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R^i-SETS AND CONTRACTIBILITY

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ABSTRACT. We introduce R^i-sets and give various relations between R^i-sets and prove that the hyperspace of a metric continuum containing any one of the R^i-sets also contains R^i-set and hence is not contractible.

1. Introduction

In 1980 Czuba [4] introduced three types of R^i-continua ($i = 1, 2, 3$) in the class of dendroids, and proved that any dendroid containing any one of the R^i-continua is not contractible. In 1986 Charatonik [2] claimed to have extended Czuba's results to the class of metric continua and also to have attempted to prove that the hyperspaces of a metric continuum containing an R^i-continuum also contains a certain R^i-continuum. In this paper we introduce R^i-sets and give various relations between R^i-sets and prove that the hyperspace of a metric continuum containing any one of the R^i-sets also contains R^i-set and hence is not contractible.

2. R^i-sets

By a *continuum* we mean a compact connected metric space. For a metric space (X, d) and $A \subset X$ and $\epsilon > 0$, let $N(A, \epsilon) = \{x \in X : d(a, x) < \epsilon$ for some $a \in A\}$. Let 2^X be the collection of all nonempty closed subsets of X, and let $C(X)$ be the collection of all subcontinua of X. Then $C(X) \subset 2^X$. The Hausdorff metric for 2^X is given by $H(A, B) = inf\{\epsilon > 0 : A \subset N(B, \epsilon)$ and $B \subset N(A, \epsilon)\}$ for $A, B \in 2^X$.

DEFINITION 2.1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of subsets of a space X. *Limit superior* of the sequence is the set, denoted by LsX_n, of all points

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$x \in X$ such that each neighborhood of x intersects infinitely many X_n. The *limit inferior* of the sequence is defined to be the set, denoted by LiX_n, of all points $x \in X$ such that each neighborhood of x intersects all but a finite number of X_n. If $LsX_n = LiX_n$, then the *limit* of the sequence to be $LtX_n = LiX_n$, and we say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to LtX_n.

DEFINITION 2.2. A nonempty closed proper subset K of a continuum X is called ;
an R^1-*set* if there exist an open set U containing K and two sequences $\{C_n^i\}_{n=1}^{\infty}$, $i = 1, 2$, of components of U such that $K = LsC_n^1 \cap LsC_n^2$,
an R^2-*set* if there exist an open set U containing K and two sequences $\{C_n^i\}_{n=1}^{\infty}$, $i = 1, 2$, of components of U such that $K = LtC_n^1 \cap LtC_n^2$,
an R^3-*set* if there exist an open set U and a sequence $\{C_n\}_{n=1}^{\infty}$ of components of U such that $K = LiC_n$.

THEOREM 2.3. *Every R^2-set is both an R^1 and R^3-set.*

Proof. Let $K = LtC_n^1 \cap LtC_n^2 \subset U$ be an R^2-set. Then, by definition, we have $LsC_n^i = LtC_n^i$ for each $i = 1, 2$. Hence K is an R^1-set. For each natural number n, let $D_{2n+1} = C_n^1$ and $D_{2n} = C_n^2$. Then it is easy to see that $K = LiD_n$. Hence K is also an R^3-set. □

THEOREM 2.4. *Every R^1-set contains an R^2-set.*

Proof. Let $K = LsC_n^1 \cap LsC_n^2 \subset U$ be an R^1-set. Let $x \in K$. Then there exist two convergent subsequences $\{C_{n_k}^i\}_{k=1}^{\infty}$ of $\{C_n^i\}_{n=1}^{\infty}$, $i = 1, 2$, such that $x \in LtC_{n_k}^1 \cap LtC_{n_k}^2$. Since $LtC_{n_k}^i \subset LsC_n^i$ for each i, $K' = LtC_{n_k}^1 \cap LtC_{n_k}^2$ is a nonempty closed subset of K. Hence K' is an R^2-set. □

The following is an immediate consequence of Theorem 2.3 and Theorem 2.4.

COROLLARY 2.5. *Every R^1-set contains an R^3-set.*

We give examples of spaces with R^i-sets which show that R^i-sets are distinct. We denote the line segment between two points p and q in a Euclidean space by pq.

EXAMPLE A. There is a disconnected R^1-set which is not an R^3-set: Let $S = \{(x, y) : x^2 + y^2 = (\frac{1}{2})^2,\ x \geq 0\}$, $a = (0, 1)$, $u = (0, \frac{3}{4})$, $v = (0, \frac{1}{2})$, and $p = (1, 0)$. For each natural number n, let $a_n = (\frac{1}{n+3}, \frac{n+5}{2n+6})$,

$b_n = (\frac{n+5}{2n+6}, \frac{1}{n+3})$, $p_n = (\frac{n+4}{n+3}, 0)$, $c_n = (\frac{n+5}{2n+6}, -\frac{1}{n+3})$, $d_n = (\frac{1}{n+3}, -\frac{n+5}{2n+6})$, $f_n = (\frac{1}{n+3}, -\frac{3}{4})$. $D_n = \{(x,y) : x^2+y^2 = (\frac{1}{n+3})^2 + (\frac{n+5}{2n+6})^2,\ \frac{1}{n+3} \leq x \leq \frac{n+5}{2n+6}, y \geq 0\}$, and $E_n = \{(x,y) : x^2+y^2 = (\frac{1}{n+3})^2 + (\frac{n+5}{2n+6})^2,\ \frac{1}{n+3} \leq x \leq \frac{n+5}{2n+6}, y \leq 0\}$. Let $Y_1 = S \cup av \cup (\bigcup_{n=1}^{\infty}(aa_n \cup D_n \cup b_n p_n \cup p_n c_n \cup E_n \cup d_n f_n))$, and let Y_2 be the image of Y_1 under the symmetry map s with respect to the origin, and let $X = Y_1 \cup Y_2$. Then X is a continuum.
Let $U = X \setminus \{a, s(a)\}$, and let, for each natural number n, C_n^1 be the component of U containing p_n, and C_n^2 the component of U containing $s(p_n)$. Then $K = LtC_n^1 \cap LtC_n^2$ is the union of the segments uv and $s(u)s(v)$.
Let $U' = \{(x,y) \in X : -\frac{9}{10} < x < \frac{9}{10}$ and $-\frac{9}{10} < y < \frac{9}{10}\}$. Let, for each natural number n, C_{2n+1}^1 be the component of U' containing a_n, C_{2n}^1 the component of U' containing c_n, C_{2n+1}^2 the component of U' containing $s(a_n)$, and C_{2n}^2 the component of U' containing $s(c_n)$. Then $K = LsC_n^1 \cap LsC_n^2 = uv \cup s(u)s(v)$ is an R^1-set. But one can see that K is not an R^3-set.

EXAMPLE B. There is a disconnected R^3-set which is not R^1-set: First we define two continua A and B in the plane. Then we attach these two in certain way to get our space X.

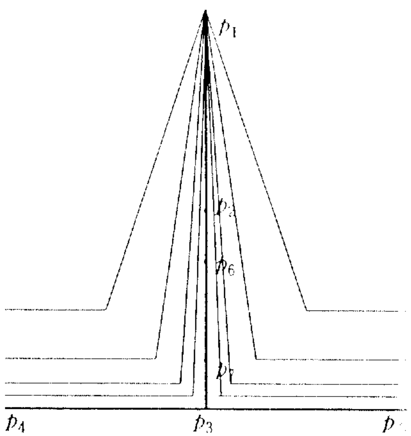

Fig. 1.

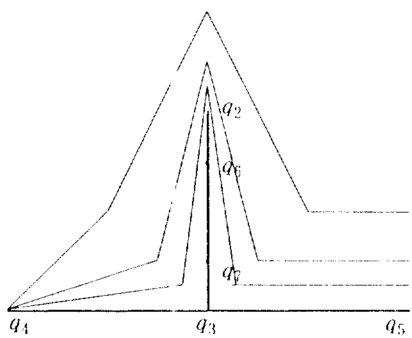

Fig. 2.

Let $p_1 = (0,2)$, $p_2 = (0,1)$, $p_3 = (0,0)$, $p_4 = (-1,0)$, and $p_5 = (1,0)$. For each natural number n, let $a_n = (-\frac{1}{n}, \frac{1}{n})$, $b_n = (-1, \frac{1}{n})$, $c_n = (\frac{1}{n}, \frac{1}{n})$, and $d_n = (1, \frac{1}{n})$. We put $A = p_1p_3 \cup p_4p_5 \cup (\bigcup_{n=1}^{\infty}(p_1 a_n \cup$

$a_nb_n)) \cup (\bigcup_{n=1}^{\infty}(p_1c_n \cup c_nd_n))$(Fig.1). For the continuum B, let $q_i = p_i$, for each $i \in \{2,3,4,5\}$. Let, for each natural number n, $e_n = (-\frac{1}{n}, \frac{1}{n})$, $f_n = (0, 1+\frac{1}{n})$, $g_n = (\frac{1}{n}, \frac{1}{n})$ and $h_n = (1, \frac{1}{n})$. We put $B = q_2q_3 \cup q_4q_5 \cup (\bigcup_{n=1}^{\infty}(q_4e_n \cup e_nf_n \cup f_ng_n \cup g_nh_n))$(Fig.2). Let $p_6 = q_6 = (0, \frac{3}{4})$ and $p_7 = q_7 = (0, \frac{1}{4})$. Let $f : A \times \{0\} \longrightarrow B \times \{1\}$ be the attaching map such that $f : (p_4p_5 \cup p_2p_6 \cup p_7p_3) \times \{0\} \longrightarrow (q_4q_5 \cup q_2q_6 \cup q_7q_3) \times \{1\}$ is given by $f(x,0) = (x,1)$. Let $X = (A \times \{0\}) \bigcup_f (B \times \{1\})$. Then X is a continuum. We let $U = X \setminus \{(p_1, 0), (q_4, 1)\}$. Then U is an open set. Let, for each natural number n, C_{3n+2} be the component of U containing $(a_n, 0)$, C_{3n+1} the component of U containing $(c_n, 0)$, and C_{3n} the component of U containing $(e_n, 1)$. Then we can see that $LiC_n = f((p_2p_6 \cup p_7p_3) \times \{0\}) = (q_2q_6 \cup q_7q_3) \times \{1\}$ is the union of two disjoint arcs q_2q_6 and q_7q_3, which is R^3-set but not an R^1-set.

EXAMPLE C There is a disconnected set which is both an R^1 and R^3-set but not an R^2-set: Let Y be the space in EXAMPLE 4 of [4](see Fig.3).

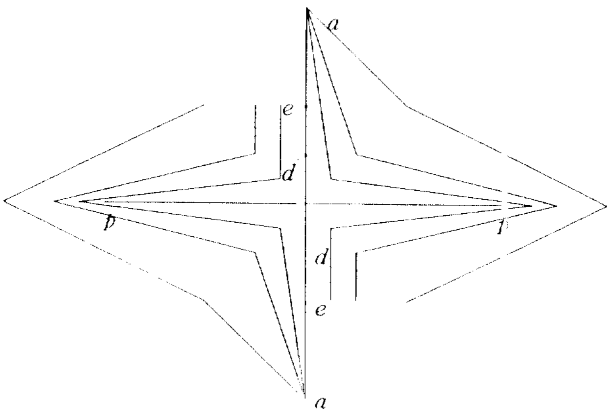

Fig. 3.

Let b_n', c_n', d_n', e_n', a', e' denote the points in Y which are images of b_n, c_n, d_n, e_n, a, e under the symmetry map with respect to the origin respectively. Let X be the space in the above Example B. Let $d' = (0, \frac{1}{4})$ and $d = (0, -\frac{1}{4})$. Let $g : (ed \cup e'd') \times \{0\} \longrightarrow X \times \{1\}$ be the attaching map defined by $g(0,x,0) = (0, x+\frac{1}{2}, 1)$. Let $Z = (Y \times \{0\}) \bigcup_g (X \times \{1\})$. Then Z is a continuum. Let $U = \{(x,y,0) \in Z : -\frac{9}{10} < x < \frac{9}{10}, -\frac{9}{10} <$

$y < \frac{9}{10}\} \cup ((X \times \{1\}) \setminus \{(p_1, 0, 1), (q_4, 1, 1)\})$. Then U is an open set in Z. Let, for each natural number n, C_n^1 be the component of U containing the point $(e_n, 1, 1)$, C_{2n}^2 the component of U containing the point $(b_n', 0)$, and C_{2n+1}^2 the component of U containing the point $(d_n', 0)$. Then the image of the union of the arcs ed and $e'd'$ is an R^1-set.
On the other hand, if we let C_{3n+2} be the component of U containing $(e_n, 1, 1)$, C_{3n+1} the component of U containing $(a_n, 0, 1)$, and C_{3n} the component of U containing $(c_n, 0, 1)$ for each natural number n, then the image $ed \cup e'd'$ of the disjoint two arcs ed and $e'd'$ is the R^3-set, which is clearly not an R^2-set.

EXAMPLE D. Let X be the space given in Example C in [16] (see Fig. 4).

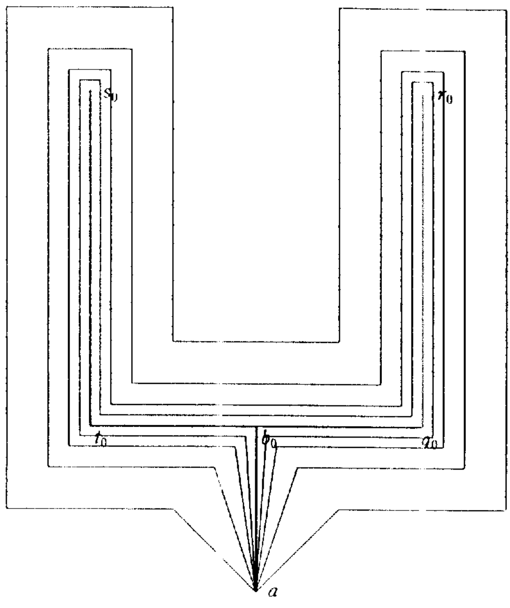

Fig. 4

As noted, this space contains R^1-continuum which is neither R^2-continuum nor R^3-continuum.

We shall find all R^i-sets of X with respect to an open set U and all its components. Let f be a map defined by $f(x, y, z) = (-y, x, z+1)$ for each $(x, y, z) \in X_0$. Let f^0 be the identity, $f^1 = f, f^2 = f \circ f$ and $f^3 = f^2 \circ f$.

Let $U = X \setminus \{f^0(a), f^1(a), f^2(a), f^3(a)\}$. For each natural number n and each $i \in \{0,1,2,3\}$, let C_n^i be the component of U containing the point $f^i(p_n^+)$. Let $b_0 = (1,0,0)$, $b_1 = (0,1,0)$, $b_2 = (-1,0,0)$ and $b_3 = (0,-1,0)$.

We will adopt the following notations: Let $E(i) = \{C_n^i\}_{n=1}^\infty$, $LsE(i) = LsC_n^i$ and $LiE(i) = LiC_n^i$ for each $i \in \{0,1,2,3\}$. For $0 \le i,\ j,\ k \le 3$, let $E(i,j) = \{D_n\}_{n=1}^\infty$ be the sequence obtained by $D_{2n+1} = C_n^i$ and $D_{2n} = C_n^j$ for each natural number n, and let $E(i,j,k) = \{F_n\}_{n=1}^\infty$ be the sequence formed by $F_{3n+2} = C_n^i$, $F_{3n+1} = C_n^j$ and $F_{3n} = C_n^k$ for each natural number n. Since each $E(i)$ converges, for each i, j we have $LsE(i,j) = LsE(j,i), LiE(i,j) = LiE(j,i)$, and $LsE(i,j,k)$ equals the limit superior of E of any rearrangement of i, j, k. Similarly $LiE(i,j,k)$ equals the inferior of E of any rearrangement of $i,\ j\ \ k$.

We find $LsE(i)$ for each i: Notice that $LsE(i) = LiE(i)$ for each i. $LtE(0) = LtC_n^0 = r_0q_0 \cup q_0t_0 \cup t_0s_0 \cup b_0f^0(a)$, $LtE(1) = LtC_n^1 = s_0r_0 \cup r_0q_0 \cup q_0t_0 \cup b_1f(a)$, $LtE(2) = LtC_n^2 = q_0r_0 \cup r_0s_0 \cup s_0t_0 \cup b_2f^2(a)$, and $LtE(3) = LtC_n^3 = r_0s_0 \cup s_0t_0 \cup t_0q_0 \cup b_3f^3(a)$. And none of these limiting sets are an R^i-set.

Let us find $LsE(i) \cap LsE(j)$: $K_1 = LsE(0) \cap LsE(1) = r_0q_0 \cup q_0t_0 \cup \{s_0\}$, $K_2 = LsE(1) \cap LsE(2) = q_0r_0 \cup r_0s_0 \cup \{t_0\}$, $K_3 = LsE(2) \cap LsE(3) = r_0s_0 \cup s_0t_0 \cup \{q_0\}$, $K_4 = LsE(3) \cap LsE(0) = s_0t_0 \cup t_0q_0 \cup \{r_0\}$, $K_5 = LsE(0) \cap LsE(2) = q_0r_0 \cup s_0t_0$, and $K_6 = LsE(1) \cap LsE(3) = r_0s_0 \cup q_0t_0$. All of these sets are R^1-sets, and none of them are connected.

We compute $LiE(i,j)$: $LiE(0,1) = LtE(0) \cap LtE(1) = K_1$, $LiE(1,2) = LtE(1) \cap LtE(2) = K_2$, $LiE(2,3) = LtE(2) \cap LtE(3) = K_3$, $LiE(3,0) = LtE(3) \cap LtE(0) = K_4$, $LiE(0,2) = LtE(0) \cap LtE(2) = K_5$, and $LiE(1,3) = LtE(1) \cap LtE(3) = K_6$.

Observe that $LsE(i,j) = LsE(i) \cup LsE(j)$ for each i and j. Similarly $LsE(i,j,k) = LsE(i) \cup LsE(j) \cup LsE(k)$.

We now compute $LiE(i,j,k)$: $LiE(i,j,k) = LiE(i) \cap LiE(j) \cap LiE(k)$ so that $LiE(0,1,2) = q_0r_0 \cup \{s_0, t_0\} = K_7$, $LiE(1,2,3) = r_0s_0 \cup \{q_0, t_0\} = K_8$, $LiE(2,3,0) = s_0t_0 \cup \{q_0, r_0\} = K_9$, $LiE(3,0,1) = q_0t_0 \cup \{r_0, s_0\} = K_{10}$. All of these are R^3-sets. None of them are connected.

$LiE(0,1,2,3) = \{q_0, r_0, s_0, t_0\} = K_{11}$, which is an R^3-set.

We compute $LsE(i,j) \cap LsE(k)$: Since $LsE(i,j) \cap LsE(i) = LsE(i)$ for all i, j, k and for each fixed k, $LsE(i,j) \cap LsE(k) = LsE(i',j') \cap LsE(k)$ for $k \ne i, i', j, j'$, we have only four types: $K_{12} = LsE(0,1) \cap LsE(2) = q_0r_0 \cup r_0s_0 \cup s_0t_0$, $K_{13} = LsE(0,1) \cap LsE(3) = r_0s_0 \cup s_0t_0 \cup t_0q_0$,

$K_{14} = LsE(0,2) \cap LsE(1) = s_0r_0 \cup r_0q_0 \cup q_0t_0$. $K_{15} = LsE(1,3) \cap LsE(0) = r_0q_0 \cup q_0t_0 \cup t_0s_0$. These are all R^1-continua.

Here we note that $LiE(i,j) \neq LsE(i,j)$ for all $i \neq j$.

Let us compute $LsE(i,j) \cap LsE(i',j')$: There is only one type. $K_{16} = LsE(0,1) \cap LsE(2,3) = LsE(0,2) \cap LsE(1,3) = LsE(0,3) \cap LsE(1,2) = S_0$. If, for any two sequences $E(i,j)$ and $E(i',j')$ such that $i \neq j$, $i \in \{i',j'\}$ or $j \in \{i',j'\}$, then $LsE(i,j) \cap LsE(i',j')$ is not an R^i-set. K_{15} is an R^1-continuum, and one can easily see that it is neither an R^2-continuum nor an R^3-continuum.

Finally we consider $LsE(i,j,k) \cap LsE(i',j')$ and $LsE(i,j,k) \cap LsE(i', j', k')$. None of these are R^i-sets. Since there are no other R^i-set of X, $\{K_n : n = 1,2,3,....,16\}$ is the collection of all R^i-sets of X. We note that for a disconnected R^i-set K_n, no component of K_n is an R^i-continuum.

3. Noncontractibility in hyperspaces

DEFINITION 3.1. [1] A nonempty subset A of a space X is said to be *homotopically fixed* if, for every deformation $h : X \times [0,1] \longrightarrow X$, we have $h(A \times [0,1]) \subset A$.

Clearly if a space contains a subset which is homotopically fixed, it is not contractible.

The next theorem is a generalization of Czuba's Theorem 3 in [3] which was given for the class of dendroids containing R^i-continuum. Our proof is almost identical with that of Czuba.

THEOREM 3.2. *If a continuum X contains an R^i-set K, then K is homotopically fixed.*

Proof. Let $K = LsC_n^1 \cap LsC_n^2 \subset U$ be an R^1-set, where $U = N(K,\epsilon)$ is an ϵ -neighborhood of K and C_n^i are components of U. Suppose, on the contrary, that there exists a deformation $h : X \times [0,1] \longrightarrow X$ for which $h(K \times [0,1]) \setminus K \neq \emptyset$, i.e., for which there exists a number $t' \in [0,1]$ with the condition $(*)$ $h(K \times \{t'\}) \setminus K \neq \emptyset$.

First we show that there are $t_0 > 0$ such that $h(K \times [0,t_0]) \subset \overline{N(K, \frac{\epsilon}{2})}$ and $(p_0, t') \in K \times [0,t_0]$ such that $h(p_0,t') = q \in X \setminus K$. For each point $p \in K$, let $t_p = sup\{t \in [0,1] : h(\{p\} \times [0,t]) \subset N(K, \frac{\epsilon}{2})\}$, and let $t_0 = inf\{t_p : p \in K\}$. We claim that $t_0 > 0$. In fact, suppose $t_0 = 0$. Then there is a sequnce $\{p_n\}_{n=1}^{\infty}$ of points of K such that the sequence $\{t_{p_n}\}_{n=1}^{\infty}$ of corresponding numbers contains a subsequence $\{t_{p_{n_k}}\}_{k=1}^{\infty}$ which converges to zero. It follows from the compactness of K that the sequence $\{p_n\}_{n=1}^{\infty}$ contains a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ which converges to $p_0 \in K$. Hence by the definition of t_0 if $t_p < 1$, then $h(p, t_p) \in Fr(N(K, \frac{\epsilon}{2}))$. Since the sequence $\{t_{p_{n_k}}\}_{k=1}^{\infty}$ converges to zero as a subsequence of $\{t_{p_n}\}_{n=1}^{\infty}$, we may assume that $t_{p_{n_k}} < 1$ for sufficiently large k, and hence $h(p_{n_k}, t_{p_{n_k}}) \in Fr(N(K, \frac{\epsilon}{2}))$. Since the map h is continuous and the set $Fr(N(K, \frac{\epsilon}{2}))$ is closed, we conclude that $h(p_0, 0) \in Fr(N(K, \frac{\epsilon}{2}))$. But since h is a deformation we have $h(p_0, 0) = p_0$, whence $p_0 \in Fr(N(K, \frac{\epsilon}{2}))$. which contradicts $p_0 \in K$. Thus the inequality $t_0 > 0$ is established. It follows now from the definition of t_0 that $h(K \times [0, t_0]) \subset \overline{N(K, \frac{\epsilon}{2})}$.

Secondly we claim that there is a number $t' \in [0, t_0]$ such that the condition $(*)$ is satisfied. To see it, if $t_0 = 1$, then $h(K \times [0, 1]) \subset \overline{N(K, \frac{\epsilon}{2})}$ so that there exists $t' \in [0, 1]$ such that $(*)$ is satisfied. Therefore, there exists $p_0 \in K$ such that $h(p_0, t') = q \in \overline{N(K, \frac{\epsilon}{2})} \cap (X \setminus K)$. If $t_0 < 1$, then we take $t' = t_0$. Then there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of points of K such that $t_0 < t_{p_n} < 1$ and $t_0 = lim_{n \to \infty} t_{p_n}$. As before, let $\{p_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{p_n\}_{n=1}^{\infty}$ which converges to $p_0 \in K$. Applying once more the same arguments as above, we get $h(p_{n_k}, t_{p_{n_k}}) \in Fr(N(K, \frac{\epsilon}{2}))$ for almost all k. Hence $h(p_0, t_0) \in Fr(N(K, \frac{\epsilon}{2}))$ and thus the point $h(p_0, t_0) \notin K$. So let $t' \in [0, t_0]$ be any number such thet the condition $(*)$ holds and let $p_0 \in K$ such that $h(p_0, t') \notin K$. Put $h(p_0, t') = q$ and define $t'_0 = inf\{t' \in [0, t_0] : h(p_0, t') = q\}$. Then by the continuity of h, we have $(**)$ $h(p_0, t'_0) = q \in X \setminus K$. Now let K be an R^1-set and $p_0 \in K$ and $t'_0 \in [0, t']$ which satisfy the condition $(**)$. Let $\{p^i_n\}_{n=1}^{\infty}, p^i_n \in C^i_{k_n}, i = 1, 2$, be sequences such that $lim_{n \to \infty} p^i_n = p_0$, for such $i = 1, 2$. Let $q^i_n = h(p^i_n, t'_0)$ for $i = 1, 2$. Then $lim_{n \to \infty} q^i_n = q$ for each $i = 1, 2$. By the definitions of q and q^i_n, $d(q, K) < \frac{\epsilon}{2}$ for sufficiently large n for each $i = 1, 2$. By the continuity of h and by the connectedness of the sets $\{p^i_n\} \times [0, t'_0], i = 1, 2$, the sets $h(p^i_n, [0, t'_0]), i = 1, 2$, are connected and $Lt(h(\{p^i_n\} \times [0, t'_0])) = h(\{p_0\} \times [0, t'_0]) \subset \overline{N(K, \frac{\epsilon}{2})}$, for $i = 1, 2$. So for a sufficiently large n, $h(\{p^i_n\} \times [0, t'_0]) \subset N(K, \epsilon)$. Since $p^i_n = h(p^i_n, 0) \in C^i_{k_n}$ and $h(\{p^i_n\} \times [0, t'_0])$ is a connected subset of $N(K\ \epsilon)$, $q^i_n \in C^i_{k_n}$ and $q = lim_{n \to \infty} q^i_n \in K$, but $q \in X \setminus K$. This is a contradiction. This proves the theorem for R^1-set.

We can also prove similarly that it is true for other R^i-set. □

COROLLARY 3.3. *If a metric continuum contains an R^i-set, then it is not contractible.*

We prove next that if a metric continuum contains an R^i-set, so do its hyperspaces.

For a subset D of X, let $C(D) = \{A \in C(X) : A \subset D\}$ and $2^D = \{A \in 2^X : A \subset D\}$.

The following Lemma is already known.

LEMMA 3.4. *If $\mathcal{D}$ is a connected subset of 2^X such that $\mathcal{D} \cap C(X) \neq \emptyset$, then $\cup\mathcal{D}$ is connected. In particular, if $\mathcal{D}$ is a connected subset of $C(X)$, then $\cup\mathcal{D}$ is connected.*

LEMMA 3.5. *Let X be a metric continuum. Let U be an open set in X and let C be a component of U. Then $2^U = \{A \in 2^X : A \subset U\}$ is open in 2^X and $2^C = \{A \in 2^X : A \subset C\}$ is a component of 2^U. Also $C(U) = \{A \in C(X) : A \subset U\}$ is open in the space $C(X)$ and $C(C) = \{A \in C(X) : A \subset C\}$ is a component of $C(U)$.*

Proof. Let $A \in 2^U$. Since $A \subset U$, there is an $\epsilon > 0$ such that the ϵ-neighborhood $N(A, \epsilon)$ of A is contained in U. Let $\mathcal{O}$ be the ϵ-neighborhood of A in 2^X. Then for each $B \in \mathcal{O}$ we have $B \subset N(A, \epsilon)$ so that $B \in 2^U$. Hence 2^U is open in 2^X. In similar manner, one can show that $C(U)$ is open in $C(X)$.

First we show that $C(C)$ is connected. Let $C^* = \{\{x\} : x \in C\}$. Then C and C^* are homeomorphic and $C^* \subset C(C)$. Let $A \in C(C)$ and $a \in A$. Let α_A be an order arc by (1.11) in [9] from $\{x\}$ to A. Then $\alpha_A \subset C(C)$. And $C(C) = \cup\{\alpha_A : A \in C(C)\} \cup C^*$ is connected. Now let $\mathcal{C}$ be the component of $C(U)$ containing $C(C)$. Let $A \in \mathcal{C}$. Since $\cup\mathcal{C}$ is a connected subset of U by the above Lemma which contains C and C being a component of U, $A \subset \cup\mathcal{C} = C$. So that $A \in C(C)$. This shows that $C(C)$ is $\mathcal{C}$.

In order to show that 2^C is a component of 2^U, we use the connectedness of 2^C by Theorem 4.10 in [8]. So if $\mathcal{D}$ is the component of 2^U containing 2^C and $A \in \mathcal{D}$, then $\cup 2^C$ is connected subset of U by the above Lemma and hence $\cup\mathcal{D} \subset C$. Therefore $A \subset \cup\mathcal{D}$ implies that $A \in 2^C$. 2^C is a component of 2^U. □

THEOREM 3.6. *If a continuum X contains an R^i-set, then 2^X contains an R^i-set for each $i \in \{1, 2, 3\}$.*

Proof. Let $K = LsC_n^1 \cap LsC_n^2 \subset U$ be an R^1-set, where U is an open set and C_n^i are components of U. Then 2^U is an open set in 2^X and $2^{C_n^i}$ are components of 2^U for each $i = 1, 2$. Let $\mathcal{K} = Ls2^{C_n^1} \cap Ls2^{C_n^2}$. Since $\{x\} \in \mathcal{K}$ for each $x \in K$, $\mathcal{K}$ is nonempty. Also it is closed. Since each $A \in \mathcal{K}$ is a closed subset of K, $\mathcal{K} \subset 2^U$. Hence it is an R^1-set.

If K is an R^2-set, then $\mathcal{K}$ defined as above is an R^1-set and thus 2^X contains an R^2-set by Theorem 2.4.

If $K = LiC_n \subset U$ is an R^3-set, then $\mathcal{K} = Li2^{C_n}$ is an R^3-set. □

THEOREM 3.7. *If a metric continuum X contains an R^i-set, $i \in \{1, 2, 3\}$, then $C(X)$ contains an R^i-set for $i \in \{1, 2, 3\}$, respectively.*

Proof. Let $K = LsC_n^1 \cap LsC_n^2 \subset U$ be an R^1-set of X. Then $C(C_n^i)$ are components of $C(U)$. Let $\mathcal{K} = LsC(C_n^1) \cap LsC(C_n^2)$. Then $K^* = \{\{x\} : x \in K\} \subset \mathcal{K}$ and $\mathcal{K}$ is closed. Let $A \in \mathcal{K}$ and let $\{A_{n_j}^i\}$, $A_{n_j}^i \in C(C_{n_j}^i), i = 1, 2$, be sequences such that $lim_{j\to\infty} A_{n_j}^i = A$, for each $i = 1, 2$. Then $A \subset K$ so that $A \in C(U)$. Hence $\mathcal{K} \subset C(U)$. It is an R^1-set of $C(X)$. The proofs for other R^i-set are similar. □

COROLLARY 3.8. *If a metric continuum X contains an R^i-set, then 2^X and $C(X)$ are not contractible.*

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