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ANGULAR ESTIMATIONS OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. In the present paper, we investigate some argument properties of certain analytic functions and the integral preserving properties in a sector. Our results include several previous results as special cases.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If f and g are analytic in U, we say that g is subordinate to f, written $g \prec f$ or $g(z) \prec f(z)$, if f is univalent in U, $g(0) = f(0)$ and $g(U) \subseteq f(U)$. A function f of $\mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α, if

$$Re\{\frac{zf'(z)}{f(z)}\} > \alpha \ \ (0 < \alpha \leq 1, \ z \in U).$$

The class $\mathcal{S}^*$ of starlike functions is identified by $\mathcal{S}^*(0) = \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(m, M)$ if

$$|\frac{zf'(z)}{f(z)} - m| < M \ \ (z \in U, \ |m-1| < M \leq m).$$

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The class $\mathcal{S}^*(m, M)$ was introduced by Jakubowski[3]. It is clear that $m > \frac{1}{2}$ and $\mathcal{S}(m, M) \subset \mathcal{S}^*(m - M) \subset \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha, \beta)$ if there is a starlike function g of order α such that

$$Re\{\frac{zf'(z)}{g(z)}\} > \beta \ \ (0 \leq \beta < 1, \ z \in U).$$

Kaplan[4] proved that every $f \in \mathcal{C}(0, 0)$, the class of close-to-convex functions, is univalent. Also, $\mathcal{C}(\alpha, \beta)$ provides an interesting generalization of the class of close-to-convex functions[13].

Many authors[1,7,8] have studied the integral operators of the form

$$I_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \tag{1.2}$$

where c is a suitably chosen real constant and f belongs to some favoured classes of univalent functions. In particular, Kumar and Shukla[6] showed that the integral operator $I_c(f)$ defined by (1.2) maps $\mathcal{S}(m, M)$ into itself for $c \geq -(m - M)$.

In the present paper, we give some argument properties of certain analytic functions and the integral operator defined by (1.2). We also generalize the previous results of Bulboacă[2], Libera[7], Owa and Srivastava[11] and Sakaguchi[12].

2. Main results

In proving our main results, we shall need the following lemmas.

LEMMA 1 ([9]). *Let $h \in \mathcal{K}$, the class of convex functions in U and let $\lambda(z)$ be analytic in U with $Re\lambda(z) \geq 0$. If $p(z)$ is analytic in U and $p(0) = h(0)$, then*

$$p(z) + \lambda(z) z p'(z) \prec h(z) \quad (z \in U)$$

implies

$$p(z) \prec h(z) \quad (z \in U).$$

LEMMA 2 ([10]). *Let $p(z)$ be analytic in U, $p(0) = 1$, $p(z) \neq 0$ in U. Suppose that there exists a point $z_0 \in U$ such that*

$$\left|arg\ p(z)\right| < \frac{\pi\beta}{2} \text{ for } |z| < |z_0|$$

and

$$\left|arg\ p(z_0)\right| = \frac{\pi\beta}{2},$$

where $\beta > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \text{ when } arg\ p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \text{ when } arg\ p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia \ (a > 0).$$

LEMMA 3 ([5,6]). *The function f of the form (1.1) belongs to $\mathcal{S}(m, M)$ if and only if there exists a function w regular in U which satisfies $w(0) = 0, |w(z)| < 1$ for $z \in U$ and*

$$\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 - Bw(z)} \quad (z \in U), \tag{2.1}$$

where $A = (M^2 - m^2 + m)/M$ and $B = (m-1)/M$.

With the help of Lemma 1 and Lemma 2, we now derive

THEOREM 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(m, M)$. If*

$$\left|arg\left((1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \ (\gamma \geq 0,\ 0 \leq \beta < 1,\ 0 < \delta \leq 1),$$

then

$$\left|arg\left(\frac{f(z)}{g(z)} - \beta\right)\right| < \frac{\pi\eta}{2},$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}Tan^{-1}\left(\frac{\gamma\eta sin\frac{\pi}{2}(1 - \frac{2}{\pi}Sin^{-1}\frac{M}{m})}{m + M + \gamma\eta cos\frac{\pi}{2}(1 - \frac{2}{\pi}Sin^{-1}\frac{M}{m})}\right). \tag{2.2}$$

Proof. Let us put

$$p(z) = \frac{1}{1-\beta}\Big(\frac{f(z)}{g(z)} - \beta\Big).$$

Then $p(z)$ is analytic in U with $p(0) = 1$. By a simple calculation, we have

$$\frac{1}{1-\beta}\Big(\frac{f'(z)}{g'(z)} - \beta\Big) = p(z) + \frac{g(z)}{zg'(z)} zp'(z).$$

Therefore we obtain

$$(1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)} - \beta = (1-\beta)\Big(p(z) + \frac{\gamma g(z)}{zg'(z)} zp'(z)\Big).$$

Applying the assumption and Lemma 1 with $\lambda(z) = \gamma g(z)/zg'(z)$, we see that $Rep(z) > 0$ in U and hence $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that

$$\Big|arg\ p(z)\Big| < \frac{\pi\eta}{2} \text{ for } |z| < |z_0|$$

and

$$\Big|arg\ p(z_0)\Big| = \frac{\pi\eta}{2},$$

then, from Lemma 2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$k \geq \frac{1}{2}\Big(a + \frac{1}{a}\Big) \text{ when } arg\ p(z_0) = \frac{\pi\eta}{2}$$

and

$$k \leq -\frac{1}{2}\Big(a + \frac{1}{a}\Big) \text{ when } arg\ p(z_0) = -\frac{\pi\eta}{2}$$

where

$$p(z_0)^{\frac{1}{\eta}} = \pm ia \ (a > 0).$$

Since $g \in \mathcal{S}^*(m, M)$, we have

$$\frac{zg'(z)}{g(z)} = \rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} m - M < \rho < m + M \\ -\frac{2}{\pi}Sin^{-1}\frac{M}{m} < \phi < \frac{2}{\pi}Sin^{-1}\frac{M}{m}. \end{cases}$$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = ia (a > 0)$. Then we obtain

$$\begin{aligned} &arg\left((1-\gamma)\frac{f(z_0)}{g(z_0)} + \gamma\frac{f'(z_0)}{g'(z_0)} - \beta\right) \\ &= arg\Big((1-\beta)p(z_0)\Big(1 + \frac{\gamma g(z_0)}{zg'(z_0)}\frac{zp'(z_0)}{p(z_0)}\Big)\Big) \\ &= arg\ p(z_0) + arg\left(1 + \frac{\gamma g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \frac{\pi\eta}{2} + arg\left(1 + \gamma(\rho e^{i\frac{\pi\phi}{2}})^{-1} i\eta k\right) \\ &= \frac{\pi\eta}{2} + Tan^{-1}\Big(\frac{\gamma\eta k sin\frac{\pi}{2}(1-\phi)}{\rho + \gamma\eta k cos\frac{\pi}{2}(1-\phi)}\Big) \\ &\geq \frac{\pi\eta}{2} + Tan^{-1}\Big(\frac{\gamma\eta sin\frac{\pi}{2}(1 - \frac{2}{\pi}Sin^{-1}\frac{M}{m})}{m + M + \gamma\eta cos\frac{\pi}{2}(1 - \frac{2}{\pi}Sin^{-1}\frac{M}{m})}\Big) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ is given by (2.2). This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\eta}} = -ia\ (a > 0)$. Applying the same method as the above, we have

$$\begin{aligned} arg\left(\frac{f'(z_0)}{g'(z_0)} - \beta\right) &\leq -\frac{\pi\eta}{2} - Tan^{-1}\Big(\frac{\gamma\eta sin\frac{\pi}{2}(1 - \frac{2}{\pi}Sin^{-1}\frac{M}{m})}{m + M + \gamma\eta cos\frac{\pi}{2}(1 - \frac{2}{\pi}Sin^{-1}\frac{M}{m})}\Big) \\ &= -\frac{\pi}{2}\delta, \end{aligned}$$

where δ is given by (2.2), which contradicts the assumption. This completes the proof of our theorem. □

Let us choose $m = N - \alpha(N-1)$ and $M = (1-\alpha)$, where $N \geq 1$ and $0 \leq \alpha < 1$. Then $|m-1| < M \leq m$, $A = \alpha/N + (1-2\alpha)$ and

$B = 1 - 1/N$ in Lemma 3. Now as $N \to \infty, A \to 1 - 2\alpha$ and $B \to 1$. In this case, the relation (2.1) reduces to

$$\frac{zf'(z)}{f(z)} = \frac{1 + (1 - 2\alpha)w(z)}{1 - w(z)} \quad (z \in U),$$

which is a necessary and sufficient condition for f to be in $\mathcal{S}^*(\alpha)$. Hence we have the following

COROLLARY 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(\alpha)$. If*

$$\left|arg\left((1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \; (\gamma \geq 0, \; 0 \leq \beta < 1, \; 0 < \delta \leq 1),$$

then

$$\left|arg\left(\frac{f(z)}{g(z)} - \beta\right)\right| < \frac{\pi\delta}{2}.$$

REMARK 1. (i) Putting $\alpha = 0$ and $\delta = 1$ in Theorem 1, we obtain the result of Bulboacă[2].

(ii) For the case $\alpha = \beta = 0$ and $\gamma = \delta = 1$, Corollary 1 is the result by Sakaguchi[12].

Taking $m = 1, \; M \to 0, \; \gamma = 1, \; \beta = 0$ and $g(z) = z$ in Theorem 1, we have

COROLLARY 2. *Let $f \in \mathcal{A}$. If*

$$|arg\, f'(z)| < \frac{\pi\delta}{2} \; (0 < \delta \leq 1),$$

then

$$\left|arg\, \frac{f(z)}{z}\right| < \frac{\pi\eta}{2},$$

where η $(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} Tan^{-1}\eta.$$

By using the same technique in the proof of Theorem 1, we have

THEOREM 2. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(m, M)$. If*

$$\left|arg\left(\beta - \left((1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)}\right)\right)\right| < \frac{\pi\delta}{2} \quad (\gamma \geq 0,\ \beta > 1,\ 0 < \delta \leq 1),$$

then

$$\left|arg\left(\beta - \frac{f(z)}{g(z)}\right)\right| < \frac{\pi\eta}{2},$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equation (2.2).

Letting $m = M$, $m \to \infty$ and $\delta = 1$ in Theorem 2, we have

COROLLARY 3. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$. If*

$$Re\left\{(1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)}\right\} < \beta \quad (\gamma \geq 0,\ \beta > 1),$$

then

$$Re\left\{\frac{f(z)}{g(z)}\right\} < \beta.$$

Next, we prove

THEOREM 3. *Let c be a real number with $c \geq 0$ and let $f \in \mathcal{A}$. If*

$$\left|arg\left(\frac{zf'(z)}{g(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \quad (0 \leq \beta < 1,\ 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}^*(m, M)$, then

$$\left|arg\left(\frac{z(I_c(f))'}{I_c(g)} - \beta\right)\right| < \frac{\pi\eta}{2},$$

where I_c is the integral operator defined by (1.2) and $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$(2.3) \quad \delta = \eta + \frac{2}{\pi}Tan^{-1}\left(\frac{\eta sin\frac{\pi}{2}\left(1 - \frac{2}{\pi}Sin^{-1}\left(\frac{M}{c+m}\right)\right)}{c + m + M + \eta cos\frac{\pi}{2}\left(1 - \frac{2}{\pi}Sin^{-1}\left(\frac{M}{c+m}\right)\right)}\right).$$

Proof. Putting

$$p(z) = \frac{T(z)}{S(z)},$$

where

$$T(z) = \frac{1}{1-\beta}\Big\{z^c f(z) - c\int_0^z t^{c-1}f(t)dt - \beta\int_0^z t^{c-1}g(t)dt\Big\}$$

and

$$S(z) = \int_0^z t^{c-1}g(t)dt,$$

we see that $p(z)$ is analytic in U with $p(0) = 1$. By a simple calculation, we have

$$\begin{aligned}\frac{T'(z)}{S'(z)} &= p(z) + \frac{S(z)}{zS'(z)}zp'(z)\\ &= \frac{1}{1-\beta}\Big(\frac{zf'(z)}{g(z)} - \beta\Big).\end{aligned}$$

Since $g \in \mathcal{S}(m, M)$, $I_c(g) \in \mathcal{S}(m, M)$ [5] and hence $S(z)$ is (possibly many-sheeted) starlike function with respect to the orign. Therefore, from our assumption and Lemma 1, $p(z) \neq 0$ in U. Since $I_c(g) \in \mathcal{S}(m, M)$, we have

$$\frac{zS'(z)}{S(z)} = \frac{z(I_c(g))'}{I_c(g)} + c = \rho e^{i\frac{\pi\phi}{2}}$$

where

$$\begin{cases} c+m-M < \rho < c+m+M,\\ -\frac{2}{\pi}Sin^{-1}(\frac{M}{c+m}) < \phi < \frac{2}{\pi}Sin^{-1}(\frac{M}{c+m}).\end{cases}$$

The remaining part of the proof is similar to that of Theorem 1 and so we omit it. □

Taking $m = N - \alpha(N-1), M = N(1-\alpha)(0 \leq \alpha < 1), N \to \infty$ and $\delta = 1$ in Theorem 3, we obtain the following result of Owa and Srivastava[11].

COROLLARY 4. *If the function f defined by (1.1) is in the class $\mathcal{C}(\alpha, \beta)$, then the integral operator $I_c(f)$ $(c \geq 0)$ defined by (1.2) is also in the class $\mathcal{C}(\alpha, \beta)$.*

REMARK 2. Taking $\alpha = \beta = 0$ and $c = 1$ in Corollary 4, we obtain the result given earlier by Libera[7].

By using the same technique as in proving Theorem 3, we have

THEOREM 4. *Let c be a real number with $c \geq 0$ and let $f \in \mathcal{A}$. If*

$$\left| arg \left(\beta - \frac{zf'(z)}{g(z)} \right) \right| < \frac{\pi\delta}{2} \ (\beta > 1, \ 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}(m, M)$, then

$$\left| arg \left(\beta - \frac{z(I_c(f))'(f)}{I_c(g)} \right) \right| < \frac{\pi\eta}{2},$$

where I_c is the integral operator defined by (1.2) *and $\eta (0 < \eta \leq 1)$ is the solution of the equation* (2.3).

Putting $m = N - \alpha(N-1), M = N(1-\alpha)(0 \leq \alpha < 1), N \to \infty$ and $\delta = 1$ in Theorem 4, we have the following result by Owa and Srivastava[11].

COROLLARY 5. *Let $c \geq 0$ and $f \in \mathcal{A}$. If*

$$Re \left\{ \frac{zf'(z)}{g(z)} \right\} < \beta \ (\beta > 1)$$

for some $g \in \mathcal{S}^*(\alpha)$, then

$$Re \left\{ \frac{z(I_c(f))'}{I_c(g)} \right\} < \beta,$$

where I_c is the integral operator defined by (1.2).

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