \leq 1, \ \ i = 1, \cdots, n \tag{7}$$

where

$$\mathbf{e}^i_\mathbf{S} = \begin{cases} 0, & i \notin \mathbf{S} \\ 1, & i \in \mathbf{S}. \end{cases}$$

Let $\mathbf{y}^*$ be an optimal solution of (6) – (7). Then due to the duality theorem the condition (5) is equivalent to

$$\sum_{\substack{\mathbf{S} \subset \mathbf{N} \\ \mathbf{S} \neq \mathbf{N}}} \mathbf{y}^*_\mathbf{S}\ v(\mathbf{S}) \leq v(\mathbf{N}). \tag{8}$$

Denote by $\mathbf{y}^1, \cdots, \mathbf{y}^r$ the extreme points of the attainable set (7) such that

$$\sum_{\substack{\mathbf{S} \subset \mathbf{N} \\ \mathbf{S} \neq \mathbf{N}}} \mathbf{e}^i_\mathbf{S}\ \mathbf{y}^j_\mathbf{S} = 1, \ j = 1, \cdots, r \tag{9}$$

and consider the collection of coalitions $\mathcal{W}^1, \cdots, \mathcal{W}^r$ such that

$$\mathcal{W}^j = \{\mathbf{S} : \mathbf{y}^j_\mathbf{S} \neq 0\}, \ 1 \leq j \leq r.$$

Then denoting $\delta^j_\mathbf{S} = \mathbf{y}^j_\mathbf{S}$ for $\mathbf{S} \in \mathcal{W}^j$ and substituting $\delta^j_\mathbf{S}$ to (9), we get

$$\sum_{\substack{\mathbf{S} \in \mathcal{W}^j \\ \mathbf{S} \ni i}} \delta^j_\mathbf{S} = 1, \ i = 1, 2, \cdots, n.$$

Thus $\delta^j_{\mathbf{S}} > 0$ are weights for balanced collection $\mathcal{W}^j$, $(1 \leq j \leq r)$.

Therefore we have

$$\sum_{\substack{\mathbf{S}\in\mathcal{W}^j \\ \mathbf{S}\ni i}} \delta^j_{\mathbf{S}}\, v(\mathbf{S}) \leq \sum_{\substack{\mathbf{S}\subset\mathbf{N} \\ \mathbf{S}\neq\mathbf{N}}} \mathbf{y}^*_{\mathbf{S}}\, v(\mathbf{S}) \leq v(\mathbf{N}), \tag{10}$$

for all balanced collections $\mathcal{W}^j$, $(1 \leq j \leq r)$.

Conversely, if (10) holds for all balanced coalitions, then the inequality (8) is fullfilled also. It means an optimal solution ξ^0 of the problem (3) – (4) satisfies the condition (5). This completes the proof. □

Thus we can formulate a new definition of balanced game

DEFINITION 4. A TU-cooperative game $(\mathbf{N}, v)$ is called *balanced* if for arbitrary $\xi^0 \in X^0(v)$, the inequality (5) is fulfilled.

4. Consistency property of the subcore

In this section we consider the consistency property of the subcore due to Davis-Mashler, Moulin and Funaki.

Suppose we are given a TU-cooperative game $(\mathbf{N}, v)$, a player $j \in \mathbf{N}$ and payoff $\xi \in \mathbf{R}^n$, the *reduced game* with respect to j and ξ is the game $(\mathbf{N}\setminus\{j\}, v^S_\xi)$ with the characteristic function in the form

$$v^S_\xi(\mathbf{S}) = \begin{cases} 0, & \text{if } \mathbf{S} = \emptyset \\ v(\mathbf{N}) - \xi_j, & \text{if } \mathbf{S} = \mathbf{N}\setminus\{j\} \\ v(\mathbf{S}), & \text{if } \mathbf{S} \in \mathcal{S}(\mathbf{N}, v, j, \xi) \\ v(\mathbf{S}\cup\{j\}) - \xi_j, & \text{otherwise} \end{cases}$$

where $\mathcal{S}(\mathbf{N}, v, j, \xi)$ is a set of proper subcoalitions in $\mathbf{N}\setminus\{j\}$. That is,

$$\mathcal{S}(\mathbf{N}, v, j, \xi) \in \mathcal{P}^{\mathbf{N}} = \{\mathbf{S}\subset\mathbf{N} : \mathbf{S}\neq\mathbf{N},\ \emptyset\}.$$

We call $\mathcal{S}$ a reduced game structure for the TU-cooperative game $(\mathbf{N}, v)$.

DEFINITION 6. We call the reduced game with

$$\mathcal{S}(\mathbf{N}, v, j, \xi) = \{\mathcal{S} \in \mathcal{P}^{\mathbf{N}\setminus\{j\}} : v(\mathbf{S} \cup \{j\}) - \xi_j \leq v(\mathbf{S})\}$$

the *DM-reduced game* introduced by Davis and Mashler.

The characteristic function in the DM-reduced game is in the form

$$v_\xi^S(\mathbf{S}) = \begin{cases} 0, & \text{if } \mathbf{S} = \emptyset \\ v(\mathbf{N}) - \xi_j, & \text{if } \mathbf{S} = \mathbf{N} \setminus \{j\} \\ \max\{v(\mathbf{S} \cup \{j\}) - \xi_j,\ v(\mathbf{S})\}, & \text{otherwise} \end{cases}$$

DEFINITION 6. The reduced game with $\mathcal{S}(\mathbf{N}, v, j, \xi) = \emptyset$ is called the *M-reduced game* which proposed by Moulin.

In the M-reduced game, the characteristic function is considered in the form

$$v_\xi^S(\mathbf{S}) = \begin{cases} 0, & \text{if } \mathbf{S} = \emptyset \\ v(\mathbf{S} \cup \{j\}) - \xi_j, & \text{otherwise.} \end{cases}$$

DEFINITION 7. The reduced game with

$$\mathcal{S}(\mathbf{N}, v, j, \xi) = \mathcal{P}^{\mathbf{N}\setminus\{j\}}$$

is called the *SIM-reduced game* which proposed by Funaki(1995).

In the SIM-reduced game, the characteristic function is considered in the form

$$v_\xi^S(\mathbf{S}) = \begin{cases} 0, & \text{if } \mathbf{S} = \emptyset \\ v(\mathbf{S}), & \text{otherwise.} \end{cases}$$

The general consistency property with respect to reduced games is a simple generalization of the usual consistency property. Let $\Phi(\mathbf{N}, v)$ be a solution in the game $(\mathbf{N}, v)$ and $\mathcal{S}$ be a reduced game structure.

DEFINITION 8. A solution Φ of the game $(\mathbf{N}, v)$ is said to be $\mathcal{S}$-*consistent* if the payoff

$$\xi^j = (\xi, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_n) \in \Phi(\mathbf{N} \setminus \{j\}, v_\xi^S)$$

for arbitrary $\xi \in \Phi(\mathbf{N}, v)$ and all $j \in \mathbf{N}$.

Let $\Gamma_{\mathbf{SC}} = \{(\mathbf{N}, v) : \mathbf{SC}(v) \neq \emptyset\}$, i.e., $\Gamma_{\mathbf{SC}}$ is the class of balanced TU-games. Obviously $\Gamma_{\mathbf{SC}} = \Gamma_{\mathbf{C}}$. Consider the following:
SC-*condition*:

For all $j \in \mathbf{N}$ and a given $\xi^0 \in X^0(v)$, there exists the optimal solution $\xi^{*j} = (\xi_1^*, \cdots, \xi_{j-1}^*, \xi_{j+1}^*, \cdots, \xi_n^*)$ of the problem

$$\text{(11)} \qquad \text{Minimize} \sum_{i \in \mathbf{N} \setminus \{j\}} \xi_i$$

$$\text{(12)} \qquad \text{subject to} \sum_{i \in \mathbf{S}} \xi_i \geq v_\xi^S(\mathbf{S}),\ \mathbf{S} \subset \mathbf{N} \setminus \{j\}$$

such that the inequality $\xi_i^0 \geq \xi_i^*$ holds for all $i \in \mathbf{N} \setminus \{j\}$.

The following theorem gives a sufficient condition for the subcore $\mathbf{SC}(v, \xi^0)$ to be $\mathcal{S}$-consistent for any reduced game structure $\mathcal{S}$ for the class $\Gamma_{\mathbf{C}}$.

THEOREM 5. *If the* **SC**-*condition holds for a given $\xi^0 \in X^0(v)$, then for all reduced game structures $\mathcal{S}$ for $\Gamma_{\mathbf{C}}$ the subcore $\mathbf{SC}(v, \xi^0)$ is $\mathcal{S}$-consistent.*

Proof. Let ξ^0 be an arbitrary optimal solution of the problem (3) – (4) and $\mathbf{SC}(v, \xi^0)$ be not empty. Notice that for an arbitrary player $j \in \mathbf{N}$, an imputation $\xi \in \mathbf{SC}(v, \xi^0)$ and a coalition $\mathbf{S} \subset \mathbf{N} \setminus \{j\}$, the following inequalities hold:

$$\sum_{i \in \mathbf{S}} \xi_i^0 \geq v(\mathbf{S}),$$
$$\sum_{i \in \mathbf{S}} \xi_i^0 \geq v(\mathbf{S} \cup \{j\}) - \xi_j^0 \geq v(\mathbf{S} \cup \{j\}) - \xi_j.$$

Then we also have

$$\sum_{i \in \mathbf{S}} \xi_i^0 \geq \max\{v(\mathbf{S} \cup \{j\}) - \xi_j^0,\ v(\mathbf{S})\}.$$

Thus for all reduced game structures $\mathcal{S}(\mathbf{N}, v, j, \xi)$ the following inequality is fulfiled:

$$\sum_{i \in \mathbf{S}} \xi_i^0 \geq v_\xi^S(\mathbf{S}),\ \mathbf{S} \subset \mathbf{N} \setminus \{j\},\ \mathbf{S} \neq \mathbf{N} \setminus \{j\}.$$

Let $\xi^{*j} = (\xi_1^*, \cdots, \xi_{j-1}^*, \xi_{j+1}^*, \cdots, \xi_n^*)$ be an optimal solution of the problem (11) – (12). Then, by **SC**-condition, we have $\xi_i \geq \xi_i^0 \geq \xi_i^*$ for $i \in \mathbf{N} \setminus \{j\}$. On the other hand

$$\sum_{i \in \mathbf{N} \setminus \{j\}} \xi_i = v(\mathbf{N}) - \xi_j = v(\mathbf{N} \setminus \{j\}).$$

Thus $\xi^j = (\xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_n)$ belongs to the subcore of the reduced game with the structure $\mathcal{S}(\mathbf{N}, v, j, \xi)$ with respect to ξ^{*j}. This completes the proof. □

For the following consideration it would be useful to prove one important lemma. We denote the *set of active restrictions* with respect to ξ^0 by

$$J(\xi^0) = \{j : a^j \xi^0 = b_j\},$$

where ξ^0, $a^j \in \mathbf{R}^n$, $b_j \in \mathbf{R}$, $j = 1, \cdots, m$.

LEMMA 1. *Let ξ^0 be a finite optimal solution of the linear programming problem*

(13) *Minimize* $c\xi$

(14) *subject to* $a^j \xi \geq b_j, \ j = 1, \cdots, m,$

where ξ, c, $a^j \in \mathbf{R}^n$, $c \neq \mathbf{0}$, $j = 1, \cdots, m$. Then an arbitrary attainable solution of (14) *which satisfies the system*

(15) $$a^j \xi = b_j, \ j \in I(\xi^0)$$

is an optimal solution of the problem (13) – (14).

Proof. Let ξ^*be an arbitrary solution of the system (14) – (15). Suppose that ξ^* is not optimal, that is,

$$\sum_{i=1}^n \xi_i^* > \sum_{i=1}^n \xi_i^0.$$

If ξ^0 is an extreme point of the attainable solutions set(14), then the system (15) has unique solution. It means $\xi^0 = \xi^*$. Let ξ^0 be a nonextreme point of (14). It is an internal point of the solution set of (14) – (15). Then there exists a number $\epsilon > 0$ such that the point $\xi^0 + \lambda(\xi^* - \xi^0)$ satisfies (14) – (15) for any $\lambda \in (-\epsilon, \epsilon)$. Choose $\lambda_1 \in (-\epsilon, 0)$. Then, due to (16), the following inequality holds:

$$(16) \qquad \sum_{i=1}^{n} \xi_i^0 + \lambda_1 (\sum_{i=1}^{n} \xi_i^* - \sum_{i=1}^{n} \xi_i^0) < \sum_{i=1}^{n} \xi_i^0.$$

But, it contradicts to the optimality of the point ξ^0. This completes the proof. □

We denote $\mathcal{A}(\xi^0)$ as the set of *active coalitions* in the game $(\mathbf{N}, v)$ with respect to ξ^0 by

$$\mathcal{A}(\xi^0) = \{\mathbf{S} \subset \mathbf{N} : \sum_{i \in \mathbf{S}} \xi_i^0 = v(\mathbf{S}), \ \mathbf{S} \neq \mathbf{N}\}.$$

We also denote $\mathcal{A}(\mathcal{S}, \xi^{*j})$ as the set of active coalitions in reduced games with structure $\mathcal{S}(\mathbf{N}, v, j, \xi)$, where ξ^{*j} is an arbitrary optimal solution of (11) – (12). The following theorem provides a more constructive condition for the subcore $\mathbf{SC}(v, \xi^0)$ to be $\mathcal{S}$-consistent for DM-reduced and SIM-reduced games.

THEOREM 6. *Suppose that there exists an optimal solution ξ^{*j} of (11) – (12) with $\mathcal{A}(\mathcal{S}, \xi^{*j}) \subset \mathcal{A}(\xi^0)$ for DM-reduced and SIM-reduced game structures $\mathcal{S}$. Then the subcore $\mathbf{SC}(v, \xi^0)$ is DM-consistent and also SIM-consistent.*

Proof. Consider an arbitrary $\mathbf{S} \in \mathcal{A}(\mathcal{S}, \xi^{*j})$ in DM-reduced or SIM-reduced games. Then $\mathbf{S} \in \mathcal{A}(\xi^0)$ and

$$v(\mathbf{S}) = \sum_{i \in \mathbf{S}} \xi_i^0 \geq v(\mathbf{S} \cup \{j\}) - \xi_j.$$

Therefore $v(\mathbf{S}) = v_\xi^S(\mathbf{S})$ in both reduced games. Thus $\sum_{i \in \mathbf{S}} \xi_i^0 = v_\xi^S(\mathbf{S})$ for all $\mathcal{S} \in \mathcal{A}(\mathcal{S}, \xi^{*j})$. If the problem (11) – (12) has a unique optimal solution or ξ^{*j} is an extreme point of the attainable solutions set (12), then $\xi^0 = \xi^*$. Otherwise, since ξ^{0j} is an available solution of (11) – (12) and due to lemma 1, we have

$$\sum_{i \in \mathbf{N} \setminus \{j\}} \xi_i^0 = \sum_{i \in \mathbf{N} \setminus \{j\}} \xi_i^*.$$

Thus the vector ξ^{0j} is an optimal solution of (11) – (12) and ξ^j belongs to the subcore in reduced games due to Davis-Mashler and Funaki with respect to ξ^{0j}. This completes the proof. □

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