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STRONG-EQUIASYMPTOTIC STABILITY OF PERTURBED LINEAR DIFFERENTIAL EQUATIONS WHEN THE PERTURBATION IS LARGE AND DEPENDS ON TIME

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ABSTRACT. We consider a system of perturbed linear differential equation $x' = A(t)x + f(t,x)$ and obtain conditions to ensure the strong-equiasymptotic stability of the zero solution.

1. Introduction

The qualitative investigation of solutions of linear differential equations and their perturbed linear differential equations plays a very important role for handling various problems on mechanical, electronic, control engineering, or economic, and other practical problems. In such qualitative theory, Liapunov's and Perron's stability theorems are most important and popular. Thus a number of authors have been studying many problems concerned with them and presenting numerous properties [1, 2, 4, 6, 9, 10], etc.

Among such properties, the following theorems in [9] are considered for the differential equation

$$x' = A(t)x + f(t,x). \tag{1.1}$$

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THEOREM A. *Let the following conditions hold for the differential equation* (1.1)*:*

(i) $\|f(t,x)\| \leq a(t)\|x\|$

(ii) $a(t)$ *is a continuous nonnegative real-valued function on the interval* $[0,\infty)$ *with* $\int_0^\infty a(t)dt < \infty$.

If the zero solution of $x' = A(t)x$ *is uniformly asymptotically stable, then the zero solution of (1.1) is also uniformly asymptotically stable.*

THEOREM B. *Let the following condition hold for the differential equation* (1.1)*: there exist a sufficiently small* $L > 0$ *and a* $w > 0$ *such that*

$$\|f(t,x)\| \leq L\|x\| \quad \text{for any } x \in B_w$$

where B_w *is the open ball centered at the origin with the radius* w.

If the zero solution of $x' = A(t)x$ *is uniformly asymptotically stable, then the zero solution of* $(1,1)$ *is also uniformly asymptotically stable.*

In a sense, it is natural that, for stable differential equations, the perturbed linear differential equations by very small perturbations are also stable. However, when linear differential equations are considered, if large perturbations would be given, are their perturbed differential equations stable? It is a very important and interesting problem under what conditions the stability is ensured.

The purpose of this paper is to give answers for those questions if the perturbed part satisfies the following hypotheses:

There exist a positive integer m_0 and a positive number r such that

$$\|f(t,x)\| \leq m^p a(t)\|x\|^{\frac{m}{m+1}} \tag{1.2}$$

for all $t \in [0,\infty)$, all $x \in B_r$, and all positive integers $m \geq m_0$, where $-\infty < p < 1$ and $a(t)$ is a continuous nonnegative real-valued function on $[0,\infty)$ and

$$\int_0^\infty a(t)dt < \infty. \tag{1.3}$$

The result is that the zero solution of (1.1) is uniformly stable (stable) if the zero solution of $x' = A(t)x$ is uniformly stable (stable) and if the perturbed part $f(t,x)$ of (1.1) satisfies the hypotheses (1.2) and (1.3). In addition, if the function $a(t)$ satisfies the following condition:

$$\int_0^\infty (t+1)^\gamma a(t)dt < \infty \tag{1.4}$$

for sufficiently small $\gamma > 0$, then the zero solution of (1.1) is unformly stable and quasi- strong- equiasymptotically stable when the zero solution of $x' = A(t)x$ is uniformly asymptotically stable.

We also introduce the definition of finitely uniform stability of zero solution for the equation (1.1) and prove the stability if we replace the condition (1.3) for the function $a(t)$ by the following condition:

$$\lim_{m\to\infty} \frac{m^p}{m+1} \int_0^m a(s)\,ds = 0 \tag{1.5}$$

and if the zero solution of $x' = A(t)x$ is uniformly stable.

The most important result of this paper is following: If the function f satisfies that there are a positive integer m_0 and a positive number γ such that

$$\|f(t,x)\| \leq a(t)\|x\|^{\frac{m}{m+1}} \tag{1.6}$$

for all $t \in [0,\infty)$, all $x \in B_r$, and all positive integers $m \geq m_0$, where $a(t)$ is a bounded continuous function on $[0,\infty)$ such that

$$\lim_{m\to\infty} \frac{1}{m+1} \int_0^m a(s)ds = 0. \tag{1.7}$$

then the zero solution of (1.1) is strong- equiasymptotically stable when the zero solution of $x' = A(t)x$ is uniformly asymptotically stable. Comparing the above result with Theorem A in Section 1, one of the main different points is the integrability conditon (1.3) for the function $a(t)$ on $[0,\infty)$. We have a large class of bounded continuous functions $a(t)$ on $[0,\infty)$ which are not integrable but satisfy the condition (1.7).

To prove the main theorems which we present we use the family of the solutions of the corresponding scalar differential equation to the given differential equation (1.1), and the comparision theorem of integral inequalities.

2. Definitions and preliminaries

Let R^n be the $n-$dimensional Euclidean space $C[X, Y]$ denotes the set of all continuous functions from a topological space X into a topological space Y. Set $B_\delta = \{x \in R^n : \|x\| < \delta\}$ for a positive number δ where $\|\cdot\|$ denotes the usual norm in R^n.

Throughout this paper we assume the following conditions :

- $A(t)$ is a continuous $n \times n$ matrix function on $[0, \infty)$,
- $f(t, x) \in C[[0, \infty) \times R^n, R^n]$,
- $f(t, 0) = 0$ for all $t \in [0, \infty)$.

We denote the solution $x(t) = x(t; t_0, x_0)$ of (1.1) with the initial condition (t_0, x_0).

DEFINITION 1. The zero solution of (1.1) is said to be

$\underline{stable}$ if for any $\epsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that if $\|x_0\| < \delta$, then $\|x(t; t_0, x_0)\| < \epsilon$ for all $t \geq t_0$,

$\underline{uniformly\ stable}$ if the δ in the above definition is independent of t_0,

$\underline{quasi - equiasymptotically\ stable}$ if given any $\epsilon > 0$ and any $t_0 \geq 0$, there exist a $\delta = \delta(t_0) > 0$ and a $T = T(t_0, \epsilon) > 0$ such that if $\|x_0\| < \delta$, then $\|x(t; t_0, x_0)\| < \epsilon$ for all $t \geq t_0 + T$ [cf. 10],

$\underline{quasi - strong - equiasymptotically\ stable}$ if the T in the above definition is independent of t_0,

$\underline{quasi - uniformly\ asymptotically\ stable}$ if there exists a $\delta > 0$ and for any $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that if $\|x_0\| < \delta$, then $\|x(t; t_0, x_0)\| < \epsilon$ for all $t \geq t_0 + T$ and any $t_0 \geq 0$,

$\underline{equiasymptotically\ stable}$ if it is stable and quasi- equiasymptotically stable [cf. 10],

$\underline{strong - equiasymptotically\ stable}$ if it is stable and quasi- strong-equiasymptotically stable,

$\underline{uniformly\ asymptotically\ stable}$ if it is uniformly stable and quasi-uniformly asymptotically stable.

REMARK. From the above definitions first we note that the zero solution of (1.1) is equiasymptotically stable if the zero solution of (1.1) is strong- equiasymptotically stable. Secondly, we see that the zero solution of (1.1) is strong- equiasymptotically stable if the zero solution of (1.1) is uniformly asymptotically stable.

DEFINITION 2. The zero solution of (1.1) is said to be

finitely uniformly stable if for any number t_1 and for any $\epsilon > 0$ there is a $\delta = \delta(t_1, \epsilon) > 0$ which depends only on ϵ and t_1 such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for all t_0 and t with $t_0 \le t \le t_1$.

At the end of this section, we present well known lemmas for integral inequalities and Gronwall inequality.

LEMMA 1. [5] *Let the following condition* (a) *or* (b) *hold for functions* $f(t), g(t) \in C[[t_0, \infty), [0, \infty)]$, *and* $F(t, u) \in C[[t_0, \infty) \times [0, \infty), [0, \infty)]$:

(a) $f(t) - \int_{t_0}^{t} F(s, f(s))ds \le g(t) - \int_{t_0}^{t} F(s, g(s))ds, t \ge t_0$ *and* $F(s, u)$ *is strictly increasing in* u *for each fixed* $s \ge 0$,

(b) $f(t) - \int_{t_0}^{t} F(s, f(s))ds < g(t) - \int_{t_0}^{t} F(s, g(s))ds, t \ge t_0$ *and* $F(s, u)$ *is monotone nondecreasing in* u *for each fixed* $s \ge 0$.

If $f(t_0) < g(t_0)$, *then* $f(t) < g(t)$ *for all* $t \ge t_0$.

LEMMA 2. (Gronwall Inequality)[3] *If* u *and* α *are real valued continuous functions on* $[a, b]$, *and* β *is a nonnegative real valued integrable function on* $[a, b]$ *with*

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)ds$$

for $a \le t \le b$, *then*

$$u(t) \le \alpha(t) + \int_a^t \beta(s)\alpha(s)\left[\exp\left(\int_s^t \beta(\tau)d\tau\right)\right]ds$$

for $a \le t \le b$.

3. Main theorems

THEOREM 1. *Let the differential equation* (1.1) *satisfy the conditions* (1.2) *and* (1.3). *If the zero solution of* $x' = A(t)x$ *is uniformly stable(stable), then the zero solution of* (1.1) *is also uniformly stable(stable), respectively.*

Proof. We only show that the zero solution of (1.1) is uniformly stable. Let $x(t) = x(t; t_1, x_1)$ be a solution of (1.1) with the initial value (t_1, x_1). Then, by the variation of constant formula, we have

$$x(t) = U(t)U^{-1}(t_1)x_1 + \int_{t_1}^{t} U(t)U^{-1}(s)f(s, x(s))ds,$$

where $U(t)$ is the fundamental $n \times n$ solution matrix of $x' = A(t)x$. Since the zero solution of $x' = A(t)x$ is uniformly stable, there is a constant $K \geq 1$ such that

$$\|U(t)U^{-1}(s)\|_n \leq K, \quad \text{for all } t \geq s \geq 0,$$

where $\| \cdot \|_n$ is the $n \times n$ matrix norm. Thus we have

(3.1)

$$\begin{aligned}\|x(t)\| &\leq \|U(t)U^{-1}(t_1)\|_n \|x_1\| + \int_{t_1}^{t} \|U(t)U^{-1}(s)\|_n \|f(s, x(s))\| ds \\ &\leq K\|x_1\| + K \int_{t_1}^{t} \|f(s, x(s))\| ds.\end{aligned}$$

Let ϵ be an arbitrary positive number. We note that

$$\lim_{m \to \infty} \epsilon^{\frac{1}{m+1}} = 1 \quad \text{and} \quad \lim_{m \to \infty} \frac{m^p K \int_0^\infty a(s)ds}{m+1} = 0,$$

since $-\infty < p < 1$ and $\int_0^\infty a(s)ds < \infty$. We also note that there is a positive integer $m_\epsilon (\geq m_0)$ so that

$$\epsilon^{\frac{1}{m_\epsilon+1}} - \frac{(m_\epsilon)^p K \int_0^\infty a(s)ds}{m_\epsilon + 1} > 0$$

and by the assumption (1.2), we can let

$$\|f(t,x)\| \leq (m_\epsilon)^p a(t) \|x\|^{\frac{m_\epsilon}{m_\epsilon+1}}$$

for all $t \in [0, \infty)$ and all $x \in B_r$.

From (3.1), if $\|x(t)\| < r$, then

$$\|x(t)\| - K(m_\epsilon)^p \int_{t_1}^{t} a(s)\|x(s)\|^{\frac{m_\epsilon}{m_\epsilon+1}} ds \le K\|x_1\|. \tag{3.2}$$

Let $y_{m_\epsilon}(t) = y_{m_\epsilon}(t; t_1, y_{m_\epsilon}(t_1))$ be the solution of the scalar differential equation

$$y' = K(m_\epsilon)^p a(t) y^{\frac{m_\epsilon}{m_\epsilon+1}} \tag{3.3}$$

with the initial value $y_{m_\epsilon}(t_1)$. By simple calculation

$$\begin{aligned} y_{m_\epsilon}(t) &= \left\{ \{y_{m_\epsilon}(t_1)\}^{\frac{1}{m_\epsilon+1}} + \frac{K(m_\epsilon)^p}{m_\epsilon+1} \int_{t_1}^{t} a(s)ds \right\}^{m_\epsilon+1} \\ &\le \left\{ \{y_{m_\epsilon}(t_1)\}^{\frac{1}{m_\epsilon+1}} + \frac{(m_\epsilon)^p K}{m_\epsilon+1} \int_{0}^{\infty} a(s)ds \right\}^{m_\epsilon+1}. \end{aligned}$$

Now we take $\delta_0 = \delta_0(\epsilon) > 0$ so that

$$0 < \delta_0 \le \left\{ \epsilon^{\frac{1}{m_\epsilon+1}} - \frac{(m_\epsilon)^p K}{m_\epsilon+1} \int_{0}^{\infty} a(s)ds \right\}^{m_\epsilon+1}.$$

If $y_{m_\epsilon}(t_1) < \delta_0$ we have

$$\begin{aligned} y_{m_\epsilon}(t) &< \left\{ \delta_0^{\frac{1}{m_\epsilon+1}} + \frac{(m_\epsilon)^p K}{m_\epsilon+1} \int_{0}^{\infty} a(s)ds \right\}^{m_\epsilon+1} \\ &\le \epsilon. \end{aligned} \tag{3.4}$$

Now let us take $K\|x_1\| < y_{m_\epsilon}(t_1)$. From (3.2) and (3.3),

$$\begin{aligned} &\|x(t)\| - (m_\epsilon)^p K \int_{t_1}^{t} a(s)\|x(s)\|^{\frac{m_\epsilon}{m_\epsilon+1}} ds \\ &\quad \le K\|x_1\| \\ &\quad < y_{m_\epsilon}(t_1) = y_{m_\epsilon}(t) - (m_\epsilon)^p K \int_{t_1}^{t} a(s)\{y(s)\}^{\frac{m_\epsilon}{m_\epsilon+1}} ds. \end{aligned}$$

Since the function $(m_\epsilon)^p a(t)\|x\|^{\frac{m_\epsilon}{m_\epsilon+1}}$ is monotone non-decreasing in $\|x\|$, if $\|x(t)\| < r$, by Lemma 1 in Section 2,

$$\|x(t)\| < y_{m_\epsilon}(t) \tag{3.5}$$

for all $t \geq t_1 \geq 0$.

To show the uniform stability, we let $\delta = \delta(\epsilon) \equiv \dfrac{\delta_0}{K}$. It follows from (3.4) and (3.5) that if $\|x_1\| < \delta$, then

$$\|x(t)\| < y_{m_\epsilon}(t) < \epsilon, \quad t \geq t_1 \geq 0.$$

□

THEOREM 2. *Let the differential equation* (1.1) *satisfy the conditions* (1.2) *and* (1.4). *If the zero solution of* $x' = A(t)x$ *is uniformly asymptotically stable, then the zero solution of* (1.1) *is uniformly stable and quasi- strong- equiasymptotically stable.*

Proof. The proof of the uniform stability of zero solution of (1.1) follows from the proof of Theorem 1. Now we will show that the zero solution of (1.1) is quasi- strong- equiasymptotically stable.

Let $x(t)$ be a solution of (1.1), $g(t) = (t+1)^\gamma$ for some $\gamma > 0$, and let $z(t) = g(t)x(t)$. Then we have the differential equation

$$z' = \left(A(t) + \frac{\gamma}{t+1}\right) z + g(t) f\left(t, \frac{z}{g(t)}\right). \tag{3.6}$$

Let $U(t)$ be the fundamental solution matrix of the differential equation

$$z' = A(t)z + \frac{\gamma}{t+1} z.$$

If $\gamma > 0$ is sufficiently small, by Theorem B in Section 1, the zero solution of the above equation is uniformly asymptotically stable. Then there is a constant $K \geq 1$ such that $\|U(t)U^{-1}(t_1)\|_n \leq K$ for $t \geq t_1 \geq 0$.

Let $z(t) = z(t; t_1, z_1)$ be a solution of (3.6). Then, by the variation of constant formula,

$$\|z(t)\| \leq K\|z_1\| + K \int_{t_1}^{t} g(s) \left\| f\left(s, \frac{z(s)}{g(s)}\right) \right\| ds. \tag{3.7}$$

We can choose a positive integer $m_1 \geq m_0$ such that $\gamma > (m_1+1)^{-1}$ and

$$1 - \frac{(m_1)^p K \int_0^\infty \{g(s)\}^{\frac{1}{m_1+1}} a(s)ds}{m_1+1} > 0.$$

From (3.7) and by the assumption (1.2), if $\|\frac{z(s)}{g(s)}\| < r$, then

$$\|z(t)\| \leq K\|z_1\| + K\int_{t_1}^{t} (m_1)^p \{g(s)\}^{\frac{1}{m_1+1}} a(s) \|z(s)\|^{\frac{m_1}{m_1+1}} ds. \tag{3.8}$$

Let $y_{m_1}(t) = y_{m_1}(t; t_1, y_{m_1}(t_1))$ be a solution of the scalar differential equation

$$y' = K(m_1)^p \{g(t)\}^{\frac{1}{m_1+1}} a(t) y^{\frac{m_1}{m_1+1}} \tag{3.9}$$

with the initial value $(t_1, y_{m_1}(t_1))$. If $K\|z_1\| < y_{m_1}(t_1)$, from (3.8) and (3.9),

(3.10)

$$\begin{aligned} &\|z(t)\| - K\int_{t_1}^{t} (m_1)^p \{g(s)\}^{\frac{1}{m_1+1}} a(s)\|z(s)\|^{\frac{m_1}{m_1+1}} ds \\ &\quad \leq K\|z_1\| \\ &\quad < y_{m_1}(t_1) = y_{m_1}(t) - \int_{t_1}^{t} K(m_1)^p \{g(s)\}^{\frac{1}{m_1+1}} a(s)\{y(s)\}^{\frac{m_1}{m_1+1}} ds. \end{aligned}$$

From the monotonicity of $(m_1)^p\{g(t)\}^{\frac{1}{m_1+1}} a(t)\|x\|^{\frac{m_1}{m_1+1}}$ in $\|x\|$, Lemma 1 in Section 2 and the inequality (3.10)

$$\|z(t)\| < y_{m_1}(t), \quad t \geq t_1 \geq 0 \tag{3.11}$$

if $K\|z_1\| < y_{m_1}(t_1)$. Let us take $\delta_0 = \delta_0(1) > 0$ such that

$$0 < \delta_0 \leq \left\{1 - \frac{(m_1)^p K \int_0^\infty \{g(s)\}^{\frac{1}{m_1+1}} a(s)ds}{m_1+1}\right\}^{m_1+1}$$

and let $y_{m_1}(t_1) < \delta_0$. Then we have

$$
\begin{aligned}
y_{m_1}(t) &< \left\{ \delta_0^{\frac{1}{m_1+1}} + \frac{(m_1)^p K}{m_1+1} \int_0^{\infty} \{g(s)\}^{\frac{1}{m_1+1}} a(s)\, ds \right\}^{m_1+1} \\
&\leq 1. \qquad (3.12)
\end{aligned}
$$

Let $K\|z_1\| < y_{m_1}(t_1) < \delta_0$. Then, from (3.11) and (3.12),

$$\|z(t)\| < 1, \quad t \geq t_1 \geq 0.$$

Now we note that for any $\epsilon > 0$ there is a $T = T(\epsilon) > 0$ such that $\frac{1}{(T+1)^\gamma} < \epsilon$. Consider $\delta = \frac{\delta_0}{K(1+t_1)^\gamma} > 0$, where δ depends on t_1. Then $\|x_1\| < \delta$ implies that $\|z_1\| < \frac{\delta_0}{K}$, that is, $K\|z_1\| < \delta_0$. Now we may take $y_{m_1}(t_1)$ such that $K\|z_1\| < y_{m_1}(t_1) < \delta_0$. Thus we have

$$\|x(t)\| < \epsilon \quad \text{for any} \quad t \geq t_1 + T$$

since

$$\|x(t)\| < \frac{1}{(t+1)^\gamma} \leq \frac{1}{(t_1+T+1)^\gamma} \leq \frac{1}{(T+1)^\gamma} < \epsilon.$$

Hence, the zero solution of (1.1) is uniformly stable and quasi-strong- equiasymptotically stable. □

THEOREM 3. *Let the differential equation* (1.1) *satisfy the conditions* (1.2) *and* (1.5). *If the zero solution of* $x' = A(t)x$ *is uniformly stable, then the zero solution of* (1.1) *is finitely uniformly stable.*

Proof. Let us choose any two numbers t_1 and t_2 with $0 \leq t_1 \leq t_2$, and let $x(t) = x(t; t_1, x_1)$ be the solution of (1.1) with the initial value (t_1, x_1). Then, by the variation of constant formula, we have

$$\|x(t)\| \leq K\|x_1\| + K \int_{t_1}^{t} \|f(s, x(s))\| ds \qquad (3.13)$$

for some constant $K > 0$.

Let ϵ be an arbitrary positive number. Since

$$\lim_{m\to\infty} \epsilon^{\frac{1}{m+1}} = 1 \quad \text{and} \quad \lim_{m\to\infty} \frac{m^p K \int_0^m a(s)ds}{m+1} = 0,$$

there is a positive integer m_ϵ with $m_\epsilon \geq m_0$ so that

$$\epsilon^{\frac{1}{m_\epsilon+1}} - \frac{(m_\epsilon)^p K \int_0^{m_\epsilon} a(s)ds}{m_\epsilon + 1} > 0.$$

Without loss of generality we assume that

$$\epsilon^{\frac{1}{m+1}} - \frac{m^p K \int_0^m a(s)ds}{m+1} > 0 \qquad \text{if } m \geq m_\epsilon.$$

By the assumption (1.2) we can also let

$$\|f(t,x)\| \leq m^p a(t) \|x\|^{\frac{m}{m+1}}$$

for all $t \in [0,\infty)$, all $x \in B_r$, and all positive integers $m \geq m_\epsilon$.

From (3.13), if $\|x(t)\| < r$, then

$$\|x(t)\| - m^p K \int_{t_1}^t a(s)\|x(s)\|^{\frac{m}{m+1}} ds \leq K\|x_1\| \qquad \text{if } m \geq m_\epsilon \tag{3.14}$$

Let $y_m(t) = y_m(t; t_1, y_m(t_1))$ be a solution of the scalar differential equation

$$y' = K m^p a(t) y^{\frac{m}{m+1}}, \tag{3.15}$$

for $m \geq m_\epsilon$. By simple calculation

$$y_m(t) \leq \left\{ \{y_m(t_1)\}^{\frac{1}{m+1}} + \frac{m^p K}{m+1} \int_0^m a(s)ds \right\}^{m+1}$$

if $t \leq m$ and $m \geq m_\epsilon$.

We choose m so that $t_2 < m$ and $m_\epsilon \leq m$, and we can take $\delta_0 = \delta_0(\epsilon, t_2) > 0$ so that

$$0 < \delta_0 \leq \left\{ \epsilon^{\frac{1}{m+1}} - \frac{m^p K}{m+1} \int_0^m a(s)ds \right\}^{m+1}.$$

Let $y_m(t_1) < \delta_0$. Then we have

$$y_m(t) < \left\{ \delta_0^{\frac{1}{m+1}} + \frac{m^p K}{m+1} \int_0^m a(s)ds \right\}^{m+1}$$

$$\leq \epsilon \tag{3.16}$$

if $t < m$.

If $K\|x_1\| < y_m(t_1)$ with $m \geq m_\epsilon$, from (3.14) and (3.15),

$$\begin{aligned} \|x(t)\| - m^p K \int_{t_1}^t a(s) \|x(s)\|^{\frac{m}{m+1}} ds \\ &\leq K\|x_1\| \\ &< y_m(t_1) = y_m(t) - m^p K \int_{t_1}^t a(s) \{y(s)\}^{\frac{m}{m+1}} ds\,. \end{aligned}$$

From the monotonicity of $m^p a(t) \|x\|^{\frac{m}{m+1}}$ in $\|x\|$ and Lemma 1 in Section 2, if $\|x(t)\| < r$,

$$\|x(t)\| < y_m(t), \tag{3.17}$$

for all t with $t_1 \leq t \leq m$.

Let $\delta = \delta(\epsilon, t_2) \equiv \dfrac{\delta_0(\epsilon, t_2)}{K}$ and $\|x_1\| < \delta$. Then we have, from (3.16) and (3.17),

$$\|x(t)\| < \epsilon$$

for all t with $t_1 \leq t \leq t_2$. Hence the zero solution of (1.1) is finitely uniformly stable. □

THEOREM 4. *Let the differential equation* (1.1) *satisfy the conditions* (1.6) *and* (1.7). *If the zero solution of* $x' = A(t)x$ *is uniformly asymptotically stable, then the zero solution of* (1.1) *is strong-equiasymptotically stable.*

Proof. Let $x(t)$ be a solution of (1.1), and let $Z(t) = g(t)x(t)$, where $g(t) = (t+1)^\gamma$ for some $\gamma > 0$ and all $t \in [0, \infty)$. Then we have the differential equation

$$Z' = \left(A(t) + \frac{\gamma}{t+1} \right) Z + g(t) f\left(t, \frac{Z}{g(t)}\right). \tag{3.18}$$

Consider the differential equation

$$Z' = A(t)Z + \frac{\gamma}{t+1}Z. \tag{3.19}$$

Then there exists a $T_\gamma > 0$ such that the zero solution of (3.19) is uniformly asymptotically stable on $[T_\gamma, \infty)$ by Theorem B in Section 1. Therefore, there exist positive constants K and β such that

$$\|U(t)U^{-1}(t_1)\|_n \le Ke^{-\beta(t-t_1)}$$

for all $t \ge t_1 \ge T_\gamma$, where $U(t)$ is the fundamental solution matrix of (3.19).

Let $Z(t) = Z(t; t_1, Z_1)$ be a solution of (3.18). Then, by the variation of constant formula,

(3.20)

$$\begin{aligned} \|Z(t)\| &\le Ke^{-\beta(t-t_1)}\|Z_1\| + K\int_{t_1}^{t} e^{-\beta(t-s)}(1+s)^{\gamma}\left\| f\left(s, \frac{Z(s)}{g(s)}\right)\right\| ds \\ &\le Ke^{-\beta(t-t_1)}\|Z_1\| + K\int_{t_1}^{t} e^{-\beta(t-s)}(1+s)^{\frac{\gamma}{m+1}} a(s)\|Z(s)\|^{\frac{m}{m+1}} ds \end{aligned}$$

for all $t \ge s \ge t_1 \ge T_\gamma$, and all $m \ge m_0$.

Now we show that the zero solution of (3.18) is uniformly stable on $[T_\gamma, \infty)$. Taking m as ∞ on the inequality (3.20), we obtain

$$\|Z(t)\| \le Ke^{-\beta(t-t_1)}\|Z_1\| + K\int_{t_1}^{t} e^{-\beta(t-s)}a(s)\|Z(s)\|ds$$

for all $t \ge s \ge t_1 \ge T_\gamma$. By the Gronwall's inequality

$$\begin{aligned} \|Z(t)\| &\le Ke^{-\beta(t-t_1)}\|Z_1\| \\ &\quad + K^2e^{-\beta(t-t_1)}\int_{t_1}^{t} a(s)\|Z_1\|\exp\left\{K\int_{s}^{t} e^{-\beta(t-u)}a(u)du\right\} ds \end{aligned}$$

for all $t \ge s \ge t_1 \ge T_\gamma$. By the condition of boundedness for the function $a(t)$ on $[0, \infty)$ there exists a constant $C > 0$ such that

$$\|Z(t)\| \le C\|Z_1\|$$

for all $t \geq t_1 \geq T_\gamma$, where $Z_1 = Z(t_1)$. Therefore, the zero solution of (3.18) is uniformly stable on $[T_\gamma, \infty)$.

Also we note that

$$\|x(t)\| \leq \|Z(t)\|$$

on $[T_\gamma, \infty)$, where $x(t)$ is a solution of (1.1) and $Z(t)$ is a solution of (3.18). That is, the zero solution of (1.1) is stable on $[T_\gamma, \infty)$.

Since the zero solution of (1.1) is finitely uniformly stable from the condition (1.7) and the Theorem 3, the zero solution of (1.1) is stable on $[0, \infty)$.

Finally we show that the zero solution of (1.1) is quasi- strong-equiasymptotically stable. Let $\epsilon > 0$ be given. Since the zero solution of (3.18) is uniformly stable on $[T_\gamma, \infty)$, for $\epsilon = 1$ there exists a $\mu > 0$ such that $\|Z(T_\gamma)\| < \mu$ implies $\|Z(t)\| \leq 1$ for all $t \geq T_\gamma$. From the finite uniform stability of the zero solution of (1.1) there exists a $\delta_1 > 0$ such that $0 \leq t_0 \leq t \leq T_\gamma$ and $\|x(t_0)\| < \delta_1$ imply $\|x(t)\| < \frac{\mu}{(T_\gamma+1)^\gamma}$. Let $\|x(t_0)\| < \delta_1$ with $0 \leq t_0 \leq T_\gamma$. Then

$$\|x(T_\gamma)\| < \frac{\mu}{(T_\gamma + 1)^\gamma},$$

that is, $\|Z(T_\gamma)\| < \mu$. Thus

$$\|x(t)\| \leq \frac{1}{(1+t)^\gamma}$$

for all $t \geq T_\gamma$.

Now we can take $T = T(\epsilon) > 0$ such that $\frac{1}{(1+T)^\gamma} < \epsilon$. Let $\|x(t_0)\| < \delta_1$ with $0 \leq t_0 \leq T_\gamma$. Then

$$\|x(t)\| < \epsilon$$

for all $t \geq t_0 + T$.

Consider the case of $t_0 > T_\gamma$. Since the zero solution of (3.18) is uniformly stable on $[T_\gamma, \infty)$, for $\epsilon = 1$ there exists $\delta_0 > 0$ such that $t_0 > T_\gamma$ and $\|Z(t_0)\| < \delta_0$ imply $\|Z(t)\| < 1$ for all $t \geq t_0$. Let $\delta_2 = \frac{\delta_0}{(1+t_0)^\gamma}$ and let $\|x(t_0)\| < \delta_2$ with $t_0 > T_\gamma$. Then $\|Z(t_0)\| < \delta_0$, and so

$$\|x(t)\| < \frac{1}{(1+t)^\gamma}$$

if $T_\gamma < t_0 \leq t$. Thus we have

$$\|x(t)\| \leq \frac{1}{(1+T)^\gamma} < \epsilon$$

for all $t \geq t_0 + T$ if $\|x(t_0)\| < \delta_2$ and $T_\gamma < t_0$.

If we take $\delta = \min\{\delta_1, \delta_2\}$, the above results complete the proof. □

THEOREM 5. *Let the following conditions hold for the differential equation* (1.1):

(i) *There is a positive integer m_0 such that*

$$\|f(t,x)\| \leq a(t)\|x\|^{\frac{m_0}{m_0+1}}$$

for all $t \in [0,\infty)$ and all $x \in R^n$, where $a(t)$ is continuous on $[0,\infty)$.

(ii) $\lim_{t\to\infty}(1+t)^{\frac{\gamma}{m_0+1}}a(t) = 0$ *for some $\gamma > 0$.*

If the zero solution of $x' = A(t)x$ is uniformly asymptotically stable, then any solution of (1.1) *approaches to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (1.1), and let $Z(t) = g(t)x(t)$, where $g(t) = (t+1)^\gamma$ for some $\gamma > 0$ and all $t \in [0,\infty)$. Then, as before, we have the differential equation

$$Z' = \left(A(t) + \frac{\gamma}{t+1}\right)Z + g(t)f\left(t, \frac{Z}{g(t)}\right). \tag{3.21}$$

Consider the equation

$$Z' = A(t)Z + \frac{\gamma}{t+1}Z. \tag{3.22}$$

Then there exists a $T_\gamma > 0$ such that the zero solution of (3.22) is uniformly asymptotically stable $[T_\gamma, \infty)$ by Theorem B in Section 1. Therefore, there exist positive constants K and β such that

$$\|U(t)U^{-1}(t_0)\|_n \leq Ke^{-\beta(t-t_0)}$$

for all $t \geq T_\gamma$, where $U(t)$ is the fundamental solution matrix of (3.22).

Let $Z(t) = Z(t; t_0, Z_0)$ be a solution of (3.21). Then, by the variation of constant formula,
(3.23)

$$\begin{aligned} \|Z(t)\| &\le Ke^{-\beta(t-t_0)}\|Z_0\| + K\int_{t_0}^{t} e^{-\beta(t-s)}(1+s)^{\gamma}\left\|f\left(s, \frac{Z(s)}{g(s)}\right)\right\| ds \\ &\le Ke^{-\beta(t-t_0)}\|Z_0\| + K\int_{t_0}^{t} e^{-\beta(t-s)}(1+s)^{\frac{\gamma}{m_0+1}} a(s)\|Z(s)\|^{\frac{m_0}{m_0+1}} ds \end{aligned}$$

for all $t \ge s \ge t_0 \ge T_\gamma$.

Now we show that all solution of (3.21) are bounded on $[T_\gamma, \infty)$. Suppose not. Then there exist a solution $Z(t)$ of (3.21) and an increasing sequence $\{t_n\}$ in $[T_\gamma, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \|Z(t_n)\| = \infty$ and $\|Z(s)\| \le \|Z(t_n)\|$ for all $s \in [t_{n-1}, t_n]$ with $n = 2, 3, 4, \ldots$.

Dividing the inequality (3.23) by $\|z(t_n)\|$ on the both sides, we have

$$\begin{aligned} 1 \le\ & Ke^{-\beta(t_n-t_0)}\frac{\|Z_0\|}{\|Z(t_n)\|} \\ & + K\int_{t_0}^{t_n} e^{-\beta(t_n-s)}(1+s)^{\frac{\gamma}{m_0+1}} a(s)\frac{\|Z(s)\|^{\frac{m_0}{m_0+1}}}{\|Z(t_n)\|} ds, \end{aligned}$$

and so, by the condition (ii),

$$\begin{aligned} 1 \le\ & \lim_{n\to\infty} Ke^{-\beta(t_n-t_0)}\frac{\|Z_0\|}{\|Z(t_n)\|} \\ & + \lim_{n\to\infty} K\int_{t_0}^{t_n} e^{-\beta(t_n-s)}(1+s)^{\frac{\gamma}{m_0+1}} a(s)\frac{\|Z(s)\|^{\frac{m_0}{m_0+1}}}{\|Z(t_n)\|} ds \\ & = 0, \end{aligned}$$

which implies contradiction. Hence any solution of (3.21) on $[T_\gamma, \infty)$ is bounded.

Let $x(t)$ be a solution of (1.1). Since the solution $Z(t) = g(t)x(t)$ is bounded on $[T_\gamma, \infty)$, so

$$\lim_{t\to\infty} \|x(t)\| = \lim_{t\to\infty} \frac{\|Z(t)\|}{(1+t)^{\gamma}} = 0$$

Hence the proof is completed. □

4. Applications

Consider a model equation described from the nonlinear oscillation of a material around the equilibrium $(x, x') = (0, 0)$:

$$x'' + a(t)g(x, x')x' + b(t)x = 0, \tag{4.1}$$

where the functions $g : R^2 \to R^1, a : [0, \infty) \to R^1$, and $b : [0, \infty) \to (0, \infty)$, are continuous.

By a classical theorem of A. M. Liapunov [7] the equilibrium of (4.1) is stable if $b(t)$ is bounded for $t \in [0, \infty)$ and

$$2a(t)g(x, x') + \frac{b'(t)}{b(t)} \geq 0 \tag{4.2}$$

for all $|x| \leq \rho, |x'| \leq \rho, t \geq 0$. Here ρ is a positive constant.

From the theorem of L. Hatvani [4], we know that, without the boundness of the function b but with the condition (4.2) and with that the damping coefficient $a(t)$ majorizes coefficient $b(t)$, the equilibrium of (4.1) is stable if the following conditions hold;

(i) for every $v_0 \neq 0$, $|v_0| \leq \rho$ there is an $\eta = \eta(v_0) > 0$ such that $|u| \leq \eta$ implies $g(u, v_0) \neq 0$;

(ii) $2a(t)b(t)g(u, v) + b'(t) \geq 0$ for all $t \in [0, \infty), |u| \leq \rho, |v| \leq \rho$;

(iii) there is a $\theta > 0$ such that $a(t) \geq \theta b(t)$ holds for all $t \in [0, \infty)$.

If $\lim_{t\to\infty} a(t) = 0$ and $b'(t) < 0$ for all $t \in [0, \infty)$, then the result of Hatvani could not apply to the problem (4.1) because the inequalities (ii) and (iii) may not be true for all $t \in [0, \infty)$. In this case, with some additional conditions on the functions a and b and with some order condition for the function g, but without the conditions (ii) and (iii), our stability result shows that the equilibrium of $(4.1)'$ is uniformly stable.

We state the result for the following system

$$x'' + a(t)f(x, x') + b(t)x = 0 \tag{4.1'}$$

The equation (4.1) is the special case of $f(x, x') = g(x, x')x'$.

THEOREM 6. *Suppose that the following conditions are satisfied :*

(i)$'$ *there are a positive integer* m_0 *and a positive number* r *such that*

$$|f(x_1, x_2)| \leq m^p (x_1^2 + x_2^2)^{\frac{m}{2(m+1)}}$$

for all $(x_1, x_2) \in B_r$ *and all positive integers* $m \geq m_0$, *where* $-\infty < p < 1$;

(ii)$'$ $0 < \alpha \leq b(t)$ *and* $b'(t) \leq 0$ *for all* $t \in [0, \infty)$ *where* α *is constant;*

(iii)$'$ $\int_0^\infty |a(t)| dt < \infty$.

Then the equilibrium, $x = x' = 0$ *of* $(4.1)'$ *is uniformly stable.*

Proof. To prove the theorem, we convert the equation $(4.1)'$ to the system. Let $x_1 = x$ and $x_2 = x'$. Then

$$(4.3) \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -a(t) f(x_1, x_2) \end{pmatrix}$$

First, we solve the linear part of (4.3)

$$(4.4) \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and let $v(t, x_1, x_2) = \frac{b(t)}{2} x_1^2 + \frac{1}{2} x_2^2$ which is the total mechanical energy of (4.3).

Since $b(t) \geq \alpha > 0$ and $b'(t) \leq 0$ for all $t \in [0, \infty)$, v is positive definite and decrescent. From the condition $b'(t) \leq 0$ for all $t \in [0, \infty)$, the derivative v' along the solution,

$$v'(t, x_1, x_2) = \frac{b'(t)}{2} x_1^2 \leq 0$$

for all $t \in [0, \infty)$. This means that v' is negative semidefinite. Hence the equilibrium of (4.4) is uniformly stable [7].

From the condition (i)$'$, there are a positive number r and a positive integer m_0 so that

$$|a(t) f(x_1, x_2)| \leq m^p (x_1^2 + x_2^2)^{\frac{m}{2(m+1)}} |a(t)|$$

for all $(x_1, x_2) \in B_r$, all $t \in [0, \infty)$, and all positive integers $m \geq m_0$. With the condition (iii)$'$ and our stablity result of Theorem 1, the equilibrium $(0, 0)$ of $(4.1)'$ is uniformly stable. □

THEOREM 7. *Assume the hypotheses* (i)′ *and* (ii)′ *in Theorem 6. We also assume that*

(iii)″ $\lim_{m\to\infty} \frac{m^p}{m+1} \int_0^m |a(s)|ds = 0$.

Then the equilibrium, $x = x' = 0$ *of (4.1)′ is finitely uniformly stable.*

Proof. We note from Theorem 6 that the zero solution of (4.4) is uniformly stable.

From the condition (i)′ and the condition (iii)″, our stablity result of Theorem 3 implies that the equilibrium $(0,0)$ of (4.1)′ is finitely unifomly stable. □

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