J. Korean Math. Soc. **34** (1997), No. 2, pp. 285–291

A FIXED POINT THEOREM REVISITED

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ABSTRACT. In this paper, we obtain an abstract formulation of a fixed point theorem for nonexpansive mappings. Our theorem is a *non-metric* version of Kirk's original theorem.

A fixed point theorem published approximately thirty years ago (Kirk [10]), along with two very similar theorems published at the same time (Browder [5], Göhde [9]), signaled the onset of a flourishing era of research in metric fixed point theory and related Banach space geometry (cf., e.g., [1], [8]). Our purpose here is to re-examine the central ideas of [10] and cast them in a *non-metric* framework.

We begin with a brief description of the result of [10]. The formulation there asserted that if K is a nonempty closed convex subset of a reflexive Banach space (reflexivity is not needed if it is assumed K is weakly compact) and if K has 'normal structure', then every nonexpansive mapping $T : K \to K$ has a fixed point.

The assertion that T is *nonexpansive* means that for each $x, y \in K$, $\|T(x) - T(y)\| \leq \|x - y\|$. The *normal structure* assumption on K, a condition introduced by Brodskii and Milman in [4], means that every convex subset H of K which contains more than one point contains a *nondiametral point*, i.e., a point $x_0 \in H$ exists such that

$$\sup\{\|x_0 - y\| : y \in H\} \leq \sup\{\|x - y\| : x, y \in H\}.$$

In 1977 Penot ([12]) reformulated the result of [10] in a more abstract setting by observing that the original argument carries over if K is

Received September 9, 1994.

1991 Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: nonexpansive mapping, normal structure, convexity structure, generalized semimetric.

The second author was supported in part by Korean Ministry of Education, 1993-1994.

replaced with a bounded metric space possessing a compact normal convexity structure. A family Σ of subsets of a metric space (M, d) is said to be a *convexity structure* if Σ contains the closed balls of M and if Σ is closed under intersections. Σ is *compact* if every subfamily of Σ which has the finite intersection property has nonempty intersection, and *normal* if every member of Σ containing more than one point has a nondiametral point. The advantage of this more abstract formulation is that it permits wider application of the result, specifically to Banach spaces which have topologies relative to which the norm closed balls are compact.

Here we show that an even more abstract formulation is possible. The triangle inequality is dropped from the distance axioms, and rather than the reals, we assume the range of the resulting "distance" function is a linearly ordered set having a smallest element.

Let X be a set and S a linearly ordered set (with its usual order topology) having a smallest element, which we denote 0. We call a mapping $D_S : X \times X \to S$ a *generalized semimetric* if

(1) $D_S(x, y) = 0 \Leftrightarrow x = y$, $(x, y \in X)$;

(2) $D_S(x, y) = D_S(y, x)$, $(x, y \in X)$.

If S is the set of nonnegative reals then, as usual, we replace D_S with D and call D a *semimetric.*

The function D_S generates a topology on X analogous to the one generated by a semimetric in the usual sense. (See e.g., Blumenthal [3; p.10].) A point $p \in X$ is said to be a *limit point* (or *accumulation point*) of a subset E of X if given any $\alpha \in S$, $\alpha \neq 0$, there is a point $q \in E$ such that $D_S(p, q) \in (0, \alpha)$. A set E in X is said to be *closed* iff it contains all of its limit points, and a set U in X is said to be *open* iff $X \backslash U$ is closed. (Of course if $(0, \alpha) = \emptyset$ for some nonzero $\alpha \in S$, then this definition merely generates the discrete topology on S.) If D_S is a continuous mapping from $X \times X$ to S (with the topology on X induced by D_S, then D_S is said to be a *continuous generalized semimetric.* While continuity of D_S is not needed for our central observations, it is relevant to our comments in Remark 4.

For $x \in X$ and $\alpha \in S$ define the *closed ball* $B(x; \alpha)$ centered at x with radius α as usual:

$$B(x; \alpha) = \{u \in X : D_S(x, u) \leq \alpha\}.$$

We say a set $D \subseteq X$ is *spherically bounded* if there exist $x \in X$ and $\alpha \in S$ such that $D \subseteq B(x;\alpha)$. For a such a set $D \subseteq X$ define

$$cov(D) = \cap\{B : B \text{ is a closed ball containing } D\}.$$

Now suppose X has a topology relative to which the closed balls are compact, and let Σ be the family of all "ball intersections" in X, i.e., $D \in \Sigma$ iff $D = \cap_{i \in I} B_i$ where each B_i is a closed ball in X.

LEMMA 1. *Suppose $D \in \Sigma$ is nonempty, suppose $T : D \to D$, and let $D^* \subseteq D$ be minimal with respect to being nonempty, T-invariant, and in Σ. Then $cov(T(D^*)) = D^*$.*

Proof. Since $T(D^*) \subseteq D^*$, and since D^* is the intersection of a family of closed balls in X, it follows that $cov(T(D^*)) \subseteq D^*$. But $cov(T(D^*)) \in \Sigma$ and $T(cov(T(D^*))) \subseteq T(D^*) \subseteq cov(T(D^*))$, so by minimality of D^*, $cov(T(D^*)) = D^*$. □

We now make the additional assumption that S has the *least upper bound* (lub) property. (The lub property is the usual one: Each set in S which is bounded above has a smallest upper bound. Dually, this implies that S has the *greatest lower bound* (glb) property.)

If $D \in \Sigma$ is nonempty and contained in some ball *centered at a point of* D, the set

$$R(D) := \{\alpha \in S : (\cap_{u \in D} B(u;\alpha)) \cap D \neq \emptyset\}$$

is nonempty. Define $r(D) = \text{glb } R(D)$, and let

$$C(D) = \{z \in D : z \in \cap_{u \in D} B(u; r(D))\}.$$

Note that if S is connected relative to its order topology, then S has the lub property.

LEMMA 2. *With D and S as above, if S is connected, then $C(D)$ is a nonempty member of Σ.*

Proof. By assumption, if $\alpha > r(D)$ then

$$C_\alpha(D) := (\cap_{u\in D} B(u;\alpha)) \cap D \neq \emptyset.$$

We show that $C(D) = \cap_{\alpha>r(D)} C_\alpha(D)$. The conclusion will then follow from the compactness of the balls of X. So, suppose $z \in \cap_{\alpha>r(D)} C_\alpha(D)$ and suppose $D_S(u,z) > r(D)$ for some $u \in D$. Then since S is connected there exists $\alpha \in S$ such that $r(D) < \alpha < D_S(u,z)$. But this contradicts the fact that $z \in C_\alpha(D)$. It follows that $C(D) \neq \emptyset$ and that

$$C(D) \supseteq \cap_{\alpha>r(D)} C_\alpha(D).$$

Since the reverse inclusion is immediate, we are done. □

We now assume that *each* nonempty member D of Σ is contained in a ball centered at one of its own points, and we say that Σ is *normal* if for each $D \in \Sigma$, either D is a singleton or $C(D)$ is a proper subset of D. Also, we say that a mapping $T : D \to D$ is *nonexpansive* if $D_S(T(x), T(y)) \leq D_S(x,y)$ for each $x, y \in D$.

The following is the analog to the theorem of [10].

THEOREM 3. *Suppose X is a nonempty set, suppose S is a linearly ordered set which is connected relative to its order topology, and suppose $D_S : X \times X \to S$ is a generalized semimetric. Suppose X has a topology relative to which its closed balls are compact, and suppose the family Σ of ball intersections in X is normal. Then if $\emptyset \neq D \in \Sigma$ and if $T : D \to D$ is nonexpansive, T has a fixed point.*

Proof. Let $D^* \in \Sigma$ be minimal with respect to being a nonempty T-invariant member of Σ. Since the members of Σ are compact, such a set exists by a simple application of Zorn's lemma. By Lemma 1 $cov(T(D^*)) = D^*$of T, $D_S(T(z), T(x)) \leq D_S(z,x) \leq r(D^*)$ from which $T(D^*) \subseteq B(T(z); r(D^*))$. In turn, it follows that $D^* = cov(T(D^*)) \subseteq B(T(z); r(D^*))$, whence $D^* \subseteq B(T(z); r(D^*))$, i.e., $T(z) \in \cap_{u\in D^*} B(u; r(D^*)) = C(D^*)$. This proves $T : C(D^*) \to C(D^*)$. Since $C(D^*) \in \Sigma$ by Lemma 2, minimality of D^* implies $C(D^*) = D^*$, and since Σ is normal, D^* must be a singleton. □

REMARK. 1. The proof of Theorem 3 is similar to its Banach/metric counterpart (Kirk [10], Goebel and Kirk [8]; the metric version is given in Penot [12] and Kirk [11]). Also, although Zorn's Lemma is used in the proof, a constructive proof can be given following the method of Kirk [11] (see also Büber-Kirk [6]).

A question raised in [10] asked whether the result would remain valid without the normal structure assumption. This was answered in the negative by Alspach in 1981 [2].

2. Note in particular that the topology on X in Theorem 3 is not assumed to be the topology generated by D_S. Indeed, in this theorem we do not even assume D_S is continuous.

3. A nontrivial example of a linearly ordered space satisfying the hypotheses of Theorem 3, but which cannot be imbedded in the reals, is the topologists' 'long line' (see, e.g., [13; p. 191]. An easy example of a continuous semimetric which is not a metric is given by letting $X = S = [0,1]$ and defining $D(x,y) = |x-y|^2$, $x, y \in X$.

4. The above setting of this paper can also be used to reformulate a theorem for 'contractive' mappings generally attributed to Edelstein [7; Remark 3.1]. Let (M,d) be a metric space, and recall that a mapping $T : M \to M$ is said to be *contractive* if $d(T(x), T(y)) < d(x,y)$ for all $x, y \in M$, $x \neq y$. Edelstein's result asserts that a contractive mapping defined on a compact metric space always has a unique fixed point, and that the Picard iterates beginning at any point of the space always converge to this fixed point. This fact is a special case of the following.

THEOREM 4. *Let X be a set, let S be a linearly ordered set having a smallest element (denoted 0), and let $D_S : X \times X \to S$ be a continuous generalized semimetric. Suppose (X, D_S) is compact, and suppose $f : X \to X$ satisfies:*

$$D_S(f(x), f(y)) < D_S(x,y) \text{ for } x, y \in X,\ x \neq y.$$

Then f has a unique fixed point $\bar{x} \in X$. If, in addition, S has the lub property, then $D_S(f^n(x), \bar{x}) \to 0$ for each $x \in X$.

Proof. Define $\Psi : X \to S$ by setting $\Psi(x) = D_S(x, f(x))$, $x \in X$, and observe that Ψ is continuous; hence by compactness of X there exists $\bar{x} \in X$ such that $\Psi(\bar{x}) \leq \Psi(x)$ for each $x \in X$. If $\Psi(\bar{x}) \neq 0$, then by (1) $\bar{x} \neq f(\bar{x})$ so

$$\Psi(f(\bar{x})) = D_S(f(\bar{x}), f^2(\bar{x})) < D_S(\bar{x}, f(\bar{x})) = \Psi(\bar{x})$$

and this contradicts minimality of $\Psi(\bar{x})$. Therefore $\Psi(\bar{x}) = 0$ and again by (1), $\bar{x} = f(\bar{x})$. Uniqueness is immediate from the contractive condition.

Now suppose S has the lub property, and let $x \in X$. If $f^n(x) = \bar{x}$ for some n the conclusion is immediate. Otherwise, let

$$u = glb\{D_S(\bar{x}, f^n(x)) : n = 1, 2, \cdots\}.$$

Since $\{D_S(\bar{x}, f^n(x))\}$ is strictly decreasing in S, $D_S(\bar{x}, f^n(x)) \to u$. Also, by compactness, $\{f^n(x)\}$ has a subnet, say $\{f^{n_\alpha}(x)\}$ which converges to a point $z \in X$, from which, by continuity of D_S, $D_S(\bar{x}, z) = u$. Suppose $u \neq 0$. Then $D_S(\bar{x}, f(z)) < u$ and $(0, u)$ is a neighborhood of $D_S(\bar{x}, f(z))$ in S. Since $D_S(\bar{x}, f^{n+1}(x)) \to u$, for α sufficiently large, $D_S(\bar{x}, f^{n+1}(x)) < u$ contradicting the fact that u is a lower bound of the set $\{D_S(\bar{x}, f^n(x)) : n = 1, 2, \cdots\}$. It follows that $D_S(\bar{x}, z) = 0$ and by (1), $z = \bar{x}$. Thus any convergent subnet of $\{f^n(x)\}$ converges to $\bar{x}$ from which the conclusion follows. □

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