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MULTIPLICITY-FREE ACTIONS OF THE ALTERNATING GROUPS

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ABSTRACT. A transitive permutation representation of a group G is said to be multiplicity-free if all of its irreducible constituents are distinct. The character corresponding to the action is called the permutation character, given by $(1_H)^G$, where H is the stabilizer of a point. Multiplicity-free permutation characters are of interest in the study of centralizer algebras and distance-transitive graphs, and all finite simple groups are known to have such characters. In this article, we extend to the alternating groups the result of J. Saxl who determined the multiplicity-free permutation representations of the symmetric groups. We classify all subgroups H for which $(1_H)^{A_n}, n > 18$, is multiplicity-free.

1. Introduction

Let π be the permutation character associated with the action of a group G. If the action is transitive, then $\pi = (1_H)^G$, where H is the stabilizer in G of a point. π is said to be *multiplicity-free* if all of its irreducible constituents are distinct. Such characters arise in the study of centralizer algebras of transitive permutation representations. The multiplicity-free condition is equivalent to the commutativity of the centralizer algebra. It is also well known that if G is the automorphism group of a distance-transitive graph, then the permutation action of G on the vertices is multiplicity-free.

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J. Saxl started the systematic investigation of the multiplicity-free permutation representations of the symmetric groups [10], and the classification program for other groups is well underway. Recently the multiplicity-free permutation characters of the sporadic simple groups have been determined [5].

Saxl's result was contained in a conference proceedings article where he gives a sketch of the proof. The purpose of this note is to provide an extension of Saxl's result to the alternating groups. The proof is patterned after that of Saxl, but we give details when necessary. Moreover, extra arguments are needed to investigate some subgroups in the alternating group case. The case where H is a maximal subgroup of A_n was considered in [8]. Most of this work was started and is included in the author's dissertation [2].

We will prove the following main theorem.

THEOREM 1. *Let H be a subgroup of A_n, $n > 18$, and Ω a set of points on which H acts. Assume that the permutation character $(1_H)^{A_n}$ is multiplicity-free. Then one of the following holds:*

(1) $n = 2k$ *and* $A_k \times A_k \subset H \subseteq (S_k \wr S_2) \cap A_{2k}$,
(2) $n = 2k$ *and* $H \subseteq (S_2 \wr S_k) \cap A_{2k}$ *of index at most two,*
(3) $n = 2k+1$, *H fixes a point of Ω, and is one of the two groups in (1) or (2) on the rest of Ω,*
(4) $A_k \times A_{n-k} \subseteq H \subseteq (S_k \times S_{n-k}) \cap A_n$ *for some integer k with* $0 \leq k < n/2$,
(5) $F_{10} \times A_{n-5} \subseteq H \subseteq (F_{20} \times S_{n-5}) \cap A_n$ *where F_{10} and F_{20} are Frobenius groups of orders 10 and 20, respectively,*
(6) $PSL(2,5) \times A_{n-6} \subseteq H \subseteq (PGL(2,5) \times S_{n-6}) \cap A_n$, *or*
(7) $P\Gamma L(2,8) \times A_{n-9} \subseteq H \subseteq (P\Gamma L(2,8) \times S_{n-9}) \cap A_n$.

REMARK. The restriction $n > 18$ is not strictly necessary, but is imposed to allow more generality. For small n, individual arguments can be used (see for instance [8]). The rest of the paper is devoted to the proof of Theorem 1.

2. Preliminaries

From this point on, we will assume that the permutation character $(1_H)^{A_n}$ is multiplicity-free, and $H \leq A_n$, acting on a set Ω of n

elements, where $n > 18$. We fix these assumptions, unless otherwise indicated. For convenience, H will be called a *multiplicity-free subgroup*. The following lemmas are key tools in the proof.

LEMMA 2.1. *Let $\Omega_{\{k\}}$ denote the set of k-element subsets of Ω, with $0 \le k \le n/2$ and $n > 4$. Then*

(1) $|orb(H, \Omega_{\{k\}})| \le k+1$, *and*
(2) $|orb(H, \Omega_{\{k\}})| \le k$, *if H is transitive on Ω.*

Proof. These are essentially the observations in [10, p.341] and Praeger [8, p.5] adapted to the present case. □

For any group G, the multiplicity-free condition on a subgroup H gives an immediate bound on the order of H; for if $\pi = (1_H)^G$, then:

$$\pi(1) = [G:H] \le \sum_{\chi \in Irr(G)} \chi(1) := d(G).$$

If $G = A_n$, we have the following.

LEMMA 2.2. *Let a be an integer with $1 \le a \le n/2$. Then we have: $|H| > (n+2)^{-1} min_a 2^a a!(n-2a)!$.*

Proof. From the character theory of the alternating groups [7, Sec. 2.5], we have $d(A_n) = \frac{1}{2}[d(S_n) + \sum \chi(1)]$, where the sum runs over all self-associated characters of S_n. Hence $[A_n : H] \le d(A_n) < d(S_n)$. The character degree sum $d(S_n)$ can then be computed explicitly, as shown by Saxl in [10]. □

The next bounds are weaker but allows us to avoid the computations required by Lemma 2.2. They will be useful in later arguments.

LEMMA 2.3. *Let $n > 18$, and $[x]$ be the integer part of x. Then*

(1) $|H| > [n/2]!$, *and*
(2) $|H| > 2(k!)$, *if $n = 2k$.*

Proof. We show that $(n+2)^{-1} 2^a a!(n-2a)! > [n/2]!$ for all integers a with $1 \le a \le n/2$, and then apply Lemma 2.2. This can be verified in a straightforward (but tedious) manner. The bound in case (2) is done similarly. We omit the computations. □

We now carry out the proof of the main theorem, according as H is primitive (Section 3), transitive but not primitive (Section 4), or intransitive (Section 5). Basic facts on finite groups and their representations can be found in [1], [6] and [7].

3. The primitive case

LEMMA 3.1. (Praeger-Saxl, [9]) *If G is a primitive group of degree n not containing A_n, then $|G| < 4^n$.*

LEMMA 3.2. (Liebeck-Praeger-Saxl, [8]) *Let H be a multiplicity-free maximal subgroup of S_n or A_n which is primitive on Ω. Then H is doubly transitive on Ω provided $n > 6$.*

We can now prove the result for this case.

PROPOSITION 3.3. *Let G be primitive on Ω, where $G \leq A_n$, $n > 18$. Then $(1_G)^{A_n}$ is not multiplicity-free.*

Proof. It is enough to prove the statement for the case where G is maximal in A_n; for if $(1_G)^{A_n}$ is not multiplicity-free, then no subgroup of G is multiplicity-free either.

Suppose that G is a multiplicity-free maximal subgroup of A_n, primitive on Ω. Then from Lemma 2.2 and Lemma 3.1 we have

$$(n+2)^{-1}\min_a 2^a a!(n-2a)! \leq |G| < 4^n,\ a \in [1, n/2].$$

As in [9], this forces $n \leq 60$. (A computer check actually shows $n \leq 55$.) Hence by Lemma 3.2, we need to check which doubly-transitive subgroups of A_n, $n \leq 55$ are multiplicity-free. For each such subgroup G, it is shown in [8] that the inequality $[A_n : G] \leq d(A_n)$ is satisified only if $n < 12$. Hence for $n > 18$, no primitive subgroup is multiplicity-free. This proves the proposition. □

4. H is transitive but not primitive

In this section we assume that H is a multiplicity-free subgroup of A_n which is transitive but not primitive on Ω, where $|\Omega| = n$, and $n > 18$. Then Ω decomposes as a disjoint union of b non-trivial blocks of size a each, where a and b are integers with $n = ab$. Hence $H \subseteq (S_a \wr S_b) \cap A_n$, with $n = ab$. ($S_a \wr S_b$ denotes the ordinary wreath product of the symmetric groups.)

Using Lemma 2.1, it is easy to obtain the next result which shows that the values of a and b are restricted if H is multiplicity-free. This was observed in [8], with a proof given in [10].

LEMMA 4.1. (Liebeck-Praeger-Saxl, [8]) *Let H be transitive but not primitive on Ω. Then $n = 2k$, for some integer k, and one of the following holds:*

(1) *H has 2 blocks of size k, or*
(2) *H has k blocks of size 2.*

We will discuss the possibilities given in Lemma 4.1 separately. Before doing so, we need the following concept. A permutation group G on a set Ω of n elements, is called *k-homogeneous* if G permutes the set $\Omega_{\{k\}}$ of all k-element subsets of Ω transitively.

We list some results on k-homogeneous groups which will be used later. These theorems are all collected in [4, XII § 6, pp. 366-376].

LEMMA 4.2. (Livingston-Wagner [4]) *Let G be k-homogeneous of degree n. Then* (i) *if $2k \leq n$, then G is $(k-1)$-homogeneous;* (ii) *if $4 \leq 2k \leq n$, then G is $(k-1)$-transitive; and* (iii) *if $10 \leq 2k \leq n$, then G is k-transitive.*

LEMMA 4.3. (Beaumont and Peterson [4]) *Suppose that G is a permutation group of degree n and is k-homogeneous for all $k = 1, \ldots, n$. Then one of the following holds:* (i) *$G = A_n$ or S_n,* (ii) *$n = 5$, G is the Frobenius group of order 20,* (iii) *$n = 6$, $G = PGL(2,5)$, or (iv) $n = 9$, $G = PGL(2,8)$ or $P\Gamma L(2,8)$.*

LEMMA 4.4. (Wielandt [4]) *Let $k \geq 2$ and G be a k-homogeneous group of degree n. Suppose that $k + p^a - 1 \leq n$ for any prime power divisor p^a of k. Then G is $(k-1)$-homogeneous.*

LEMMA 4.5. (Kantor [4]) *If G is a k-homogeneous group which is not 4-transitive, then G is one of $PGL(2,8)$, $P\Gamma L(2,8)$, or $P\Gamma L(2,32)$ in its natural representation.*

We will also use the facts that the only 4-transitive groups are S_n $(n \geq 4)$, A_n $(n \geq 6)$, and the Mathieu groups M_{11}, M_{12}, M_{23}, and M_{24}. If $k \geq 6$, S_k and A_k are the only k-transitive groups.

We can now discuss the first case of Lemma 4.1

Case (1): H has 2 blocks of size k

We assume in this discussion the following. H is a multiplicity-free subgroup of A_{2k}, $2k > 18$. The set Ω decomposes as $\Omega = \Delta \cup \Gamma$ and $\{\Delta, \Gamma\}$ is a complete set of imprimitivity, with $|\Delta| = |\Gamma| = k$.

The action of H on Ω induces a permutation group on Δ (as well as on Γ). The elements either fix Δ or send Δ to Γ. The set H_0 of elements which leave Δ setwise invariant (and hence Γ as well) is a normal subgroup of index two in H. The pointwise stabilizer H_Δ of Δ is a normal subgroup of H_0. The factor group H_0/H_Δ is the permutation group on Δ induced from H. The groups H_Γ and H_0/H_Γ are defined similarly.

Since H is transitive, H_Δ and H_Γ are conjugate in H, i.e., there is an element which interchanges Δ and Γ. Hence $|H_\Delta| = |H_\Gamma|$. Next, since Ω is the disjoint union of Δ and Γ, any element which fixes both Δ and Γ pointwise must fix the whole of Ω pointwise. Thus $H_\Delta \cap H_\Gamma = \{1\}$. From the correspondence theorem, we have that $H_\Delta H_\Gamma / H_\Delta \trianglelefteq H_0/H_\Delta$. Also, note that $H_\Gamma \cong H_\Delta H_\Gamma / H_\Delta$. The same remarks hold if we interchange the roles of Δ and Γ.

LEMMA 4.6. *Let $k \geq 10$. Then each of H_0/H_Δ and H_0/H_Γ is either S_k or A_k.*

Proof. We divide the analysis into several cases, depending on k. The notation $\{1^i, 2^j\}$ denotes a subset of $\Omega_{\{k\}}$ with i elements from Δ and j elements from Γ.

Case (i): Let $k \geq 20$. Consider $\Omega_{\{10\}}$. Then $\theta_1 = \{1^{10}\}$, $\theta_2 = \{1^9, 2\}$, $\theta_3 = \{1^8, 2^2\}$, $\theta_4 = \{1^7, 2^3\}$, $\theta_5 = \{1^6, 2^4\}$ and $\theta_6 = \{1^5, 2^5\}$ are in distinct orbits of H. Consider $\theta_1, \theta_2, \theta_3, \theta_4$, and θ_5. One of these sets must be an H-orbit, otherwise if all split, together with θ_6, we get at least 11 orbits on $\Omega_{\{10\}}$, which contradicts Lemma 2.1(2). Hence if one of $\theta_1, \theta_2, \theta_3, \theta_4$, or θ_5 is an orbit, then the permutation group H_0/H_Δ on Δ is either 10, 9, 8, 7, or 6-homogeneous. By Lemma 4.2(iii), H_0/H_Δ is 6-transitive, and must be either S_k or A_k.

Case (ii): $13 \leq k < 20$: Consider $\Omega_{\{6\}}$. Then $\theta_1 = \{1^6\}$, $\theta_2 = \{1^5, 2\}$, $\theta_3 = \{1^4, 2^2\}$, and $\theta_4 = \{1^3, 2^3\}$ are in different orbits. Consider θ_1, θ_2 and θ_3. As in the previous case, one of these sets must remain an orbit (otherwise, including θ_4, we get at least 7 orbits, which is a contradiction). Thus H_0/H_Δ is either 6, 5, or 4-homogeneous. In any case, it is at least 4-homogeneous. A 4-homogeneous group which is not 4-transitive must have degree 9 or 33, by Lemma 4.5 (Kantor's result). Since $13 \leq k < 20$, we find that H_0/H_Δ must be 4-transitive, and hence is either S_k or A_k.

Case (iii): $k = 12$: From the analysis above, we find that H_0/H_Δ is 4-homogeneous, and again must be 4-transitive. So H_0/H_Δ either contains A_{12} or $H_0/H_\Delta = M_{12}$, the Mathieu group on 12 letters. In the former case we are done. In the latter, we get $|H| \leq 2|M_{12}|^2 = 18,065,520,320$. We show that this gives a contradiction. For if H were a multiplicity-free subgroup of A_{24}, it should satisfy: $[A_{24} : H] \leq d(A_{24}) = \frac{1}{2}[d(S_{24}) + \sum \chi(1)]$, the sum over all the 11 self-associated irreducible characters of S_{24}. The terms can be computed by hand using the Hook Formula [7, p. 56], and we obtain $[A_{24} : H] \leq 8,836,179,426,416$. Solving for H, we get $H > 35,108,400,000$. This rules out the Mathieu group. Note that the bound of Lemma 2.2 does not rule out this possibility.

Case (iv): $k = 11$: As in the previous case, we find that H_0/H_Δ must be 4-transitive. Hence H_0/H_Δ must either contain A_{11} or $H_0/H_\Delta = M_{11}$, the Mathieu group on 11 letters. Hence $|H| \leq 2|M_{11}|^2$. But $2|M_{11}|^2 < \frac{1}{24} min_a 2^a a!(22-2a)!, a \in [1, 11]$, contradicting Lemma 2.2. (The minimum is attained at $a = 9$.)

Case (v): $k = 10$: Considering $\Omega_{\{6\}}$, we find again that H_0/H_Δ is 4-homogeneous. Using Kantor's result, the only possibilities for H_0/H_Δ are S_{10} or A_{10} as asserted.

The same arguments hold for H_0/H_Γ. This completes the proof of the lemma. □

PROPOSITION 4.7. *Suppose that H has two blocks, each of size k, where $k \geq 10$. Then: $A_k \times A_k \subset H \subseteq (S_k \wr S_2) \cap A_{2k}$.*

Proof. The inclusion $H \subseteq (S_k \wr S_2) \cap A_{2k}$ is trivial. We show the other. From the previous lemma, we know that H_0/H_Δ is either S_k or A_k. Since A_k is simple (as $k \geq 10$), we have either $H_\Gamma \supseteq A_k$ or $H_\Gamma = \{1\}$. In the former case, we get (since $H_\Delta \supseteq A_k$ also) that $H \supseteq H_\Delta \times H_\Gamma \supseteq A_k \times A_k$, and we are done. In the second case, we have that $H_\Gamma = H_\Delta = \{1\}$. But then $H_0/H_\Delta \cong H_0$, so that H_0 is either S_k or A_k. Since $[H : H_0] = 2$, we get that $|H| = 2|H_0| \leq 2(k!)$. This contradicts Lemma 2.3(2).

Finally, it can be shown that the character $(1_{A_k \times A_k})^{A_{2k}}$ is not multiplicity-free (see [3]), which gives the strict inclusion in the proposition and completes the proof. □

We now consider the second case of Lemma 4.1

Case (2): H has k blocks of size 2

We now consider the case where H is a multiplicity-free subgroup of A_{2k} with k blocks of size 2.

Let $\{\Delta_1, \ldots, \Delta_k\}$ be the set of blocks, with $|\Delta_i| = 2$, $1 \leq i \leq k$. Let $K = \{k \in H | (\Delta_i)^k = \Delta_i, \text{ for each } i\}$, the subgroup of H leaving each block invariant. The action of H induces a permutation group H/K which acts on the blocks Δ_i. Let $\Delta_i = \{\alpha_i, \beta_i\}$, $1 \leq i \leq k$, and let $K_0 = \langle (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k) \rangle$ be the group generated by the k involutions (α_i, β_i). K_0 consists of elements which fix each block setwise. Then $|K_0| = 2^k$, K_0 is a direct product of k copies of the cyclic group of order 2, and $K \subseteq K_0$.

LEMMA 4.8. *Let $k \geq 10$. Then H/K is either A_k or S_k.*

Proof. As in the proof of Lemma 4.6, we use Lemma 2.1 with care. Consider the set $\Omega_{\{6\}}$. The sets $\theta_1 = \{1^2, 2^2, 3^2\}$, $\theta_2 = \{1^2, 2^2, 3, 4\}$, $\theta_3 = \{1^2, 2, 3, 4, 5\}$, and $\theta_4 = \{1, 2, 3, 4, 5, 6\}$ (with the obvious notation) are in disjoint orbits of H. By Lemma 2.1(2), the number of H-orbits on $\Omega_{\{6\}}$ is at most 6. Consider $\theta_2, \theta_3, \theta_4$. Then necessarily, at least one of the sets should remain an orbit. If θ_2 is an orbit, then H/K is 4-homogeneous; an element of H/K which sends $\{1^2, 2^2, 3, 4\}$ to another set of the form $\{h^2, i^2, j, k\}$ in the orbit sends $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ to $\{\Delta_h, \Delta_i, \Delta_j, \Delta_k\}$. Similarly, if θ_3 is an orbit, H/K is 5-homogeneous; if θ_4 is an orbit, H/K is 6-homogeneous.

The group H/K is a permutation group of degree k. By Wielandt's theorem (Lemma 4.4), if H/K is 6-homogeneous, it is 5-homogeneous. By Livingston-Wagner (Lemma 4.2(i)), if it is 5-homogeneous, then it is 4-homogeneous. In any case, H/K is 4-homogeneous. If the degree k satisfies $10 \leq k \leq 19$, by Kantor's result, all 4-homogeneous groups are 4-transitive. Hence H/K is either S_k, A_k, M_{11} or M_{12}. We show that it is not M_{12} or M_{11}.

Suppose $H/K = M_{12}$. Then $|H| = |M_{12}||K| \leq |M_{12}|2^{12} = 11 \cdot 5 \cdot 3^3 \cdot 2^{18}$, since $|K| \leq 2^{12}$. But by Lemma 2.3(2) $|H| > 2(12!) = 11 \cdot 7 \cdot 5^2 \cdot 3^5 \cdot 2^{11}$. A comparison shows that this leads to a contradiction. In the same way we find that H/K cannot equal M_{11}. Hence $H/K \supseteq A_k$ for $10 \leq k \leq 19$.

Now let $k \geq 20$. Considering the orbits of H on $\Omega_{\{10\}}$ and using similar arguments as before, we see that H/K is at least 6-homogeneous. By Livingston-Wagner (Lemma 4.2(iii)), H/K is 6-transitive, and thus is either symmetric or alternating.

Thus for all cases considered, $H/K \supseteq A_k$ as claimed. □

LEMMA 4.9. $|K| \geq 2^{k-1}$.

Proof. View K_0 as the natural permutation module of the group A_k over the finite field $GF(2)$. Since $K \subseteq K_0$, K is a submodule of K_0. From the preceding lemma, $H/K \subseteq S_k$. Thus $|H||K|^{-1} \leq k!$. So, $|K| \geq |H|/k!$. By Lemma 2.3(2), we know $|H| > 2(k!)$. Hence, $|K| > 2(k!)/k! = 2$. This shows that K is a non-trivial submodule. Hence, using the structure of the natural module [1, p.50 and 74], we have either $K = K_0$ or K is the core. In any case, $[K_0 : K] \leq 2$. Since $|K_0| = 2^k$, we have $|K| \geq 2^{k-1}$. □

We now have our result for this case.

PROPOSITION 4.10. *Suppose that H has k blocks, each of size two, where $k \geq 10$. Then H is a subgroup of $(S_2 \wr S_k) \cap A_{2k}$ of index at most two.*

Proof. The inclusion is immediate. Since $|(S_2 \wr S_k) \cap A_{2k}| = 2^{k-1}k!$, we have $2^{k-1}k! = [(S_2 \wr S_k) \cap A_{2k} : H][H : K][K : \{1\}]$, from which we get the result, since $H/K \supseteq A_k$ and $|K| \geq 2^{k-1}$ from the two previous lemmas. □

5. H is intransitive

In this section we assume that H is a multiplicity-free subgroup of A_n acting intransitively on a set Ω of n-elements.

From Lemma 2.1(1) the number of orbits of H on $\Omega_{\{1\}} \cong \Omega$ is at most two. Since H is intransitive, this number is exactly two, by Lemma 2.1(2). Let Γ and Δ be the two orbits, and take Γ to be the smaller orbit with size $|\Gamma| = k \leq n/2$. The action of H induces an action on Γ. Let H_Γ be the pointwise stabilizer of H on Γ. Then $H^\Gamma = H/H_\Gamma$ is the permutation group on Γ induced from H. Similarly, $H^\Delta = H/H_\Delta$ is the group induced on Δ. We fix these notations for the rest of the section. The key step in the analysis of this section is the determination of the groups $H^\Gamma, H^\Delta, H_\Gamma, H_\Delta$.

LEMMA 5.1. *For each $t \leq k$, H^Γ and H^Δ are t-homogeneous. Furthermore, H^Γ is one of the following: $S_k, A_k, PGL(2,5), P\Gamma L(2,8)$, or $PGL(2,8)$.*

Proof. For each integer t with $1 \leq t \leq n/2$, consider the action of H on $\Omega_{\{t\}}$. We see that the $t+1$ sets: $\{1^t\}, \{1^{t-1}, 2\}, \{1^{t-2}, 2^2\}, \cdots, \{1, 2^{t-1}\}$, and $\{2^t\}$ are in distinct orbits of H on $\Omega_{\{t\}}$. (As before $\{1^i, 2^j\}$ means a set with i elements from Δ and j elements from Γ.) By Lemma 2.1(1), these are all the orbits of H on $\Omega_{\{t\}}$. Hence for each t, $1 \leq t \leq k$, the groups H^Γ and H^Δ are t-homogeneous, since the sets $\{1^t\}$ and $\{2^t\}$ are orbits. This proves the first statement.

Since H^Γ is a permutation group of degree k such that H^Γ is t-homogeneous for all t, $t \leq k$, by Beaumont and Peterson's classification (Lemma 4.3), the possibilities for H^Γ are determined. This completes the proof. $\square$

LEMMA 5.2. $H_\Gamma \neq \{1\}$

Proof. Suppose that $H_\Gamma = \{1\}$. Then $H^\Gamma = H/H_\Gamma = H/\{1\} \cong H$. Hence $|H| = |H^\Gamma| \leq k! \leq [n/2]!$, since $H^\Gamma \hookrightarrow S_k$. But by Lemma 2.3(1), $|H| > [n/2]!$. This contradiction shows that $H_\Gamma \neq \{1\}$. $\square$

We now state and prove our result for the intransitive case.

PROPOSITION 5.3. *Let H be intransitive on Ω. Then H is one of the groups in cases* (3), (4), (5), (6), *or* (7) *of the main theorem.*

Proof. As before, let Γ and Δ be the two orbits of H, with $|\Gamma| = k \leq n/2$. Via the natural embedding,

$$H = H/(H_\Gamma \cap H_\Delta) \hookrightarrow H/H_\Gamma \times H/H_\Delta = H^\Gamma \times H^\Delta,$$

we obtain the inclusion $H \subseteq H^{\Gamma} \times H^{\Delta}$. Since H^{Γ} and H^{Δ} are permutation groups of degrees k and $n-k$ respectively, we have that $H \subseteq H^{\Gamma} \times H^{\Delta} \subseteq S_k \times S_{n-k}$. Since $H \subseteq A_n$ we further have the following inclusions:

$$H \subseteq (H^{\Gamma} \times H^{\Delta})^+ \subseteq (S_k \times H^{\Delta})^+ \subseteq (S_k \times S_{n-k})^+,$$

where G^+ denotes the subgroup of index two of all even permutations of a group G (i.e $G^+ = G \cap A_n$).

Define the map $f : (S_k \times S_{n-k})^+ \to S_{n-k}$ which sends $(\sigma, \tau) \mapsto \tau$. Then f is a homomorphism of $(S_k \times S_{n-k})^+$ onto S_{n-k} with kernel $A_k \times \{1\}$. Similarly, the map $g : (S_k \times H^{\Delta})^+ \to H^{\Delta}$ which maps $(\sigma, \tau) \mapsto \tau$ is a surjective homomorphism with kernel $A_k \times \{1\}$ also.

By transitivity of induction, we have

$$\begin{aligned}(1_H)^{A_n} &= [\{(1_H)^{(S_k \times H^{\Delta})^+}\}^{(S_k \times S_{n-k})^+}]^{A_n} \\ &= [\{1_{(S_k \times H^{\Delta})^+} + \cdots\}^{(S_k \times S_{n-k})^+}]^{A_n} \\ &= [\{1_{(S_k \times H^{\Delta})^+}\}^{(S_k \times S_{n-k})^+} + \cdots]^{A_n}.\end{aligned}$$

Since $(1_H)^{A_n}$ is multiplicity-free by assumption, the permutation character $\{1_{(S_k \times H^{\Delta})^+}\}^{(S_k \times S_{n-k})^+}$ must be multiplicity-free also.

Let $T = \{t_i\}$ be a complete set of representatives of the cosets of H^{Δ} in S_{n-k}. We can identify T with the set of coset representatives of $(S_k \times H^{\Delta})^+$ in $(S_k \times S_{n-k})^+$ via $t_i \mapsto (1, t_i)$.

The group $(S_k \times S_{n-k})^+$ acts on the cosets of $(S_k \times H^{\Delta})^+$ with trivial action on the first factor, i.e., if $(\rho, \sigma) \in (S_k \times S_{n-k})^+$, then

$$(S_k \times H^{\Delta})^+(1, t_i)(\rho, \sigma) =: S_k\rho \times H^{\Delta}t_i\sigma := S_k \times H^{\Delta}t_i\sigma.$$

Hence the $(S_k \times S_{n-k})^+$-module affording the permutation character $\{1_{(S_k \times H^{\Delta})^+}\}^{(S_k \times S_{n-k})^+}$ is inflated via the earlier defined epimorphism f from the S_{n-k}-module affording the character $(1_{H^{\Delta}})^{S_{n-k}}$. Consequently, since $\{1_{(S_k \times H^{\Delta})^+}\}^{(S_k \times S_{n-k})^+}$ is multiplicity-free, then $(1_{H^{\Delta}})^{S_{n-k}}$ is also multiplicity-free. Hence H^{Δ} is a multiplicity-free subgroup of S_{n-k} and we can apply Saxl's theorem ([10, p. 340]) to classify H^{Δ}. Let $|\Delta| = n - k = m$. Then we have the following possibilities for H^{Δ}:

(*i*) $A_{m-t} \times A_t \subseteq H^{\Delta} \subseteq S_{m-t} \times S_t$, $0 \leq t < m/2$,
(*ii*) $m = 2t$ and $A_t \times A_t \subset H^{\Delta} \subseteq S_t \wr S_2$,
(*iii*) $m = 2t$ and $H^{\Delta} \subseteq S_2 \wr S_t$ of index at most four,
(*iv*) m is odd and H^{Δ} fixes a point of Δ and is one of the groups in (ii) and (iii) on the rest of Δ,
(*v*) $A_{m-t} \times G_t \subseteq H^{\Delta} \subseteq S_{m-t} \times G_t$, where t is 5,6, or 9, and G_t is $F_{20}, PGL(2,5)$, and $P\Gamma L(2,8)$ respectively.

Since H^{Δ} is transitive on Δ, the intransitive cases (*iv*) and (*v*) are immediately excluded. Case (*i*) is possible only if $t = 0$; in this case, we have $A_{n-k} \subseteq H^{\Delta} \subseteq S_{n-k}$. We discuss this case and the remaining possibilities, i.e., (*ii*) and (*iii*) in the next step of the proof.

Case I: $A_{n-k} \subseteq H^{\Delta} \subseteq S_{n-k}$:

Recall that H_{Δ} and H_{Γ} are normal subgroups of H with $H_{\Delta} \cap H_{\Gamma} = \{1\}$. Hence $H_{\Delta}H_{\Gamma} \trianglelefteq H$, and $H_{\Delta}H_{\Gamma}/H_{\Delta} \trianglelefteq H/H_{\Delta} = H^{\Delta}$ by the correspondence theorem. From Lemma 5.2, we know that $H_{\Gamma} \neq \{1\}$. Hence $H_{\Delta}H_{\Gamma}$ strictly contains H_{Δ}. Since H^{Δ} is a subgroup of S_{n-k} containing A_{n-k}, $H_{\Delta}H_{\Gamma}/H_{\Delta}$ is either S_{n-k} or A_{n-k} (since $n - k \geq n/2 \geq 10$). In any case, we get $H_{\Delta}H_{\Gamma}/H_{\Delta} \supseteq A_{n-k}$. From the 2nd Isomorphism Theorem, we have $H_{\Gamma} \cong H_{\Gamma}/\{1\} \cong H_{\Delta}H_{\Gamma}/H_{\Delta} \supseteq A_{n-k}$. Thus, $H_{\Gamma} \supseteq A_{n-k}$.

We next determine H_{Δ}. Since $H/H_{\Delta} \subseteq S_{n-k}$ and $H_{\Delta}H_{\Gamma}/H_{\Delta} \supseteq A_{n-k}$, we see that $[H/H_{\Delta} : H_{\Delta}H_{\Gamma}/H_{\Delta}] \leq 2$. By the 3rd Isomorphism Theorem, we have $(H/H_{\Delta})/(H_{\Delta}H_{\Gamma}/H_{\Delta}) \cong H/H_{\Delta}H_{\Gamma}$. Similarly, $(H/H_{\Gamma})/(H_{\Delta}H_{\Gamma}/H_{\Gamma}) \cong H/H_{\Delta}H_{\Gamma}$ This shows that $[H/H_{\Gamma} : H_{\Delta}H_{\Gamma}/H_{\Gamma}] \leq 2$ also. Since $H_{\Delta} \cong H_{\Delta}H_{\Gamma}/H_{\Gamma}$ and $H^{\Gamma} = H/H_{\Gamma}$, we have $[H^{\Gamma} : H_{\Delta}] \leq 2$ (abusing notation). Hence to determine H_{Δ}, we only need to determine the normal subgroups of H^{Γ} of index at most two.

From Lemma 5.1, H^{Γ} can be one of the following groups: S_k, A_k, F_{20}, $PGL(2,5)$, or $P\Gamma L(2,8)$.

Suppose that $H^{\Gamma} = S_k$ or A_k. Then H_{Δ} is either S_k or A_k. In any case, $H_{\Delta} \supseteq A_k$.

Let $H^{\Gamma} = F_{20}$. In this case the degree $k = 5$. The Frobenius group of order 5 has a normal subgroup of index two which is the product of a Frobenius kernel of order 5 and a cyclic group of order 2. We denote this by F_{10}. Hence $H_\Delta \supseteq F_{10}$.

Next, let $H^\Gamma = PGL(2,5)$. Here $k = 6$. $PSL(2,5)$ is a normal subgroup of index two. Hence $H_\Delta \supseteq PSL(2,5)$.

Finally, $P\Gamma L(2,8)$, the projective semi-linear group of degree 9, has no subgroup of index two. Hence $H_\Delta = P\Gamma L(2,8)$.

Since H_Δ and H_Γ commute ($H_\Delta \cap H_\Gamma = \{1\}$), we have the following inclusions:

$$H_\Delta \times H_\Gamma \subseteq H \subseteq (H^\Gamma \times H^\Delta)^+.$$

Substituting the information on H^Γ and H_Δ we obtained above, and the fact that for this case $H^\Delta \subseteq S_{n-k}$ and $H_\Gamma \supseteq A_{n-k}$, we obtain cases (4), (5), (6), and (7) of the main theorem. Note that the statement of Case (4) includes $k = 0$ since $(1_{A_n})^{S_n}$ is clearly multiplicity-free.

We now consider the remaining two possibilities for H^Δ (cases (*ii*) and (*iii*) in our earlier discussion). We will discuss them in a single case below.

Case II. $H^\Delta \subseteq S_t \wr S_2$ or $S_2 \wr S_t$.

Suppose first that the other orbit Γ has more than one element. Then $|\Gamma| > 1$ and so $|\Delta| = n - k > 1$ also, since $k \leq n/2$. By Lemma 2.1(1), the number of orbits of H on $\Omega_{\{2\}}$ is at most three. The sets $\{1^2\}$ and $\{2^2\}$, consisting of pairs of unordered points of Δ and Γ, respectively, as well the sets $\{1,2\}$ consisting of a point each from Δ and Γ are in disjoint orbits. Hence these are exactly the three orbits of H on $\Omega_{\{2\}}$. In particular, since $\{1^2\}$ is an orbit, the group H^Δ is 2-homogeneous on Δ. It is well known that this implies that H^Δ is primitive on Δ. However, this contradicts the assumption that H^Δ is contained in one of the imprimitive groups $S_t \wr S_2$ or $S_2 \wr S_t$. The contradiction arises from our assumption that Γ contains more than one element. Hence $|\Gamma| = 1$.

Thus for Case II, the group H is the stabilizer of one point, the degree of Ω is odd, and on the rest of Ω, H is one of the groups obtained in Section 4 of this paper. This is precisely the statement of Case (3) in our main theorem.

This completes the proof of Proposition 5.3 and gives the result for the intransitive case. □

6. Conclusion

We now collect the results of Sections 3, 4, and 5 and give a summary of the proof of our main theorem.

Proof of Theorem 1. Proposition 3.3 shows that no primitive subgroup is multiplicity-free. Propositions 4.7 and 4.10 give the results for the transitive but imprimitive case, i.e. cases (1) and (2) of the main theorem. Finally, the intransitive case is given by cases (3), (4), (5), (6), and (7) which we obtain in Proposition 5.3. □

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References

[1] M. Aschbacher, *Finite Group Theory*, Cambridge Studies in Adv. Math. 10, Cambridge Univ. Press, Cambridge, 1986.

[2] J. M. P. Balmaceda, *Multiplicity-free permutation representations of the alternating groups*, Ph. D. Thesis, University of Illinois, Urbana-Champaign, Il, 1991.

[3] J. M. P. Balmaceda, *On some permutation characters of the alternating groups*, submitted.

[4] N. Blackburn and B. Huppert, *Finite Groups III, Grundlehren der Math. Wiss. Vol. 243*, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[5] T. Breuer and K. Lux, *The multiplicity-free permutation characters of the sporadic simple groups and their automorphism groups*, Comm. in Alg. **24(7)** (1996), 2293–2316.

[6] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York-San Francisco-London, 1976.

[7] G. James and A. Karber, *The Representation Theory of the Symmetric Group, Encyclopedia of Math. and its Applic. Vol. 16*, Addison-Wesley, Reading, Mass., 1981.

[8] M. W. Liebeck, C. E. Praeger, and J. Saxl, *Distance-transitive graphs with symmetric or alternating automorphism group*, Bull. Austral. Math. Soc. **35** (1987), 1–25.

[9] C. E. Praeger and J. Saxl, *On the orders of primitive permutation groups*, Bull. London Math. Soc. **12** (1980), 303–307.

[10] J. Saxl, *On multiplicity-free permutation representations*, in *Finite Geometries and Designs*, London Math. Soc. Lecture Notes **49** (1981), 337–353.

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