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LOW TYPE PSEUDO-RIEMANNIAN SUBMANIFOLDS

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ABSTRACT. We study low type submanifolds in pseudo-Euclidean space which is especially of 2-type pseudo-umbilical. We also determine full null 2-type surfaces with parallel mean curvature vector in 4-dimensional Minkowski space-time.

1. Introduction

The general notion of finite type submanifolds of Euclidean space was introduced by B.-Y. Chen about ten years ago ([1]). Later on, he extended this notion to a pseudo-Riemannian version ([2], [3]) as follows: Let E_s^{m+1} be an $(m+1)$-dimensional pseudo-Euclidean space with metric tensor given by

$$\tilde{g} = -\sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{m+1} dx_i^2,$$

where $(x_1, x_2, \cdots, x_{m+1})$ is a Cartesian coordinate system of E_s^{m+1}. Let M be a connected n-dimensional pseudo-Riemannian submanifold of E_s^{m+1} with signature $(r, n-r)$. Then we have the Laplacian Δ of M acting on the space of smooth functions defined on M. M is said to be of k-type if the position vector x of M in E_s^{m+1} can be decomposed in the following way

$$(1.1) \qquad x = x_0 + x_{i_1} + \cdots + x_{i_k}, \quad \Delta x_{i_j} = l_{ij} x_{ij}, l_{i_1} < \cdots < l_{i_k}$$

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for some natural number k, where x_0 is a constant map and $x_{i_1}, \cdots, x_{i_k}$ are non-constant maps. A k-type submanifold is said to be of null k-type if one of $l_{i_1}, \cdots, l_{i_k}$ is zero. Since it is well known that $\Delta x = -nH$ where H is the mean curvature vector of M in E_s^{m+1}, the submanifold M is of null 1-type if and only if M is minimal in E_s^{m+1}.

We now introduce typical pseudo-Riemannian manifolds. Let $\mathbf{c}$ be a point in E_s^{m+1} and $r > 0$. We put

$$S_s^m(\mathbf{c}, r) = \{x \in E_s^{m+1} | \langle x - \mathbf{c}, x - \mathbf{c} \rangle = r^2\},$$

$$H_{s-1}^m(\mathbf{c}, r) = \{x \in E_s^{m+1} | \langle x - \mathbf{c}, x - \mathbf{c} \rangle = -r^2\}$$

where $\langle , \rangle$ denotes the indefinite scalar product on E_s^{m+1}. It is well known that $S_s^m(\mathbf{c}, r)$ and $H_{s-1}^m(\mathbf{c}, r)$ are complete pseudo-Riemannian manifolds with constant sectional curvatures $\frac{1}{r^2}$ and $-\frac{1}{r^2}$ respectively. Such $S_s^m(\mathbf{c}, r)$ and $H_{s-1}^m(\mathbf{c}, r)$ are respectively called the pseudo-Riemannian sphere and the pseudo-hyperbolic space and $\mathbf{c}$ is called the center. B.-Y. Chen ([2]) classified 1-type submanifolds of pseudo-Euclidean space E_s^{m+1}.

In this article, we study 2-type pseudo-umbilical pseudo-Riemannian submanifolds of E_s^{m+1} and null 2-type surfaces with unit parallel mean curvature vector. Throughout this paper, submanifolds of pseudo-Euclidean space are always assumed to be pseudo-Riemannian.

2. Preliminaries

Let M be a submanifold of a pseudo-Euclidean space E_s^{m+1} with indefinite scalar product $\langle , \rangle$. Let X be a vector in E_s^{m+1}. X is called spacelike if $\langle X, X \rangle \geq 0$ (respectively, timelike or null if $\langle X, X \rangle < 0$ or $\langle X, X \rangle = 0$ and $X \neq 0$). We denote by A, h, ∇, D and $\tilde{\nabla}$ the Weingarten map, the second fundamental form, the Levi-Civita connection of M, the normal connection and the Levi-Civita conection of E_s^{m+1}, respectively. A submanifold M is said to be pseudo-umbilical if $A_H = \rho I$ for some smooth function ρ on M and $\langle H, H \rangle \neq 0$. For the second fundamental form h, we define its covariant derivative, denoted by $\bar{\nabla} h$, as follows :

$$(2.1) \qquad (\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for vector fields X, Y and Z tangent to M.

Let $e_1, e_2, \cdots, e_n$ be an orthonormal basis of the tangent bundle TM. If ϕ is an endomorphism of T_xM, then the trace of ϕ is defined by

$$tr\phi = \sum_{i=1}^{n} \epsilon_i \langle \phi e_i, e_i \rangle, \quad \epsilon_i = \langle e_i, e_i \rangle = \pm 1. \tag{2.2}$$

Thus, the mean curvature vector H of M is given by

$$H = \frac{1}{n} trh = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i h(e_i, e_i). \tag{2.3}$$

Let $e_{n+1}, e_{n+2}, \cdots, e_{m+1}$ be an orthonormal normal basis of the normal bundle $T^{\perp}M$. B.-Y. Chen ([3]) gave the formula for ΔH:

$$\begin{aligned} \Delta H = \Delta^D H &+ \frac{n}{2} grad\langle H, H\rangle + 2trA_{DH} \\ &+ \sum_{r=n+1}^{m+1} \epsilon_r tr(A_H A_{e_r}) e_r, \end{aligned} \tag{2.4}$$

where Δ^D denotes the Laplacian of the normal bundle given by $\Delta^D = -\sum_{i=1}^{n} \epsilon_i (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i})$ and A_r stands for the shape operator associated with e_r. Let ω^A be the dual 1-form of e_A $(A = 1, 2, \cdots, m+1)$ defined by $\omega^A(X) = \epsilon_A \langle e_A, X\rangle$. The connection forms ω_A^B are defined by

$$de_A = \sum_B \omega_A^B e_B, \qquad \omega_A^B + \epsilon_A \epsilon_B \omega_B^A = 0. \tag{2.5}$$

Then, the structure equations of E_s^{m+1} are obtained as follows:

$$d\omega^A = \sum_{B=1}^{m+1} \omega^B \wedge \omega_B^A, \tag{2.6}$$

$$d\omega_B^A = \sum_{C=1}^{m+1} \omega_B^C \wedge \omega_C^A. \tag{2.7}$$

We now define the curvature forms Ω^i_j and Ω^s_t on M :

$$\Omega^i_j = \frac{1}{2}\sum_{k,l} R^i_{jkl}\omega^k \wedge \omega^l, \quad \Omega^s_t = \frac{1}{2}\sum_{k,l} R^s_{tkl}\omega^k \wedge \omega^l,$$

where $R^i_{jkl} = \epsilon_i\langle R(e_k, e_l)e_j, e_i\rangle$ and

$$R^s_{tkl} = \epsilon_t \sum_i (h^t_{ik}h^s_{il} - h^t_{il}h^s_{ik}) \tag{2.8}$$

with $h^t_{ij} = \langle h(e_i, e_j), e_t\rangle$ where $i, j, k, l \in \{1, 2, \cdots, n\}$ and $t, s \in \{n+1, \cdots, m+1\}$. The equation (2.8) is called the equation of Ricci. We then have the structure equations for M

$$d\omega^i = \sum_{j=1}^{n} \omega^j \wedge \omega^i_j, \quad \omega^i_j + \epsilon_j\epsilon_i\omega^j_i = 0, \tag{2.9}$$

$$d\omega^i_j = \sum_{k=1}^{n} \omega^k_j \wedge \omega^i_k + \Omega^i_j, \tag{2.10}$$

$$d\omega^s_t = \sum_{u=n+1}^{m+1} \omega^u_t \wedge \omega^s_u + \Omega^s_t. \tag{2.11}$$

Fore later uses, we introduce some examples of null 2-type.

EXAMPLE 1 ([5]). Let $f : E^3_1 \to R$ be a real function defined by

$$f(x, y, z) = -\delta_1 x^2 + y^2 + \delta_2 z^2,$$

where δ_1 and δ_2 belong to the set $\{0, 1\}$ and they do not vanish at the same time. Taking $r > 0$ and $\epsilon = \pm 1$, the set $f^{-1}(\epsilon r^2)$ is a surface in E^3_1 provided that $(\delta_1, \delta_2, \epsilon) \neq (0, 1, -1)$. Then, the unit normal vector is given by $\frac{1}{r}(\delta_1 x, y, \delta_2 z)$ and the principal curvatures are easily derived as $-\frac{\delta_1}{r}$ and $-\frac{\delta_2}{r}$. According to the choice of numbers 0, 1 and -1 for δ_1, δ_2 and ϵ, we have several examples of null 2-type surfaces : $E^1_1 \times S^1(r)$ for $(\delta_1, \delta_2, \epsilon) = (0, 1, 1)$, $R \times H^1(r)$ for $(\delta_1, \delta_2, \epsilon) = (1, 0, -1)$ and $S^1_1 \times R$ for $(\delta_1, \delta_2, \epsilon) = (1, 0, 1)$.

EXAMPLE 2 (B-SCROLL IN E^3, [6]). Let $\gamma(s)$ be a null curve in Minkowski 3-space E_1^3 with Cartan frame $\{A, B, C\}$, that is, A, B, C are vector fields along $\gamma(s)$ satisfying the following :

$$\langle A, A\rangle = \langle B, B\rangle = 0, \qquad \langle A, B\rangle = -1,$$

$$\langle A, C\rangle = \langle B, C\rangle = 0, \qquad \langle C, C\rangle = 1,$$

and

$$\dot{x} = A,$$
$$\dot{A} = k(s)C,$$
$$\dot{B} = a_0 C, a_0 \text{being a nonzero constant},$$
$$\dot{C} = a_0 A + k(s)B.$$

If we consider an immersion $x(s,t) = x(s) + tB(s)$, then x defines a Lorentz surface called a *B-scroll.* In this surface, we can have the unit normal vector field $N = -a_0 tB(s) - C(s)$ and the mean curvature vector $H = a_0 N$. It is a null 2-type surface in e_1^3 with minimal polynomial $(x - a_0)^2$.

EXAMPLE 3 (EXTENDED B-SCROLL IN E_1^4). Let γ be a null curve in E_1^4 and let $A(s), B(s), C(s), D(s)$ be a Cartan frame along γ such that

$$\langle A, A\rangle = \langle B, B\rangle = 0, \qquad \langle A, B\rangle = -1,$$

$$\langle A, C\rangle = \langle A, D\rangle = \langle B, C\rangle = \langle B, D\rangle = 0,$$

$$\langle C, C\rangle = \langle D, D\rangle = 1, \qquad \langle C, D\rangle = 0, \qquad \dot{\gamma}(s) = A(s).$$

Let $X(s)$ be the matrix of $(A(s) \quad B(s) \quad C(s) \quad D(s))$ consisting of column vectors of A, B, C, D with respect to standard coordinates of E_1^4. Then, $X(s)$ satisfies

$$X^t(s) E X(s) = T(s),$$

where $E = \mathrm{diag}(-1, 1, 1, 1)$ and

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $X^t(s)$ denotes the transpose of $X(s)$.

We now consider a system of ordinary differential equations :

$$\begin{aligned}\dot{A}(s) &= k_1(s)A(s) - k_2(s)C(s) - k_3(s)D(s),\\ \dot{B}(s) &= -k_1(s)B(s) - aC(s),\\ \dot{C}(s) &= -aA(s) - k_2(s)B(s),\\ \dot{D}(s) &= -k_3(s)B(s),\end{aligned}$$

where $k_1(s), k_2(s)$ and $k_3(s)$ are smooth functions and a is a constant. In other words, we may write

$$\dot{X}(s) = X(s)M(s),$$

where

$$M(s) = \begin{pmatrix} k_1(s) & 0 & -a & 0 \\ 0 & -k_1(s) & -k_2(s) & -k_3(s) \\ -k_2(s) & -a & 0 & 0 \\ -k_3(s) & 0 & 0 & 0 \end{pmatrix}.$$

For a given $X(0) = (A(0) \quad B(0) \quad C(0) \quad D(0))$ satisfying $X^t(0)EX(0) = T$, or, equivalently $X(0)TX^t(0) = E$, there is a unique solution to above matrix equation with initial condition $X(0)$. Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X(s)TX^t(s)) = 0$ and hence $X(s)TX^t(s) = E$ which is equivalent to

$$X^t(s)EX(s) = T.$$

Therefore, $A(s), B(s), C(s), D(s)$ form the null frame along a null curve γ in E^4_1.

Let $x(s,t) = \gamma(s) + tB(s)$. Then, it defines a Lorentz surface M in E^4_1. The mean curvature vector field H is given by

$$H(s,t) = -ta^2B(s) + aC(s),$$

which implies

$$\Delta H = 2a^2 H.$$

If $a \neq 0$, then we can easily see that M is of null 2-type. And, the shape operator A_H associated with the mean curvature vector field H has the form

$$\begin{pmatrix} a^2 & 0 \\ ak_2 & a^2 \end{pmatrix}$$

in the coordinate frame $\{x_s, x_t\}$.

3. 2-type pseudo-umbilical submanifolds

In this section we deal with 2-type pseudo-umbilical submanifolds of E_s^{m+1}. First of all, if we follow the similar argumetnt in [7], we have

LEMMA 3.1 ([7]). *Let M be a pseudo-umbilical submanifold of E_s^{m+1}. Then we have*
(1) $\sum_r \epsilon_r tr(A_H A_r)e_r = n\epsilon\alpha^2 H$,
(2) $trA_{DH} = \frac{2-n}{2} grad\langle H, H\rangle$,
where $\epsilon = sgn\langle H, H\rangle$ *and* $\alpha = |\langle H, H\rangle|^{\frac{1}{2}}$.

LEMMA 3.2. *Let M be an n-dimensional pseudo-umbilical submanifold of E_s^{m+1}. Then, $(\Delta H)^T$ vanishes if and only if the mean curvature is constant or $n = 4$, where $(\Delta H)^T$ denotes the tangential component of ΔH.*

Proof. By Lemma 3.1, (2.4) can be reduced to

$$\Delta H = \Delta^D H + n\epsilon\alpha^2 H + \frac{4-n}{2} grad\langle H, H\rangle.$$

Thus, $(\Delta H)^T = \frac{4-n}{2} grad\langle H, H\rangle$ which gives the lemma. □

LEMMA 3.3. *Let M be an n-dimensional submanifold of E_s^{m+1}. If M is of 2-type, then $(\Delta H)^T$ vanishes if and only if one of the following occurs*

(a) *M lies in a pseudo-Riemannian sphere, a pseudo-hyperbolic space or a null cone.*
(b) *M is of null 2-type.*

Proof. Since M is of 2-type, the position vector x of M can be decomposed into

$$x = x_0 + x_p + x_q, \quad \Delta x_p = \lambda x_p, \quad \Delta x_q = \mu x_q \tag{3.1}$$

for some constants λ and μ. Then we have

$$\Delta^2 x + b\Delta x + c(x - x_0) = 0 \tag{3.2}$$

where $b = -(\lambda+\mu)$ and $c = \lambda\mu$. Since $\Delta x = -nH$, (3.2) can be rewritten as

(3.3)

$$\Delta^D H + \frac{n}{2} grad\langle H, H\rangle + 2tr A_{DH} + \sum \epsilon_r tr(A_H A_{e_r}) e_r = -bH + \frac{c}{n}(x - x_0).$$

Therefore, $(\Delta H)^T$ vanishes if and only if $c = 0$ or $x - x_0$ is normal to M. If $c = 0$, M is of null 2-type. Suppose $x - x_0$ is normal to M. Let X be a vector field tangent to M. Then,

$$X\langle x - x_0, x - x_0\rangle = \langle X, x - x_0\rangle = 0,$$

that is, $\langle x-x_0, x-x_0\rangle$ is constant. Thus, M lies in a pseudo-Riemannian sphere, a pseudo-hyperbolic space or a null cone depending upon $\langle x - x_0, x - x_0\rangle > 0, < 0$ or equals zero. □

THEOREM 3.4. *Let M be an $n(\neq 4)$-dimensional 2-type pseudo-umbilical submanifold of E_s^{m+1}. Then, M has constant mean curvature if and only if one of the following holds:*

(a) M lies in a pseudo-Riemannian sphere, a pseudo-hyperbolic space or a null cone.

(b) M is of null 2-type.

Proof. If M has a constant mean curvature, then Lemma 3.2 and Lemma 3.3 give (a) or (b). Conversely, if (a) or (b) holds, then $(\Delta H)^T$ vanishes if we take account of (3.3). Then, Lemma 3.2 implies that the mean curvature is constant. □

THEOREM 3.5. *Let M be a 4-dimensional 2-type pseudo-umbilical submanifold of E_s^{m+1}. Then, (a) or (b) described in Theorem 3.4 holds.*

Proof. By Lemma 3.2, $(\Delta H)^T$ always vanishes. If we make use of (3.3) and the argument developed in Lemma 3.3, we obtain the theorem. □

4. Null 2-type surfaces with unit parallel mean curvature vector

A surface M of pseudo-Euclidean space E_s^{m+1} is said to have the unit parallel mean curvature vector if the mean curvature vector H is non-null and its normalized vector field is parallel in the normal bundle. In this section we assume that the surface M of pseudo-Euclidean space E_s^{m+1} have the unit parallel mean curvature vector and is of null 2-type. We choose a local adapted orthonormal frame $e_1, e_2, e_3, \cdots, e_{m+1}$ such that $H = \alpha e_3$. Then the last term of (2.4) can be rewritten as

$$\sum_{r=3}^{m+1} \epsilon_r tr(A_H A_r)e_r = \epsilon_3 \alpha tr A_3^2 e_3 + \alpha a(H)$$

where $a(H)$ stands for the allied mean curvature vector field of H defined by $a(H) = \sum_{r=4}^{m+1} \epsilon_r tr(A_3 A_r)e_r$. A submanifold in E_s^{m+1} is called Chen submanifold if the allied mean curvature vector field $a(H)$ vanishes identically. As is well known we have the following proposition.

PROPOSITION 4.1 ([4]). *Let M be a null 2-type submanifold of E_s^m. Then*

$$tr\bar{\nabla} A_H = tr\nabla A_H + tr A_{DH} = 0 \tag{4.1}$$

and

$$\Delta^D H + \sum_{i=1}^{n} \epsilon_i h(A_H e_i, e_i) = \lambda H \tag{4.2}$$

for some nonzero constant λ, where $tr\nabla A_H = \sum_{i=1}^{2} \epsilon_i (\nabla_{e_i} A_H) e_i$.

LEMMA 4.2. *Let M be a surface of E_s^{m+1} with unit parallel mean curvature vector. If M is of null 2-type, then M is a Chen surface and on the open subset $\mathcal{U} = \{p \in M | (\nabla\alpha)(p) \neq 0\}$ we have*

$$A_3 \nabla\alpha = -\epsilon_3 \alpha \nabla\alpha$$

and

$$\Delta\alpha = (\lambda - \epsilon_3 tr A_3^2)\alpha,$$

where $\nabla\alpha$ stands for the gradient of α.

Proof. Sine e_3 is parallel in the normal bundle, (4.2) implies

$$(\Delta\alpha)e_3 + \epsilon_3\alpha tr A_3^2 e_3 + \alpha a(H) = \lambda\alpha e_3.$$

It follows that $a(H) = 0$ and thus M is a Chen surface. We also have $\Delta\alpha = (\lambda - \epsilon_3 tr A_3^2)\alpha$. Since $tr A_{DH} = A_3\nabla\alpha$, we obtain from (2.4) and (4.1)

$$A_3\nabla\alpha = -\epsilon_3\alpha\nabla\alpha$$

on $\mathcal{U}$. □

LEMMA 4.3. *Under the same hypothesis of Lemma 4.2, the mean curvature vector field H is parallel in the normal bundle.*

Proof. We choose a local orthonormal frame $\{e_1, e_2, e_3, \cdots, e_{m+1}\}$ of E_s^{m+1} such that e_1 and e_2 are tangent to M, $e_3, \cdots, e_{m+1}$ are normal to M and $H = \alpha e_3$. Suppose the open subset $\mathcal{U} = \{p \in M | (\nabla\alpha)(p) \neq 0\}$ is not empty. By Lemma 4.2, we see that $-\epsilon_3\alpha$ is a principal curvature of A_3 on $\mathcal{U}$ and thus the other principal curvature is $3\epsilon_3\alpha$ on $\mathcal{U}$.

If M is spacelike, we choose e_1 in the direction of $\nabla\alpha$ on $\mathcal{U}$. We now show that $\nabla\alpha$ cannot be null even if M is Lorentzian. Let M be Lorentzian. Then, the Weingarten map A_3 has one of the following forms ([9]):

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \mu & 0 \\ 1 & \mu \end{pmatrix}, \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for some functions $\lambda_1, \lambda_2, \mu, a$ and b, where the first and the last representations are induced by an orthonormal frame and the second one is obtained by an pseudo-orthonormal frame $\{e_1, e_2\}$ on M satisfying $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = 1$. Since one principal curvature of M is $-\epsilon_3\alpha$, A_3 cannot have the second and third forms. Hence, M is diagonalizable on $\mathcal{U}$. Thus, we may assume that the unit vector field e_1 tangent to M is parallel to $\nabla\alpha$ on $\mathcal{U}$ if M is spacelike or Lorentzian. Therefore, we may put

$$\omega_3^1 = \epsilon_3\alpha\omega^1, \qquad \omega_3^2 = -3\epsilon_3\alpha\omega^2, \tag{4.3}$$

$$(4.4)\qquad d\alpha = (e_1\alpha)\omega^1.$$

Taking the exterior differentiation of the first equation of (4.3) and making use of the structure equations, we obtain

$$(4.5)\quad d\omega_3^1 = \omega_3^2 \wedge \omega_2^1 + \sum_{r=4}^{m+1} \omega_3^r \wedge \omega_r^1 = -3\epsilon_3\alpha\omega^2 \wedge \omega_2^1 = -3\epsilon_3\alpha d\omega^1.$$

If we use (4.4), we have

$$d(\alpha\omega^1) = \alpha d\omega^1.$$

The last two equations give

$$d\omega^1 = 0.$$

By the Poincaré lemma, ω^1 can be locally written as

$$(4.6)\qquad \omega^1 = du$$

where u is a smooth function. Similarly, by taking the exterior differentiation of the second equaton of (4.3) and using $De_3 = 0$, we may get

$$(4.7)\qquad \omega_1^2(e_2) = -\frac{(3e_1\alpha)}{4\alpha}.$$

We also have from (4.6)

$$(4.8)\qquad \omega_2^1(e_1) = 0.$$

On the other hand, we see from (4.4) and (4.6)

$$(4.9)\qquad d\alpha \wedge du = 0.$$

Thus, α is a function of u, that is, $\alpha = \alpha(u)$. Therefore, (4.4) and (4.7) give rise to

$$(4.10)\qquad d\alpha = \alpha'(u)du, \qquad e_1(\alpha) = \alpha'$$

and

$$4\alpha\omega_1^2 = -3\alpha'\omega^2. \tag{4.11}$$

Taking the exterior differentiation of (4.11), we get

$$21(\alpha')^2 - 48\epsilon_1\epsilon_3\alpha^4 + 16\epsilon_2\alpha^2 K - 12\alpha\alpha'' = 0 \tag{4.12}$$

where K denotes the Gaussian curvature of M. Since $H = \alpha e_3$ and M is a Chen surface by Lemma 4.2, we have

$$trA_r = 0 \quad \text{and} \quad trA_3A_r = 0 \quad (r = 4, \cdots, m+1).$$

It follows that

$$h_{11}^r = h_{22}^r = 0 \tag{4.13}$$

where we have put $A_r = (h_{ij}^r)$ for $r = 4, \cdots, m+1$.

On the other hand, the equation (2.8) of Ricci is rewritten as

$$\langle K^N(X,Y)e_r, e_s\rangle = \langle [A_r, A_s]X, Y\rangle$$

for $r, s = 3, 4, \cdots, m+1$, where K^N denotes the normal curvature tensor field associated with the normal connection D and X, Y are tangent vector fields to M. Since $De_3 = 0$, we have from the last equation

$$[A_3, A_r] = 0 \tag{4.14}$$

for all $r = 3, \cdots, m+1$. If we make use of (4.3), (4.13) and (4.14), then we obtain

$$h_{12}^r = 0$$

for $r = 4, \cdots, m+1$. Thus we have $A_r = 0$ for $r = 4, \cdots, m+1$ and

$$K = \sum_3^{m+1} det(h_{ij}^r) = det(h_{ij}^3) = -3\alpha^2.$$

Consequently, (4.12) becomes

$$7(\alpha')^2 - 16\alpha^4(\epsilon_1\epsilon_3 + \epsilon_2) - 4\alpha\alpha'' = 0. \tag{4.15}$$

Let $y = (\alpha')^2$. Then the equation (4.15) is reduced to the following ordinary differential equation of first order :

$$7y - 16\alpha^4(\epsilon_1\epsilon_3 + \epsilon_2) - 2\alpha y' = 0, \tag{4.16}$$

where y' stands for the derivative of y with respect to the variable α. Then, the solution of this differential equation is given by

$$y = (\alpha')^2 = C\alpha^{\frac{7}{2}} - 16\alpha^4(\epsilon_1\epsilon_3 + \epsilon_2) \tag{4.17}$$

where C is a constant.

Lemma 4.2 and (4.3) imply

$$\Delta\alpha = (\lambda - 10\epsilon_3\alpha^2)\alpha. \tag{4.18}$$

On the other hand, if we use the definition of $\Delta\alpha$ together with (4.4), (4.7) and (4.8), we can obtain

$$4\alpha\Delta\alpha = -4\epsilon_1\alpha\alpha'' + 3\epsilon_1(\alpha')^2. \tag{4.19}$$

Combining (4.18) and (4,19), we get

$$4\epsilon_1\alpha\alpha'' - 3\epsilon_1(\alpha')^2 + 4\alpha^2(\lambda - 10\epsilon_3\alpha^2) = 0. \tag{4.20}$$

Together with (4.15) and (4.20) we get

$$\epsilon_1(\alpha')^2 + \alpha^2(\lambda - 10\epsilon_3\alpha^2) - 4\epsilon_1\alpha^4(\epsilon_1\epsilon_3 + \epsilon_2) = 0. \tag{4.21}$$

Comparing (4.17) and (4.21), we can conclude that α is constant on each component of $\mathcal{U}$ which is contrary to our assumption. Therefore, the open subset $\mathcal{U}$ is empty and hence the mean curvature of M is constant. Automatically, the mean curvature vector H is parallel in the normal bundle. It completes the proof. □

We now prove

THEOREM 4.4. *Let M be a surface of E_1^4. If M is of null 2-type with unit parallel mean curvature vector, then M is an open portion of a B-scroll in E_1^3, $E_1^1 \times S^1(r)$, $S_1^1(r) \times R$, $H^1(r) \times R$ or an extended B-scroll in E_1^4.*

Proof. Let M be Lorentzian in E_1^4. As before, we choose one of unit normal vector fields e_3 as $H = \alpha e_3$ which is the mean curvature vector field of M in E_1^4. Suppose that the shape operator A_3 associated with e_3 is diagonalizable. Thus, we may choose an orthonormal frame $\{e_1, e_2\}$ so that

$$A_3 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \qquad A_4 = \begin{pmatrix} \mu & \delta \\ -\delta & -\mu \end{pmatrix},$$

where A_4 is the shape operator of M associated with the normal vector field e_4 orthogonal to e_3. We obtain from the above A_3 and A_4

$$\begin{aligned} \omega_3^1 &= -\beta\omega^1, \quad \omega_3^2 = -\gamma\omega^2, \\ \omega_4^1 &= -\mu\omega^1 - \delta\omega^2, \quad \omega_4^2 = \delta\omega^1 + \mu\omega^2. \end{aligned}$$

Since e_3 is parallel, we have

$$(\beta - \gamma)\delta = 0.$$

Suppose that an open subset $\mathcal{O} = \{p \in M | (\beta - \gamma)(p) \neq 0\} \neq \phi$. Then, $\delta = 0$ on $\mathcal{O}$. Since M is of null 2-type, $\Delta H = \lambda H$ for some real number $\lambda \neq 0$. That implies

$$\lambda = \beta^2 + \gamma^2 \qquad \text{and} \qquad \mu = 0.$$

Consequently, A_4 is identically zero on $\mathcal{O}$. Thus, the reduction theorem can be applied in such a way that $\mathcal{O}$ lies in E_1^3. According to Theorm 3.1 in [6], one of β and γ must be zero. Hence, each component of $\mathcal{O}$ lies in $E_1^1 \times S^1(r)$ or $S_1^1(r) \times \mathbb{R}$ for some constant r. By continuity, a component C of $\mathcal{O}$ must be M if $C \neq \phi$.

If $\mathcal{O} = \phi$, that is, $\beta = \gamma = \alpha$, then M is pseudo-umbilical in E_1^4. Since the mean curvature vector is parallel in the normal bundle, M is minimal in a hypersphere $S_1^3(c, r)$ for some $c \in E_1^4$ and $r > 0$ due to B.-Y. Chen ([2]). Therefore, M is of 1-type and hence this case cannot occur.

We now suppose that there is a point $p \in M$ such that A_3 is not diagonalizable at p. The, we can choose a pseudo-orthonormal frame $\{A, B\}$ on a neighborhood W of p so that A_3 takes the form

$$\begin{pmatrix} \alpha & 0 \\ k & \alpha \end{pmatrix} \quad (k \neq 0).$$

We put

$$e_1 = \frac{A+B}{\sqrt{2}}, \quad e_2 = \frac{A-B}{\sqrt{2}}.$$

Then, $\{e_1, e_2\}$ is an orthonormal frame on W. A_3 and A_4 take the form

$$A_3 = \begin{pmatrix} \alpha + \frac{k}{2} & \frac{k}{2} \\ -\frac{k}{2} & \alpha - \frac{k}{2} \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} \beta & \gamma \\ -\gamma & -\beta \end{pmatrix}$$

in the frame $\{e_1, e_2\}$ for some functions β and γ on W. Since the normal connection is flat, we may assume that $\beta = \gamma$.

Since $\nabla_A B$ and $\nabla_B A$ are parallel to B and A respectively, the equation of Codazzi

$$\nabla_B(A_3 A) - A_3 \nabla_B A = \nabla_A(A_3 B) - A_3 \nabla_A B$$

gives

$$k \nabla_B B = A_3 \nabla_B A - \alpha \nabla_B A - B(k) B$$

which is parallel to B. Therefore, we have

$$\tilde{\nabla}_B B = \nabla_B B = f B$$

for some function f because of $\langle A_3 B, B \rangle = \langle A_4 B, B \rangle = 0$. Thus, the integral curves of B are straight lines. Let $\gamma(s)$ be an integral curve of A and let $C(s) = e_3(\gamma(s))$ and $D(s) = e_4(\gamma(s))$. Since the normal connection is flat, we get

$$\dot{A}(s) = k_1(s) A(s) - k C(s) - 2\beta(s) D(s),$$

$$\dot{B}(s) = -k_1(s) B(s) - \alpha C(s)$$

$$\dot{C}(s) = \tilde{\nabla}_A e_3 = -\alpha A(s) - kB(s)$$

and

$$\dot{D}(s) = \tilde{\nabla}_A e_4 = -2\beta(s)B(s).$$

Therefore, $x(s,t) = \gamma(s) + tB(s)$ is a parametrization of each connected component of W which is an extended B-scroll in E_1^4.

For the space-like surface of null 2-type with unit parallel mean curvature vector in E_1^4, M is given by an open portion of $H^1(r) \times R$ if we use the result of [5]. It completes our proof. □

From this theorem, we have

THEOREM 4.5. *The only full null 2-type surface with unit parallel mean curvature vector in 4-dimensional Minkowski space-time is an open portion of an extended B-scroll.*

References

[1] Chen, B.-Y., *Total mean curvature and submanifolds of finite type*, World Scientific, 1984.

[2] Chen, B.-Y., *Finite-type submanifolds in pseudo-Euclidean spaces and applications*, Kodai Math. J. **8** (1985), 358–374.

[3] Chen, B.-Y., *Finite type pseudo-Riemannian submanifolds*, Tamkang J. Math. **17** (1986), 137–151.

[4] Chen, B. -Y., and Song, H., *Null 2-type surfaces in Minkowski space-time*, Algebras, Groups and Geom. **6** (1989), 333-352.

[5] Férrandez, A. and Lucas, P., *On Surfaces in the 3-dimensional Lorentz-Minkowski space*, Pacific J. Math. **152** (1992), 93-100.

[6] Graves, L. K. Codimension one isemetric immersions beween Lorentz spaces, Trans. Amer. Math. Soc. **252** (1979), 367-392.

[7] Li, S.-J., *2-type pseudo-umbilical submanifolds*, Bull. Soc. Math.Belg-Tijdschr. Belg. Gen. **43** (1991), 69–74.

[8] Li, S.-J., *Null 2-type surfaces in E^m with parallel normalized mean curvature vector*, Math. J. Toyama Univ. **7** (1994), 23–30.

[9] Magid, M., *Lorentzian isoparametric hypersurfaces*, Pacific J. Math. **118** (1985), 165–190.

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