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THE MEAN-SQUARE ERROR BOUNDS FOR THE GAUSSIAN QUADRATURE OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we present the L^2-estimation for the kernel K_n of the remaider term for the Gaussian quadrature with respect to one of four Chebyshev weight functions and the error bound of the type on the contour

$$|R_n(f)| \le \frac{\sqrt{l(\Gamma)}}{2\pi} \max_{z\in\Gamma} |f(z)| \left(\int_\Gamma |K_n(z)|^2 |dz|\right)^{\frac{1}{2}},$$

where $l(\Gamma)$ denotes the length of the contour Γ.

1. Introduction

Consider the Gaussian quadrature with respect to the nonnegative weight function $w(x)$ defined on the interval $[-1,1]$ such that the moments $\int_{-1}^1 x^k w(x)\,dx$ exist for $k = 0,1,2,\cdots$. If we apply the residue theorem to the contour integral

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)\pi_n(x)}{(z-x)\pi_n(z)}\,dz,$$

where Γ is the closed contour which contains the simple zeros of orthogonal polynomial $\pi_n(z)$, and integrate with respect to the $w(x)$ on $[-1,1]$, we get the formula

$$(1.1)\qquad \int_{-1}^1 f(t)w(t)\,dt = \sum_{k=1}^n \lambda_k f(\tau_k) + R_n(f),$$

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where τ_k are the zeros of the nth degree orthogonal polynomial $\pi_n(\cdot : w)$ on $[-1,1]$ and λ_k are the corresponding Christoffel numbers which are defined by

$$\lambda_k = \int_{-1}^{1} \frac{\pi_n(t)}{(t-\tau_k)\pi_n'(\tau_k)} w(t)\,dt.$$

If f is a single-valued analytic function in a domain D which contains $[-1,1]$ and if Γ is a closed contour in D surrounding $[-1,1]$, then the remainder term $R_n(\cdot)$ can be represented as a contour integral

$$R_n(f) = \frac{1}{2\pi i}\int_\Gamma K_n(z) f(z)\,dz, \tag{1.2}$$

where the kernel K_n is given by

$$K_n(z) = R_n\left(\frac{1}{z-\cdot}\right), \tag{1.3}$$

or, alternatively, by

$$K_n(z) = \frac{\rho_n(z)}{\pi_n(z)}. \tag{1.4}$$

Here, $\pi_n(z)$ is the orthogonal polynomial $\pi_n(\cdot : w)$ evaluated at z, while $\rho_n(z)$ is known as the function of the second kind which is defined by

$$\rho_n(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t} w(t)\,dt. \tag{1.5}$$

See, e.g.,[3].
Gautschi and Varga [5] have shown the error estimates in the following form on the circle and ellipse

$$|R_n(f)| \le \frac{l(\Gamma)}{2\pi} \max_{z\in\Gamma} |K_n(z)| \max_{z\in\Gamma} |f(z)|, \tag{1.6}$$

where $l(\Gamma)$ denotes the length of Γ. And they [5] have given an explicit representation for the kernel K_n on Γ, and from these determined the maximum points on the circle and ellipse for a Jacobi measure $w(t) =$

$(1-t)^\alpha(1+t)^\beta$ for arbitrary $\alpha > -1, \beta > -1$. Martin and Stamp [8] have shown an explicit expression for kernel $K_n(z)$ by the method of the Laurent series expansion. They have given methods for computing the coefficients (in terms of the moments) for the Laurent series of $K_n(z)$. Our goal is to get the other estimate with the following form

$$
\begin{aligned}
|R_n(f)| &\le \frac{1}{2\pi}\int_\Gamma |K_n(z)||f(z)||dz| \\
&\le \frac{\sqrt{l(\Gamma)}}{2\pi}\max_{z\in\Gamma}|f(z)|\left(\int_\Gamma |K_n(z)|^2|dz|\right)^{\frac{1}{2}},
\end{aligned}
\tag{1.7}
$$

that is , for a Jacobi measure $w(t) = (1-t)^\alpha(1+t)^\beta$ to find the L^2 norm for the kernel and to determine error bound on the ellipse. In section 2 we find the L^2 norm of the kernel $K_n(z)$ and obtain error bound on the elliptic contour with respect to one of four Chebyshev weight functions. In section 3 we give an example.

2. Main results

In this section we consider contour Γ as an elliptic contour which is defined by

$$
\Gamma = \{z : z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}), \quad 0 \le \theta \le 2\pi\}, \quad \rho > 1.
$$

2.1 Chebyshev measures of the first kind

For the weight function $w(t) = (1-t^2)^{-\frac{1}{2}}$, we know that the orthogonal polynomial is the Chebyshev polynomial T_n of the first kind,

$$
T_n(z) = \frac{1}{2}(u^n + u^{-n}), \qquad z = \frac{1}{2}(u+u^{-1}). \tag{2.1.1}
$$

Furthermore, one finds

(2.1.2)

$$
\int_{-1}^{1}\frac{T_n(t)}{z-t}(1-t^2)^{-\frac{1}{2}}\,dt = \int_0^\pi \frac{\cos n\theta}{z-\cos\theta}\,d\theta = \frac{\pi}{\sqrt{z^2-1}}(z-\sqrt{z^2-1})^n,
$$

hence we have the functions of the second kind

$$(2.1.3)\qquad \int_{-1}^{1}\frac{T_n(t)}{z-t}(1-t^2)^{-\frac{1}{2}}\,dt=\frac{2\pi}{(u-u^{-1})u^n},\qquad z=\frac{1}{2}(u+u^{-1}).$$

It follows that

$$(2.1.4)\quad K_n(z)=\frac{\rho_n(z)}{\pi_n(z)}=\frac{4\pi}{(u-u^{-1})u^n(u^n+u^{-n})},\qquad z=\frac{1}{2}(u+u^{-1}).$$

We know that

$$(2.1.5)\qquad |K_n(z)|^2=\frac{4\pi^2}{\rho^{2n}}\frac{1}{[a_2(\rho)-\cos 2\theta][a_{2n}(\rho)+\cos 2n\theta]},\qquad z\in\Gamma$$

where

$$(2.1.6)\qquad a_j(\rho)=\frac{1}{2}(\rho^j+\rho^{-j}),\quad j=1,2,3,\cdots,\quad \rho>1.$$

For the detail, we refer to [5].

THEOREM 1. *If $w(t)=(1-t^2)^{-\frac{1}{2}}$ on $(-1,1)$, then*

$$(2.1.7)\qquad \int_\Gamma |K_n(z)|^2|dz|\le\frac{16\pi^3}{(\rho-\frac{1}{\rho})(\rho^{4n}+1)},\qquad \rho>1.$$

Proof. Since $z=\frac{1}{2}(\rho e^{i\theta}+\rho^{-1}e^{-i\theta})$, we get $|dz|\le\frac{1}{2}(\rho+\rho^{-1})\,d\theta$, and the L^2 bound for the kernel K_n

$$\int_\Gamma |K_n(z)|^2|dz|\le\frac{2\pi^2(\rho+\rho^{-1})}{\rho^{2n}}\int_0^{2\pi}\frac{1}{[a_2(\rho)-\cos 2\theta][a_{2n}(\rho)+\cos 2n\theta]}\,d\theta.$$

Setting $t=e^{i\theta}$, we see that $dt=it\,d\theta$ and

$$\int_0^{2\pi}\frac{1}{[a_2(\rho)-\cos 2\theta][a_{2n}(\rho)+\cos 2n\theta]}\,d\theta$$
$$=-\frac{4}{i}\int_C\frac{t^{2n+1}}{(t^4-2a_2(\rho)t^2+1)(t^{4n}+2a_{2n}(\rho)t^{2n}+1)}\,dt,$$

where C is the unit circle, and applying the residue theorem, we get

$$-\frac{4}{i}\int_C \frac{t^{2n+1}}{(t^2-\rho^2)(t^2-\rho^{-2})(t^{2n}+\rho^{2n})(t^{2n}+\rho^{-2n})}\,dt$$
$$= -\frac{4}{i}2\pi i\left[\operatorname{Res}(f,\rho^{-1}) + \operatorname{Res}(f,-\rho^{-1}) + \sum_{j=1}^{2n}\operatorname{Res}(f,-\rho^{-1}w_j)\right], \quad w_j^{2n} = -1$$

where $\operatorname{Res}(f,x)$ denotes the residue value of f at x.
To calculate the value $\sum_{j=1}^{2n}\operatorname{Res}(f,-\rho^{-1}w_j)$, let $t^{2n} + (\frac{1}{\rho})^{2n} = (t-z_1)(t-z_2)\cdots(t-z_{2n})$, then we have

$$\operatorname{Res}(f,-\frac{1}{\rho}w_j) = g(z_j)\frac{1}{(z_j-z_1)\cdots(z_j-z_{j-1})(z_j-z_{j+1})\cdots(z_j-z_{2n})},$$

where $g(t) = \frac{t^{2n+1}}{(t^2-\rho^2)(t^2-\frac{1}{\rho^2})(t^{2n}+\rho^{2n})}$. □

LEMMA 1. *It holds that*

$$\prod_{\substack{1\le i\le 2n\\ i\ne j}}(z_j - z_i) = 2nz_j^{2n-1}.$$

Proof. Let $w_j^{2n} = -1$ and $z_j = \frac{1}{\rho}w_j$, since

$$t^{2n} + (\frac{1}{\rho})^{2n} = t^{2n} - (\frac{1}{\rho}w_j)^{2n} = t^{2n} - z_j^{2n}$$
$$= (t-z_j)(t^{2n-1} + t^{2n-2}z_j + \cdots + z_j^{2n-1}),$$

it follows that

$$(t-z_1)(t-z_2)\cdots(t-z_j)\cdots(t-z_{2n})$$
$$= (t-z_j)(t^{2n-1} + t^{2n-2}z_j + \cdots + z_j^{2n-1}).$$

Cancelling $(t-z_j)$ on both sides, one gets

$$(t-z_1)\cdots(t-z_{j-1})(t-z_{j+1})\cdots(t-z_{2n})$$
$$= (t^{2n-1} + t^{2n-2}z_j + \cdots + z_j^{2n-1}),$$

and inserting t into z_j, we have

$$(z_j - z_1)\cdots(z_j - z_{j-1})(z_j - z_{j+1})\cdots(z_j - z_{2n}) = 2nz_j^{2n-1}. \qquad \square$$

Therefore we calculate the residue of f at $-\frac{1}{\rho}w_j$ in the form

$$\mathrm{Res}(f, -\frac{1}{\rho}w_j) = \frac{1}{2n(\rho^{2n} - \frac{1}{\rho^{2n}})}\left[\frac{z_j^2}{(z_j^2 - \rho^2)(z_j^2 - \frac{1}{\rho^2})}\right].$$

We have the residue by elementary calculation;

$$\mathrm{Res}(f, \frac{1}{\rho}) = \mathrm{Res}(f, -\frac{1}{\rho}) = -\frac{1}{4}\frac{1}{(\rho^2 - \frac{1}{\rho^2})(\rho^{2n} + \frac{1}{\rho^{2n}})}.$$

Now we have to calculate summation;

$$\frac{1}{2n(\rho^{2n} - \frac{1}{\rho^{2n}})}\sum_{j=1}^{2n}\frac{z_j^2}{(z_j^2 - \rho^2)(z_j^2 - \frac{1}{\rho^2})}, \quad z_j = \frac{1}{\rho}w_j, w_j^{2n} = -1.$$

LEMMA 2. *It holds that*

$$\sum_{j=1}^{2n}\frac{1}{w_j - x} = \frac{-2nx^{2n-1}}{1 + x^{2n}}, \quad w_j^{2n} = -1.$$

Proof. We can write in the form

$$x^{2n} + 1 = (x - w_1)(x - w_2)\cdots(x - w_{2n}).$$

If we differentiate both sides with respect to x, we have

$$\begin{aligned} 2nx^{2n-1} &= (x - w_2)\cdots(x - w_{2n}) + (x - w_1)(x - w_3)\cdots(x - w_{2n}) + \cdots \\ &\quad + (x - w_1)\cdots(x - w_{2n-1}). \end{aligned}$$

Dividing by the first equation, we have

$$\frac{2nx^{2n-1}}{1 + x^{2n}} = \sum_{j=1}^{2n}\frac{1}{x - w_j}.$$

It follows that

$$\sum_{j=1}^{2n} \frac{1}{w_j^2 - x^2} = \sum_{j=1}^{2n} \frac{1}{2x}\left(\frac{1}{w_j - x} - \frac{1}{w_j + x}\right) = \frac{-2nx^{2n-2}}{1 + x^{2n}}, \quad w_j^{2n} = -1.$$

Therefore for $z_j = \frac{1}{\rho} w_j$ and $w_j^{2n} = -1$, we have

$$\sum_{j=1}^{2n} \frac{z_j^2}{(z_j^2 - \rho^2)(z_j^2 - \frac{1}{\rho^2})} = \rho^2 \sum_{j=1}^{2n} \frac{w_j^2}{(w_j^2 - \rho^4)(w_j^2 - 1)}$$

$$= \rho^2 \sum_{j=1}^{2n}\left(\frac{a}{w_j^2 - \rho^4} + \frac{b}{w_j^2 - 1}\right), \quad a = \frac{\rho^4}{\rho^4 - 1}, b = \frac{1}{1 - \rho^4}.$$

We use Lemma 2, then it follows that

$$\sum_{j=1}^{2n} \frac{z_j^2}{(z_j^2 - \rho^2)(z_j^2 - \frac{1}{\rho^2})} = -\frac{n\rho^2(\rho^{2n} - \frac{1}{\rho^{2n}})}{(\rho^4 - 1)(\rho^{2n} + \frac{1}{\rho^{2n}})}.$$

So we have the residue;

$$\mathrm{Res}(f, \frac{1}{\rho}) + \mathrm{Res}(f, -\frac{1}{\rho}) + \sum_{j=1}^{2n} \mathrm{Res}(f, -\frac{1}{\rho} w_j) = -\frac{1}{(\rho^2 - \frac{1}{\rho^2})(\rho^{2n} + \frac{1}{\rho^{2n}})}.$$

We have proved the L^2 bound for the kernel K_n

$$\int_\Gamma |K_n(z)|^2 |dz| \leq \frac{16\pi^3}{(\rho - \frac{1}{\rho})(\rho^{4n} + 1)}, \quad \rho > 1.$$

□

Now we have to estimate error bound

$$|R_n(f)| \leq \frac{\sqrt{l(\Gamma)}}{2\pi} \max_{z \in \Gamma} |f(z)| \left(\int_\Gamma |K_n(z)|^2 |dz|\right)^{\frac{1}{2}}$$

Since the ellipse Γ has the length $l(\Gamma) = 4\epsilon^{-1} E(\epsilon)$, where

$$\epsilon = \frac{2}{\rho + \rho^{-1}}, \quad E(\epsilon) = \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \sin^2\theta}\, d\theta,$$

we have the error bound for the analytic function on the elliptic contour

$$|R_n(f)| \le 4\sqrt{\epsilon^{-1}E(\epsilon)}\sqrt{\frac{\pi}{(\rho-\frac{1}{\rho})(\rho^{4n}+1)}}\max_{z\in\Gamma}|f(z)|. \tag{2.1.8}$$

2.2 Chebyshev measures of the Second Kind

For the Chebyshev measure of the second kind, i.e., $w(t)=(1-t^2)^{\frac{1}{2}}$, the nth degree orthogonal polynomial is

$$U_n(z)=\frac{u^{n+1}-u^{-(n+1)}}{u-u^{-1}}, \qquad z=\frac{1}{2}(u+u^{-1}) \tag{2.2.1}$$

while

$$\int_{-1}^{1}\frac{U_n(t)}{z-t}(1-t^2)^{\frac{1}{2}}\,dt=\frac{\pi}{u^{n+1}}, \quad z=\frac{1}{2}(u+u^{-1}). \tag{2.2.2}$$

We know that

$$|K_n(z)|^2=\frac{\pi^2}{\rho^{2n+2}}\frac{a_2(\rho)-\cos 2\theta}{a_{2n+2}(\rho)-\cos 2(n+1)\theta}, \quad z\in\Gamma \tag{2.2.3}$$

where

$$a_j(\rho)=\frac{1}{2}(\rho^j+\rho^{-j}), \quad j=1,2,3,\cdots, \quad \rho>1. \tag{2.2.4}$$

For the detail, we refer to [5].

THEOREM 2. *If* $w(t)=(1-t^2)^{\frac{1}{2}}$ *on* $(-1,1)$, *then*

$$\int_\Gamma |K_n(z)|^2|dz| \le \frac{\pi^3(\rho+\rho^{-1})(\rho^2+\frac{1}{\rho^2})}{(\rho^{4n+4}-1)}, \quad \rho>1. \tag{2.2.5}$$

Proof. For $z \in \Gamma$, we get $|dz| \le \frac{1}{2}(\rho+\rho^{-1})d\theta$, and the L^2 bound for the kernel K_n

$$\int_\Gamma |K_n(z)|^2|dz| \le \frac{\pi^2(\rho+\rho^{-1})}{2\rho^{2n+2}} \int_0^{2\pi} \frac{a_2(\rho)-\cos 2\theta}{a_{2n+2}(\rho)-\cos 2(n+1)\theta}\, d\theta.$$

Setting $t = e^{i\theta}$, we see that $dt = itd\theta$ and

$$\int_0^{2\pi} \frac{a_2(\rho)-\cos 2\theta}{a_{2n+2}(\rho)-\cos 2(n+1)\theta}\, d\theta = \frac{1}{i}\int_C \frac{t^{2n-1}(t^4-2a_2(\rho)t^2+1)}{t^{4n+4}-2a_{2n+2}(\rho)t^{2n+2}+1}\, dt,$$

where C is the unit circle, and applying the residue theorem, we get

$$\begin{aligned}
&\frac{1}{i}\int_C \frac{t^{2n-1}(t^2-\rho^2)(t^2-\rho^{-2})}{(t^{2n+2}-\rho^{2n+2})(t^{2n+2}-\rho^{-(2n+2)})}\, dt \\
&= \frac{1}{i}2\pi i\left[\sum_{j=1}^{2n+2} \mathrm{Res}(f, \frac{1}{\rho}w_j)\right], w_j^{2n+2} = 1
\end{aligned}$$

where $\mathrm{Res}(f,x)$ denotes the residue value of f at x.
To calculate the value $\sum_{j=1}^{2n+2}\mathrm{Res}(f,\frac{1}{\rho}w_j)$, let $t^{2n+2}-(\frac{1}{\rho})^{2n+2} = (t-z_1)(t-z_2)\cdots(t-z_{2n+2})$, then we have

$$\mathrm{Res}(f,\frac{1}{\rho}w_j) = g(z_j)\frac{1}{(z_j-z_1)\cdots(z_j-z_{j-1})(z_j-z_{j+1})\cdots(z_j-z_{2n+2})},$$

where $g(t) = \frac{t^{2n-1}(t^2-\rho^2)(t^2-\rho^{-2})}{t^{2n+2}-\rho^{2n+2}}$.
Since $\prod_{\substack{1\le i\le 2n+2\\ i\ne j}}(z_j-z_i) = (2n+2)z_j^{2n+1}$, we have

$$\mathrm{Res}(f,\frac{1}{\rho}w_j) = \frac{z_j^{2n-1}(z_j^2-\rho^2)(z_j^2-\rho^{-2})}{(2n+2)z_j^{2n+1}(z_j^{2n+2}-\rho^{2n+2})},$$

and

$$\sum_{j=1}^{2n+2}\mathrm{Res}(f,\frac{1}{\rho}w_j) = \frac{1}{(2n+2)(\frac{1}{\rho^{2n+2}}-\rho^{2n+2})}\sum_{j=1}^{2n+2}\frac{(z_j^2-\rho^2)(z_j^2-\rho^{-2})}{z_j^2}.$$

Therefore for $z_j = \frac{1}{\rho} w_j$ and $w_j^{2n+2} = 1$, we have

$$\sum_{j=1}^{2n+2} [z_j^2 - (\rho^2 + \frac{1}{\rho^2}) - \frac{1}{z_j^2}] = -(2n+2)(\rho^2 + \frac{1}{\rho^2}),$$

since $\sum_{j=1}^{2n+2} w_j^2 = \sum_{j=1}^{2n+2} \frac{1}{w_j^2} = 0$. We have proved the L^2 bound for the kernel K_n

$$\int_\Gamma |K_n(z)|^2 |dz| \le \frac{\pi^3(\rho + \rho^{-1})(\rho^2 + \frac{1}{\rho^2})}{(\rho^{4n+4} - 1)}.$$

□

2.3 Chebyshev Measures of the Type $\alpha = -\beta = -\frac{1}{2}$

Let $\alpha = -\beta = -\frac{1}{2}$. The orthogonal polynomial in this case is given by

$$p_n(z) = \frac{u^{n+1} + u^{-n}}{u+1}, \qquad z = \frac{1}{2}(u + u^{-1}). \tag{2.3.1}$$

Furthermore

$$\int_{-1}^{1} \frac{p_n(t)}{z-t} \sqrt{\frac{1+t}{1-t}}\, dt = \int_0^\pi \frac{\cos(n+1)\theta + \cos n\theta}{z - \cos\theta}\, d\theta$$

$$= \frac{2\pi(u+1)}{(u - u^{-1})u^{n+1}}, \qquad z = \frac{1}{2}(u + u^{-1}). \tag{2.3.2}$$

We know that
(2.3.3)

$$|K_n(z)|^2 = \frac{4\pi^2}{\rho^{2n+1}} \frac{(a_1(\rho) + \cos\theta)^2}{[a_2(\rho) - \cos 2\theta][a_{2n+1}(\rho) + \cos(2n+1)\theta]}, \quad z \in \Gamma$$

where

$$a_j(\rho) = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j = 1, 2, 3, \cdots, \quad \rho > 1. \tag{2.3.4}$$

For the detail, we refer to [5].

THEOREM 3. *If* $w(t) = \sqrt{\frac{1+t}{1-t}}$ *on* $(-1,1)$, *then*

$$\int_\Gamma |K_n(z)|^2|dz| \leq \left[\frac{8\pi^3(\rho+\rho^{-1})^2}{(\rho-\rho^{-1})(\rho^{4n+2}+1)} - \frac{4\pi^3(\rho+\rho^{-1})}{(\rho^{4n+2}-1)}\right], \quad \rho > 1. \tag{2.3.5}$$

Proof. For $z \in \Gamma$, we get $|dz| \leq \frac{1}{2}(\rho+\rho^{-1})\,d\theta$, and the L^2 bound for the kernel K_n

$$\int_\Gamma |K_n(z)|^2|dz| \leq \frac{2\pi^2(\rho+\rho^{-1})}{\rho^{2n+1}} \int_0^{2\pi} \frac{(a_1(\rho)+\cos\theta)^2}{(a_2(\rho)-\cos 2\theta)(a_{2n+1}(\rho)+\cos(2n+1)\theta)}\,d\theta.$$

Setting $t = e^{i\theta}$, we see that $dt = itd\theta$ and

$$\int_0^{2\pi} \frac{(a_1(\rho)+\cos\theta)^2}{[a_2(\rho)-\cos 2\theta][a_{2n+1}(\rho)+\cos(2n+1)\theta]}\,d\theta$$
$$= -\frac{1}{i}\int_C \frac{t^{2n}(t^2+2a_1(\rho)t+1)^2}{(t^4-2a_2(\rho)t^2+1)(t^{4n+2}+2a_{2n+1}(\rho)t^{2n+1}+1)}\,dt,$$

where C is the unit circle, and applying the residue theorem, we get

$$-\frac{1}{i}\int_C \frac{t^{2n}(t+\rho)(t+\frac{1}{\rho})}{(t-\rho)(t-\rho^{-1})(t^{2n+1}+\rho^{2n+1})(t^{2n+1}+\rho^{-(2n+1)})}\,dt$$
$$= -\frac{1}{i}2\pi i\left[\operatorname{Res}(f,\rho^{-1}) + \sum_{j=1}^{2n+1}\operatorname{Res}(f,-\rho^{-1}w_j)\right], \quad w_j^{2n+1} = -1$$

where $\operatorname{Res}(f,x)$ denotes the residue value of f at x.
To calculate the value $\sum_{j=1}^{2n+1}\operatorname{Res}(f,-\rho^{-1}w_j)$, let $t^{2n+1}+(\frac{1}{\rho})^{2n+1} = (t-z_1)(t-z_2)\cdots(t-z_{2n+1})$, then we have

$$\operatorname{Res}(f,-\frac{1}{\rho}w_j)$$
$$= g(z_j)\frac{1}{(z_j-z_1)\cdots(z_j-z_{j-1})(z_j-z_{j+1})\cdots(z_j-z_{2n+1})},$$

where $g(t) = \frac{t^{2n}(t+\rho)(t+\frac{1}{\rho})}{(t-\rho)(t-\frac{1}{\rho})(t^{2n+1}+\rho^{2n+1})}$.

Since $\prod_{\substack{1\le i\le 2n+1 \\ i\neq j}}(z_j - z_i) = (2n+1)z_j^{2n}$, we calculate residue of f at $-\frac{1}{\rho}w_j$ in the form

$$\mathrm{Res}(f, -\frac{1}{\rho}w_j) = \frac{1}{(2n+1)(\rho^{2n+1} - \frac{1}{\rho^{2n+1}})}\left[\frac{(z_j+\rho)(z_j+\frac{1}{\rho})}{(z_j-\rho)(z_j-\frac{1}{\rho})}\right].$$

We have the residue by elementary calculation;

$$\mathrm{Res}(f, \frac{1}{\rho}) = -\frac{(\rho+\frac{1}{\rho})}{(\rho-\frac{1}{\rho})(\rho^{2n+1}+\frac{1}{\rho^{2n+1}})}.$$

For $z_j = \frac{1}{\rho}w_j$ and $w_j^{2n+1} = -1$, it follows that

$$\sum_{j=1}^{2n+1}\frac{(z_j+\rho)(z_j+\frac{1}{\rho})}{(z_j-\rho)(z_j-\frac{1}{\rho})} = \sum_{j=1}^{2n+1}\frac{(w_j+\rho^2)(w_j+1)}{(w_j-\rho^2)(w_j-1)}$$

$$= \sum_{j=1}^{2n+1}\left[1+\frac{a}{w_j-\rho^2}+\frac{b}{w_j-1}\right], \quad a = \frac{2\rho^2(\rho^2+1)}{\rho^2-1}, b = \frac{2(1+\rho^2)}{1-\rho^2}.$$

Therefore we get

$$\frac{1}{(2n+1)(\rho^{2n+1}-\frac{1}{\rho^{2n+1}})}\sum_{j=1}^{2n+1}\frac{(z_j+\rho)(z_j+\frac{1}{\rho})}{(z_j-\rho)(z_j-\frac{1}{\rho})}$$

$$= \frac{1}{(\rho^{2n+1}-\frac{1}{\rho^{2n+1}})} - \frac{(\rho+\frac{1}{\rho})}{(\rho-\frac{1}{\rho})(\rho^{2n+1}+\frac{1}{\rho^{2n+1}})},$$

where using $\sum_{j=1}^{2n+1}\frac{1}{w_j-x} = \frac{-(2n+1)x^{2n}}{1+x^{2n+1}}, w_j^{2n+1} = -1$. We have proved the L^2 bound for the kernel K_n

$$\int_\Gamma |K_n(z)|^2|dz|$$

$$\le \frac{2\pi^2(\rho+\rho^{-1})}{\rho^{2n+1}}\int_0^{2\pi}\frac{(a_1(\rho)+\cos\theta)^2}{(a_2(\rho)-\cos 2\theta)(a_{2n+1}(\rho)+\cos(2n+1)\theta)}\,d\theta$$

$$= \left[\frac{8\pi^3(\rho+\rho^{-1})^2}{(\rho-\rho^{-1})(\rho^{4n+2}+1)} - \frac{4\pi^3(\rho+\rho^{-1})}{(\rho^{4n+2}-1)}\right].$$

□

REMARK. The well-known identity for Jacobi polynomials, $\pi_n^{(\beta,\alpha)}(z) = (-1)^n\pi_n^{(\alpha,\beta)}(z)$, implies $|K_n^{(\beta,\alpha)}(z)| = |K_n^{(\alpha,\beta)}(-z)| = |K_n^{(\alpha,\beta)}(-\bar{z})|$. Thus for the case $\alpha = -\beta = \frac{1}{2}$, we have the same L^2 bound (2.3.5) for the kernel K_n.

3. Example

In this section we apply the results in section 2 to obtain the L^2 error estimate for Gaussian quadrature.

EXAMPLE.

$$\text{I)} \int_{-1}^{1} \frac{1}{2-t}\sqrt{\frac{1-t}{1+t}}\,dt, \qquad \text{II)} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{2-t}\,dt.$$

I) We consider $w(t) = \sqrt{\frac{1-t}{1+t}}$ as the Jacobi weight with parameter $\alpha = -\beta = \frac{1}{2}$. To bound f on the elliptic contour Γ, note that

$$|f(z)| = \frac{1}{|2-z|} \le \frac{1}{2-|z|} \le \frac{1}{2-\frac{1}{2}(\rho+\rho^{-1})}, \quad z \in \Gamma$$

Therefore we get the error bound in the form

$$|R_n(f)| \le 2\sqrt{\epsilon^{-1}E(\epsilon)}\sqrt{\frac{2\pi(\rho+\rho^{-1})^2}{(\rho-\rho^{-1})(\rho^{4n+2}+1)} - \frac{\pi(\rho+\rho^{-1})}{(\rho^{4n+2}-1)}}$$
$$\times \frac{1}{2-\frac{1}{2}(\rho+\rho^{-1})}, \tag{3.1}$$

where $\epsilon = \frac{2}{\rho+\rho^{-1}}$, $\quad 1 < \rho < 2+\sqrt{3}$.

II) We consider $w(t) = (1-t^2)^{\frac{1}{2}}$ as the Jacobi weight with parameter $\alpha = \beta = \frac{1}{2}$ and we get the error bound in the form

$$|R_n(f)| \le \sqrt{\epsilon^{-1}E(\epsilon)}\sqrt{\frac{\pi(\rho+\rho^{-1})(\rho^2+\rho^{-2})}{\rho^{4n+4}-1}}\,\frac{1}{2-\frac{1}{2}(\rho+\rho^{-1})}, \tag{3.2}$$

where $\epsilon = \frac{1}{\rho+\rho^{-1}}$, $\quad 1 < \rho < 2+\sqrt{3}$.

Table 3.1 (On the ellipse)

$f(x) = \frac{1}{2-x}$, $w(t) = (1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}}$				
n	bound	rho	True error	supremum norm
5	5.2100(-5)	3.3984	1.4906(-6)	5.4156(-5)
10	1.9254(-10)	3.5558	2.8436(-12)	
15	5.4507(-16)	3.6123	-1.1102(-16)	5.6433(-16)

Table 3.2 (On the ellipse)

$f(x) = \frac{1}{2-x}$, $w(t) = (1-t)^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}$				
n	bound	rho	True error	supremum norm
5	1.2187(-4)	3.4066	1.8543(-6)	1.8257(-4)
10	4.4492(-10)	3.5579	3.5374(-12)	6.5903(-10)
15	1.2544(-15)	3.6133	-6.6613(-16)	1.8507(-15)

We have expressed the error bound as a parameter of ρ. Number in parentheses of Table 3.1 and Table 3.2 indicate decimal exponents. As n incresses, the number ρ approaches to $2+\sqrt{3}$. This is due to the nature of weak singularity of denominator factor. And we have compared with the supremum norm(Gautschi and Varga [5]).

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